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h-Vectors of Generalized Associahedra and Noncrossing Partitions

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A uniform proof is given that the entries of the h-vector of the cluster complex $\Delta(\Phi)$, associated by S. Fomin and A. Zelevinsky to a finite root system Φ, count elements of the lattice $\mathbf{L}$ of noncrossing partitions of corresponding type by rank. Similar interpretations for the h-vector of the positive part of $\Delta(\Phi)$ are provided. The proof utilizes the appearance of the complex $\Delta(\Phi)$ in the context of the lattice $\mathbf{L}$ in recent work of two of the authors, as well as an explicit shelling of $\Delta(\Phi)$.

1 Introduction

Let Φ be a finite root system of rank n with corresponding finite real reflection group W. The cluster complex $\Delta(\Phi)$ was introduced by Fomin and Zelevinsky in the context of algebraic Y-systems [12] and cluster algebras [11, 13]. It is a pure $(n-1)$-dimensional simplicial complex which is homeomorphic to a sphere [12]. When Φ is crystallographic, there is a cluster algebra associated to Φ and $\Delta(\Phi)$ was realized explicitly in [9] as the boundary complex of a simplicial convex polytope, known as a (simplicial) *generalized associahedron*; see [10] for an overview of cluster complexes and generalized associahedra.

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The combinatorics of $\Delta(\Phi)$ encodes the exchange of clusters in the corresponding cluster algebra of finite type. The remarkable enumerative properties of $\Delta(\Phi)$ have provided combinatorialists with many interesting puzzles and have suggested connections with a number of seemingly unrelated objects appearing in various areas of mathematics; see [7, 12] and [10, Lecture 5]. One such object is the lattice of noncrossing partitions associated to W (see [1, 3, 5, 6, 14] and Section 2.1). This is a self-dual graded poset of rank n, denoted by $\mathbf{L}_W$, which is a lattice of considerable interest in the topology of finite-type Artin groups. It encodes the combinatorics of reduced decompositions of Coxeter elements of W with respect to the generating set of all reflections in W. At the core of the connection between $\Delta(\Phi)$ and $\mathbf{L}_W$ lies the fact that both the number of facets of $\Delta(\Phi)$ (clusters) and the number of elements of $\mathbf{L}_W$ are given by the expression

$$N(\Phi) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}, \tag{1.1}$$

known as the *Catalan number* associated to W, where h is the Coxeter number and $e_1, \ldots, e_n$ are the exponents of W. Recall (see Section 2.5 for definitions) that the h-vector of a simplicial complex Δ is a fundamental enumerative invariant which refines the number of facets of Δ. The entries of the h-vector of $\Delta(\Phi)$ are known as the *Narayana numbers* associated to W. When Φ is crystallographic, they are equal to the dimensions of the real cohomology groups of the complex projective toric variety associated to the corresponding generalized associahedron. They admit various interesting combinatorial interpretations and generalizations; see [10, Section 5.2] and the references therein. The following theorem has been verified by use of the classification of finite root systems and case-by-case computations due to various authors; see parts (i) and (ii) of [10, Theorem 5.9].

Theorem 1.1. Let Φ be a finite root system of rank n with corresponding reflection group W. The ith entry $h_i(\Delta(\Phi))$ of the h-vector of $\Delta(\Phi)$ is equal to the number of elements of $\mathbf{L}_W$ of rank i for all $0 \leq i \leq n$.

In particular, the number of facets of $\Delta(\Phi)$ is equal to the total number of elements of $\mathbf{L}_W$. □

One of the challenging problems on cluster combinatorics has been to find a conceptual proof of the previous theorem which makes no reference to the classification of finite root systems. The main objective of this paper is to provide such a proof. We use recent work [6] of two of the authors which provides a structural connection between cluster complexes and noncrossing partitions.

There are two main elements to our approach: (a) a bijection ϕ from the set of facets of $\Delta(\Phi)$ to $\mathbf{L}_W$ and (b) a shelling of $\Delta(\Phi)$ which is compatible with ϕ, in the sense that the number of vertices of the restriction face of any facet F of $\Delta(\Phi)$, with respect to this shelling, is equal to the corank of $\phi(F)$ in $\mathbf{L}_W$. The map ϕ is defined in a uniform manner in Section 5, and the remainder of that section and Section 6 are devoted to proving that ϕ is indeed a bijection. Our definition of ϕ involves partitioning the vertices of each facet F of $\Delta(\Phi)$ into *left vertices* and *right vertices*, the right vertices being used to define $\phi(F)$. This partitioning is shown to have two equivalent characterizations: (i) in terms of the subcomplexes $X(w)$ (for $w \in \mathbf{L}_W$) which is essential in establishing the bijectivity of ϕ (see Theorem 5.2) and (ii) in terms of the restriction sets of facets of our shelling (part (ii) of Theorem 7.3). The choice for our shelling is a variation of the lexicographic ordering on the facets with respect to the total ordering (see Section 2.3) of the vertices of $\Delta(\Phi)$ considered in [6].

The other sections of this paper are organized as follows. Basic definitions and background related to noncrossing partition lattices and cluster complexes are collected in Section 2. In particular, the realization of $\Delta(\Phi)$ given in [6] and other necessary material from that paper are reviewed (and extended where necessary). The description [6, Section 6] of the lexicographically first facet of the complex $X(w)$, with respect to the vertex ordering of [6], is recalled in Section 3 and an analogous description for the lexicographically last facet is provided. These constructions are used in establishing some key properties (Theorem 5.2 and Proposition 5.5) of the map ϕ. Some further definitions and results related to the structure of faces of $\Delta(\Phi)$ and the complexes $X(w)$ are given in Section 4. The proof of Theorem 1.1 is completed in Section 7 by combining bijectivity of ϕ with the explicit shelling of $\Delta(\Phi)$ mentioned above. Some combinatorial interpretations of the entries of the h-vector of the positive part of $\Delta(\Phi)$ are also derived. General background on root systems, cluster complexes, and noncrossing partition lattices can be found in [10] and references therein.

Another conceptual approach to Theorem 1.1 appears in [15, 16]. More precisely, the last statement of the theorem was proven by Reading in [15] by constructing bijections between a certain subset of W and both clusters and noncrossing partitions. Furthermore, the definitions of his bijections do not refer to the classification of root systems. A proof of Theorem 1.1, based on the constructions of [15], has been announced more recently by Reading and Speyer [16].

2 Preliminaries

Throughout this paper W is a finite real reflection group of rank n, generated by orthogonal reflections in $\mathbb{R}^n$. Unless otherwise stated we assume that W is irreducible. The reflection in the linear hyperplane in $\mathbb{R}^n$ orthogonal to a nonzero vector α is written as $R(\alpha)$.

The root system for W which we use is denoted by Φ and consists of the pairs $\{\alpha, -\alpha\}$ of normals to the reflecting hyperplanes. We will further assume that these normals have *unit* length.

2.1 The reflection length order and the lattice $\mathbf{L}_W$

To any $w \in W$ we can associate two linear subspaces of $\mathbb{R}^n$: the fixed space $\mathcal{F}(w)$ of w and the orthogonal complement $\mathcal{M}(w)$ of $\mathcal{F}(w)$ in $\mathbb{R}^n$. The subspace $\mathcal{M}(w)$ is called the *moved space* of w in [6]. The dimension $\ell(w)$ of $\mathcal{M}(w)$ is equal to the smallest integer k such that w can be written as a product of k reflections in W [5, Proposition 2.2]. Note that ℓ is *not* the usual length function associated to a simple system for W. For $a, b \in W$, we let

$$a \preceq b \quad \text{iff } \ell(a) + \ell\left(a^{-1}b\right) = \ell(b). \tag{2.1}$$

The relation $\preceq$ turns W into a graded partially ordered set of rank n, which has the identity I as its unique minimal element and rank function ℓ. The following lemma collects some useful properties of this partial order.

Lemma 2.1. Let $a, b, c \in W$.

(i) If $a \preceq b$, then $\mathcal{M}(a) \subseteq \mathcal{M}(b)$.
(ii) If $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and a is a reflection, then $a \preceq b$.
(iii) If $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and $a, b \preceq w$ for some $w \in W$, then $a \preceq b$.
(iv) If $a, b \preceq c \preceq w$ for some $w \in W$ and $ab \preceq w$, then $ab \preceq c$. □

Proof. For parts (i), (ii), and (iii), see [5, Section 2]. Part (iv) follows from part (iii) since from the assumptions we have $\mathcal{M}(a), \mathcal{M}(b) \subseteq \mathcal{M}(c)$ and hence $\mathcal{M}(ab) \subseteq \mathcal{M}(a) + \mathcal{M}(b) \subseteq \mathcal{M}(c)$. ■

If γ is a Coxeter element of W, then the interval $[I, \gamma]$ in the poset $(W, \preceq)$, denoted by $\mathbf{L}_W(\gamma)$ or $\mathbf{L}(\gamma)$, is a locally self-dual, graded lattice of rank n; see [1, 3, 5, 6] for further information and interesting properties. The isomorphism type $\mathbf{L}_W$ of $\mathbf{L}_W(\gamma)$ is independent of γ and is called the *noncrossing partition lattice* associated to W.

2.2 Coxeter elements and the operator μ

As in [3, 6], we fix an ordered simple system $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for Φ with the property that $\Pi_1 = \{\alpha_1, \ldots, \alpha_s\}$ and $\Pi_2 = \{\alpha_{s+1}, \ldots, \alpha_n\}$ are orthonormal sets for some $1 \leq s \leq n$. We work with the Coxeter element γ given by

$$\gamma = R(\alpha_1)R(\alpha_2)\cdots R(\alpha_n). \tag{2.2}$$

Following [6] we write

$$\mu(x) = 2(I - \gamma)^{-1}(x) \tag{2.3}$$

for $x \in \mathbb{R}^n$. Recall from [6, Corollary 3.3] that for $\tau \in \Phi$, $\mu(\tau)$ is the unique vector in the one-dimensional space $\mathcal{F}(R(\tau)\gamma)$ satisfying $\mu(\tau) \cdot \tau = 1$. The following lemma is implicit in [6].

Lemma 2.2. For nonparallel roots σ, τ, $R(\sigma)R(\tau) \preceq \gamma$ if and only if $\mu(\sigma) \cdot \tau = 0$. □

Proof. The statement $R(\sigma)R(\tau) \preceq \gamma$ is equivalent to $R(\tau) \preceq R(\sigma)\gamma$. By parts (i) and (ii) of Lemma 2.1, this in turn holds if and only if τ lies in the $(n-1)$-dimensional space $\mathcal{M}(R(\sigma)\gamma)$ or, in other words, if and only if $\mu(\sigma) \cdot \tau = 0$. ■

Lemma 2.3 and part (i) of Lemma 2.4 below generalize [6, Theorem 3.2] and [6, Theorem 3.7(a)], respectively, which deal with the case where $w = \gamma$.

Lemma 2.3. If $\tau \in \Phi$ and $w \in W$ satisfy $R(\tau) \preceq w \preceq \gamma$, then $w(\mu(\tau)) = \mu(\tau) - 2\tau$. □

Proof. Observe that $\mu(\tau) \in \mathcal{F}(R(\tau)\gamma) \subseteq \mathcal{F}(w^{-1}\gamma)$, since $w^{-1}\gamma \preceq R(\tau)\gamma$. Hence $w(\mu(\tau)) = \gamma(\mu(\tau)) = \mu(\tau) - 2\tau$. ■

Lemma 2.4. If $\sigma, \tau \in \Phi$ and $w \in W$ satisfy $R(\sigma), R(\tau) \preceq w \preceq \gamma$, then

(i) $w(\mu(\sigma)) \cdot \tau = -\mu(\tau) \cdot \sigma$,

(ii) $w(\mu(\sigma)) \cdot \tau = \mu(w(\sigma)) \cdot \tau$,

(iii) $\mu(w(\sigma)) \cdot w(\tau) = \mu(\sigma) \cdot \tau$. □

Proof. For part (i), we use Lemma 2.3 to write $\tau = (1/2)[\mu(\tau) - w(\mu(\tau))]$ and $\sigma = (1/2)[\mu(\sigma) - w(\mu(\sigma))]$, so that

$$\begin{aligned} w\big(\mu(\sigma)\big) \cdot \tau &= w\big(\mu(\sigma)\big) \cdot \left(\frac{1}{2}\right)\big[\mu(\tau) - w\big(\mu(\tau)\big)\big] \\ &= \left(\frac{1}{2}\right)\big[w\big(\mu(\sigma)\big) \cdot \mu(\tau) - w\big(\mu(\sigma)\big) \cdot w\big(\mu(\tau)\big)\big] \\ &= \left(\frac{1}{2}\right)\big[w\big(\mu(\sigma)\big) \cdot \mu(\tau) - \mu(\sigma) \cdot \mu(\tau)\big] \\ &= \left(\frac{1}{2}\right)\mu(\tau) \cdot \big[w\big(\mu(\sigma)\big) - \mu(\sigma)\big] \\ &= -\mu(\tau) \cdot \sigma. \end{aligned} \tag{2.4}$$

For part (ii), observe that $R(w(\sigma)) \preceq w$ and apply part (i) twice to get

$$\mu\big(w(\sigma)\big) \cdot \tau = -w\big(\mu(\tau)\big) \cdot w(\sigma) = -\mu(\tau) \cdot \sigma = w\big(\mu(\sigma)\big) \cdot \tau. \tag{2.5}$$

Similarly part (iii) follows by applying part (ii) with τ replaced by $w(\tau)$. ■

2.3 The ordering of roots

Let us denote by Φ^+ the positive system of Φ corresponding to the simple system Π and by N the cardinality of Φ^+ and let us use the notation $\Phi_{\geq -1}$ of [12] for the set $\Phi^+ \cup (-\Pi)$ of almost positive roots in Φ.

As in [6], we set $\rho_i = R(\alpha_1)R(\alpha_2)\cdots R(\alpha_{i-1})(\alpha_i)$ for $i \geq 1$, where the α_i are indexed cyclically modulo n (so that $\rho_1 = \alpha_1$ and $\rho_{-i} = \rho_{2N-i}$ for $i \geq 0$) and recall that

$$\begin{aligned} &\big\{\rho_1, \rho_2, \ldots, \rho_N\big\} = \Phi^+, \\ &\big\{\rho_{N+i} : 1 \leq i \leq s\big\} = \big\{-\rho_1, \ldots, -\rho_s\big\} = -\Pi_1, \\ &\big\{\rho_{-i} : 0 \leq i < n-s\big\} = \big\{-\rho_{N-i} : 0 \leq i < n-s\big\} = -\Pi_2. \end{aligned} \tag{2.6}$$

Thus, as in [6], we can consider the total order $<$ of the set $\Phi_{\geq -1}$ defined by

$$\rho_{-n+s+1} < \cdots < \rho_0 < \rho_1 < \cdots < \rho_{N+s}. \tag{2.7}$$

Unless otherwise stated, whenever we talk about the lexicographic order on a collection of subsets of $\Phi_{\geq -1}$, it should be understood that this is defined with respect to the total order (2.7). The set $\{\rho_{N-n+1}, \ldots, \rho_N\}$ of the last n positive roots in this order will be denoted by Ω. The following three lemmas from [6] are listed here for easy reference. Each one concerns properties of the matrix whose ijth entry is $\mu(\rho_i) \cdot \rho_j$. Lemma 2.5 characterizes ordered minimal factorizations of elements of $\mathbf{L}(\gamma)$ via certain upper triangular submatrices. Lemma 2.6 states that the matrix has nonnegative entries above the diagonal and nonpositive entries below the diagonal. Lemma 2.7 involves factorizations of length two elements of $\mathbf{L}(\gamma)$ and says that the angle between corresponding roots is nonobtuse unless the roots are the simple roots for the corresponding dihedral subgroup.

Lemma 2.5 (see [6, Lemma 3.9])**.** For a set $\{\tau_1, \tau_2, \ldots, \tau_k\}$ of positive roots satisfying $\tau_1 < \tau_2 < \cdots < \tau_k$, the following are equivalent:

(a) $R(\tau_k)\cdots R(\tau_1)$ is an element of $\mathbf{L}(\gamma)$ of length k,

(b) $\mu(\tau_j) \cdot \tau_i = 0$ for $1 \leq i < j \leq k$. □

Lemma 2.6 (see [6, Theorem 3.7(b) and (d)]). For $1 \leq i < j \leq N$,

(i) $\mu(\rho_i) \cdot \rho_j \geq 0$,

(ii) $\mu(\rho_j) \cdot \rho_i \leq 0$. □

Lemma 2.7 (see [6, Lemma 5.6]). Let τ, ρ be distinct positive roots with $R(\tau)R(\rho) \preceq \gamma$.

(i) If $\tau < \rho$, then $\tau \cdot \rho \leq 0$.

(ii) If $\tau > \rho$, then $\tau \cdot \rho \geq 0$. □

Remark 2.8. Recall from [6, Section 3] that $\gamma^{-1} = R(\rho_i)R(\rho_{i+1}) \cdots R(\rho_{i+n-1})$ and that $\gamma(\rho_i) = \rho_{i+n}$ for $i \geq 1$. In particular, Ω consists of those positive roots ρ such that $\gamma(\rho)$ is a negative root.

The following technical result gives some information about the action of an arbitrary $w \in \mathbf{L}(\gamma)$ on $\Phi \cap \mathcal{M}(w)$. The action of γ on Φ is very simple, moving roots forward by n in the ordering, whereas for general $w \in \mathbf{L}(\gamma)$ roots in $\Phi \cap \mathcal{M}(w)$ are moved forward and by at least n.

Corollary 2.9. Suppose $w \in \mathbf{L}(\gamma)$.

(i) If $R(\rho) \preceq w$ and ρ and $w(\rho)$ are positive roots, then $\rho < w(\rho)$.

(ii) If $R(\rho_i) \preceq w$ for some $i \geq 1$, then $w(\rho_i) \notin \{\rho_{i+1}, \rho_{i+2}, \ldots, \rho_{i+n-1}\}$.

(iii) If $R(\rho) \preceq w$ and $\rho \in \Omega$, then $w(\rho)$ is a negative root. □

Proof. (i) Using Lemma 2.4 we compute $\mu(w(\rho)) \cdot \rho = -\mu(\rho) \cdot \rho = -1$, so the result follows from part (i) of Lemma 2.6.

(ii) Suppose that $w(\rho_i) = \rho_j$ with $i < j \leq i + n - 1$. Then $R(\rho_j)R(\rho_i) \preceq \gamma$ by Remark 2.8 and hence $\mu(\rho_j) \cdot \rho_i = 0$ by Lemma 2.2. However $\mu(\rho_j) \cdot \rho_i = \mu(w(\rho_i)) \cdot \rho_i = -1$ as in part (i), which gives a contradiction.

(iii) This follows from parts (i) and (ii). ■

2.4 Peripheral elements

For the simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, we recall that a standard parabolic subgroup of W is a subgroup generated by a subset of $\{R(\alpha_1), \ldots, R(\alpha_n)\}$. It will be necessary for us to distinguish between those $\mathbf{L}_W$ elements which lie in one of the proper standard parabolics and those which do not. We will see in Proposition 2.11 that the former elements coincide with the peripheral elements which we now define.

Definition 2.10. An element $w \in \mathbf{L}(\gamma)$ is called peripheral if $w \preceq \gamma'$ for some

$$\gamma' \in \big\{R(\alpha_1)\gamma, \ldots, R(\alpha_s)\gamma,\ \gamma R(\alpha_{s+1}), \ldots, \gamma R(\alpha_n)\big\}. \tag{2.8}$$

Otherwise w is called nonperipheral.

Proposition 2.11. If $w \in \mathbf{L}(\gamma)$, then the following are equivalent:

(i) w is peripheral,

(ii) w lies in a proper standard parabolic subgroup,

(iii) at least one of the roots in Ω belongs to $\mathcal{M}(w^{-1}\gamma)$. □

Proof. (i)⇒(ii). If w is peripheral and $R \preceq w$ is a reflection, then R lies in the standard parabolic subgroup with simple system $\Pi \setminus \{\alpha_i\}$ for some $1 \le i \le n$. Thus w, which is a product of such reflections, must belong to the same parabolic subgroup.

(ii)⇒(i). If w belongs to a proper standard parabolic subgroup, then it belongs to one with simple system $\Pi \setminus \{\alpha_i\}$ for some $1 \le i \le n$ and $\mathcal{M}(w) \subseteq \mathcal{M}(\gamma')$, where

$$\gamma' \in \big\{R(\alpha_1)\gamma, \ldots, R(\alpha_s)\gamma,\ \gamma R(\alpha_{s+1}), \ldots, \gamma R(\alpha_n)\big\}. \tag{2.9}$$

By part (iii) of **Lemma 2.1** we have $w \preceq \gamma'$.

(i)⇔(iii). Since $R(\alpha_i)\gamma = \gamma R(\gamma^{-1}(\alpha_i))$ for $1 \le i \le s$, we can characterize peripheral elements as those w satisfying $w \preceq \gamma'$, where γ' is one of the elements

$$\gamma R(\gamma^{-1}(\alpha_1)), \ldots, \gamma R(\gamma^{-1}(\alpha_s)), \qquad \gamma R(\alpha_{s+1}), \ldots, \gamma R(\alpha_n). \tag{2.10}$$

This happens if and only if $R \preceq w^{-1}\gamma$, where R is one of the reflections

$$R(\gamma^{-1}(\alpha_1)), \ldots, R(\gamma^{-1}(\alpha_s)), \qquad R(\alpha_{s+1}), \ldots, R(\alpha_n). \tag{2.11}$$

By the definition of the total order (**2.7**), these are the reflections defined by the last n positive roots, that is, by the elements of Ω. ■

2.5 h-vectors and shellings

We will use the notion of a spherical simplicial complex in $\mathbb{R}^n$. The faces of such a complex are formed by intersecting the unit sphere S^{n-1} in $\mathbb{R}^n$ with simplicial cones, each pointed at the origin. These cones form a polyhedral fan in $\mathbb{R}^n$. If C is such a cone of dimension n and H runs through the supporting hyperplanes of the facets of C, we will refer to the intersections $S^{n-1} \cap H$ as the *walls* of the spherical simplex $S^{n-1} \cap C$.

Given a finite (abstract, geometric, or spherical) simplicial complex Δ of dimension $d-1$, let f_i denote the number of i-dimensional faces of Δ. The h-*vector* of Δ is the sequence $h(\Delta) = (h_0, h_1, \dots, h_d)$ defined by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}, \tag{2.12}$$

where $f_{-1} = 1$ unless Δ is empty. The complex Δ is *pure* if all its facets (faces which are maximal with respect to inclusion) have dimension $d-1$. Given a total ordering $F_1, F_2, \dots, F_m$ of the facets of a pure simplicial complex Δ of dimension $d-1$ and $1 \le j \le m$, we denote by $\mathcal{R}(F_j)$ the set of vertices x of F_j for which the codimension one face of F_j not containing x is contained in at least one of the facets $F_1, F_2, \dots, F_{j-1}$. Such an ordering $F_1, F_2, \dots, F_m$ is called a *shelling* of Δ if there are no indices $1 \le i < j \le m$ for which $\mathcal{R}(F_j)$ is contained in the vertex set of F_i. In that case $\mathcal{R}(F_j)$ is called the *restriction set* of F_j with respect to this shelling and the entries $h_i = h_i(\Delta)$ of $h(\Delta)$ are nonnegative integers given by

$$h_i = \#\{1 \le j \le m : \#\mathcal{R}(F_j) = i\}, \quad 0 \le i \le d, \tag{2.13}$$

where the cardinality of a finite set S is denoted by $\#S$. More information and references on shellability of simplicial complexes can be found in [**4**, Section 11].

2.6 Cluster complexes and subcomplexes

Let $\Delta(\gamma)$ denote the collection of spherical simplices in $\mathbb{R}^n$ on the vertex set $\Phi_{\ge -1}$ defined by declaring a subset $\{\tau_1, \tau_2, \dots, \tau_k\}$ of $\Phi_{\ge -1}$ satisfying $\tau_1 < \tau_2 < \cdots < \tau_k$ to be the vertex set of a simplex in $\Delta(\gamma)$ if and only if $R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_1)$ is an element of $\mathbf{L}(\gamma)$ of rank k. (Note that $\Delta(\gamma)$ is denoted by $EX(\gamma)$ in [**6**].) A simplex in $\Delta(\gamma)$ is said to be *positive* if its vertices are positive roots. As in [**6**], for $w \in \mathbf{L}(\gamma)$, let $X(w)$ denote the subcollection of $\Delta(\gamma)$ consisting of those simplices with vertex set contained in $\mathcal{M}(w) \cap \Phi^+$; in particular, $X(\gamma)$ is the subcollection of positive simplices of $\Delta(\gamma)$. The set of vertices of $X(w)$ is the positive system induced by Φ^+ on the root system $\Phi(w) = \Phi \cap \mathcal{M}(w)$ and is denoted by $\Phi^+(w)$. By parts (i) and (ii) of **Lemma 2.1**, this set coincides with the set of positive roots τ satisfying $R(\tau) \preceq w$. We note that $R(\tau)\rho \in \Phi(w)$ whenever $\tau, \rho \in \Phi(w)$. Finally, let $\Delta_+(\Phi)$ denote the induced subcomplex of $\Delta(\Phi)$ on the vertex set Φ^+ (where $\Delta(\Phi)$ is the cluster complex, discussed in the introduction). The complex $\Delta_+(\Phi)$ is referred to as the *positive part* of $\Delta(\Phi)$. The following theorem relates the combinatorics of $\Delta(\Phi)$ to that of $\mathbf{L}_W$.

Theorem 2.12 (see [**6**]). (i) The collection $\Delta(\gamma)$ is a spherical simplicial complex of dimension $n-1$ which is a realization of the cluster complex $\Delta(\Phi)$.

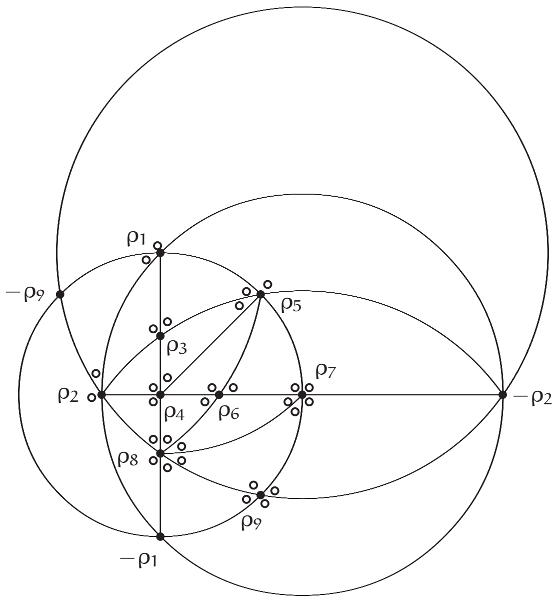

Figure 2.1

(ii) If $w \in \mathbf{L}(\gamma)$, then the collection $X(w)$ is a spherical simplicial complex of dimension $\ell(w) - 1$.
In particular, $X(\gamma)$ is a spherical simplicial complex of dimension $n - 1$ which is a realization of $\Delta_+(\Phi)$. □

It follows that $\Delta(\gamma)$ and $X(\gamma)$ are pure simplicial complexes which are homeomorphic to a sphere and a ball, respectively, of dimension $n-1$. More generally, for $w \in \mathbf{L}(\gamma)$, the complex $X(w)$ is a triangulation of a spherical simplex of dimension $\ell(w)-1$ [6, Corollary 7.7] and hence $X(w)$ is a pure simplicial complex which is homeomorphic to a ball of dimension $\ell(w) - 1$. These facts will be used repeatedly in Sections 4 and 7. Figure 2.1 shows $\Delta(\gamma)$ in the case where W is the group C_3 (or B_3) of symmetries of the cube, with the roots ordered as in (2.7), so that $\rho_0 = -\rho_9$, while $\rho_{10} = -\rho_1$ and $\rho_{11} = -\rho_2$. The meaning of the small circles close to vertices of facets is explained in Section 5.

3 Lexicographically first and last facets

In this section we fix $w \in \mathbf{L}(\gamma)$ of length k and describe the first and last facets of the complex $X(w)$ in the lexicographic order. We let $\Pi(w) = \{\delta_1, \ldots, \delta_k\}$ be the simple system for $\Phi^+(w)$, as in [6, Section 5], with $\delta_1 < \delta_2 < \cdots < \delta_k$ and $w = R(\delta_1) \cdots R(\delta_k)$. We will show that the five sets consisting of (i) the simple system, (ii) the set of vertices of the first facet of $X(w)$, (iii) the set of vertices of the last facet of $X(w)$, (iv) the set of roots in $\Phi^+(w)$ taken to negative roots by the action of w, and (v) the set of roots in $\Phi^+(w)$ taken to negative roots by the action of w^{-1} have the property that each one determines the other four.

3.1 The lexicographically first facet

We first recall the description of the first facet of $X(w)$ in the lexicographic order from [6, Section 6]. We provide proofs of statements, somewhat simplified from those of [6, Section 6], to make this paper more self-contained and since proofs of the corresponding statements about the lexicographically last facet will be similar. We define the roots

$$\epsilon_i = R(\delta_1)\cdots R(\delta_{i-1})\delta_i \tag{3.1}$$

for $1 \le i \le k$, so that $\epsilon_1 = \delta_1$. Since any two distinct elements of $\Pi(w)$ have nonnegative inner product and $R(\delta_j)x = x - 2(x\cdot\delta_j)\delta_j$ for all $x \in \mathbb{R}^n$, formula (3.1) implies that $\epsilon_i - \delta_i$ is a nonnegative linear combination of $\{\delta_1, \ldots, \delta_{i-1}\}$. In particular, the set $\{\epsilon_1, \ldots, \epsilon_k\}$ is a linearly independent subset of $\Phi^+(w)$.

Lemma 3.1. It holds that

$$\mu(\epsilon_i)\cdot\delta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.2}$$

In particular, $\mu(\epsilon_i)\cdot\tau \ge 0$ for all i and for all $\tau \in \Phi^+(w)$. □

Proof. From (3.1) we deduce that $R(\epsilon_i) = R(\delta_1)\cdots R(\delta_i)\cdots R(\delta_1)$ and hence that

$$w = R(\epsilon_i)R(\delta_1)\cdots R(\delta_{i-1})R(\delta_{i+1})\cdots R(\delta_k). \tag{3.3}$$

Since $w \preceq \gamma$ has length k, it follows that, for $i \neq j$, $R(\epsilon_i)R(\delta_j) \preceq \gamma$ and that $\mu(\epsilon_i)\cdot\delta_j = 0$, by Lemma 2.2.

Setting $w_i = R(\delta_1)\cdots R(\delta_i)$ we have $w_i(\delta_i) = -\epsilon_i$ and $\mu(\epsilon_i)\cdot\delta_i = -\mu(w_i(\delta_i))\cdot\delta_i = \mu(\delta_i)\cdot\delta_i = 1$, where the second to last equality follows by combining parts (i) and (ii) of Lemma 2.4 and observing that $\epsilon_i, \delta_i \in \mathcal{M}(w_i)$. For the last statement, note that any $\tau \in \Phi^+(w)$ can be written as a nonnegative linear combination $\tau = a_1\delta_1 + \cdots + a_k\delta_k$, so that $\mu(\epsilon_i)\cdot\tau = a_i \ge 0$. ■

Lemma 3.2. If $\tau \in \Phi^+(w)$ satisfies $\tau < \epsilon_i$, then $\mu(\epsilon_i)\cdot\tau = 0$ and $\epsilon_i\cdot\tau \ge 0$. □

Proof. From Lemma 3.1 we have $\mu(\epsilon_i) \cdot \tau \geq 0$. But $\tau < \epsilon_i$ implies $\mu(\epsilon_i) \cdot \tau \leq 0$ by Lemma 2.6(ii). Hence $\mu(\epsilon_i) \cdot \tau = 0$ or, equivalently, $R(\epsilon_i)R(\tau) \preceq \gamma$. Finally we get $\epsilon_i \cdot \tau \geq 0$ by applying Lemma 2.7 to this last relation and $\tau < \epsilon_i$. ■

Lemma 3.3. Let $\tau \in \Phi^+(w)$ and do fix $1 \leq i \leq k$. Then $\tau \in \{\epsilon_1, \ldots, \epsilon_i\}$ if and only if $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau$ is a negative root. In particular, $\tau \in \{\epsilon_1, \ldots, \epsilon_k\}$ if and only if $w^{-1}(\tau)$ is a negative root. □

Proof. If $\tau = \epsilon_j$ with $1 \leq j \leq i$, then $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau = -R(\delta_i) \cdots R(\delta_{j+1})\delta_j$ is a negative root since $\delta_a \cdot \delta_b \leq 0$. For the converse, let $\tau_0 = \tau$ and $\tau_j = R(\delta_j)\tau_{j-1}$ for $1 \leq j \leq i$ and assume that $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau = \tau_i$ is a negative root. Then there exists $1 \leq j \leq i$ such that τ_{j-1} is a positive root but τ_j is negative. Since $\tau_j = R(\delta_j)\tau_{j-1}$, we must have $\tau_{j-1} = \delta_j$. On the other hand, $\tau_{j-1} = R(\delta_{j-1}) \cdots R(\delta_1)\tau$, so that $\tau = \epsilon_j$. ■

Lemma 3.4. If $\tau \in \Phi^+(w)$ satisfies $\tau < \epsilon_i$ and $\tau \notin \{\epsilon_1, \ldots, \epsilon_{i-1}\}$, then $\epsilon_i \cdot \tau = 0$. □

Proof. From Lemma 3.2 we have $\epsilon_i \cdot \tau \geq 0$ and $\mu(\epsilon_i) \cdot \tau = 0$. Since τ is in the linear span of $\Pi(w)$, the last equation and Lemma 3.1 imply that τ lies in the linear span of $\Pi(w) - \{\delta_i\}$. Since $\tau \in \Phi^+(w)$, it must be in the positive linear span of $\Pi(w) - \{\delta_i\}$. Note that the root $\tau' = R(\delta_{i-1}) \cdots R(\delta_1)\tau$, being positive by Lemma 3.3, lies in the positive linear span of $\Pi(w) - \{\delta_i\}$ as well. We conclude that

$$\epsilon_i \cdot \tau = \big(R(\delta_1) \cdots R(\delta_{i-1})\delta_i\big) \cdot \tau = \delta_i \cdot \tau' \leq 0 \tag{3.4}$$

and hence that $\epsilon_i \cdot \tau = 0$. ■

Corollary 3.5. If $i < j$ and $\epsilon_i > \epsilon_j$, then $\epsilon_i \cdot \epsilon_j = 0$. □

Proof. Set $\tau = \epsilon_j$ in Lemma 3.4. ■

Proposition 3.6. The set $\{\epsilon_1, \ldots, \epsilon_k\}$ is the vertex set of the first facet of $X(w)$ in the lexicographic order. □

Proof. It follows from (3.1) that $R(\epsilon_k) \cdots R(\epsilon_1) = R(\delta_1) \cdots R(\delta_k) = w$. Moreover, by Corollary 3.5, and since any two orthogonal nonzero vectors give rise to commuting reflections, we may relabel, so that $\epsilon_1 < \cdots < \epsilon_k$. Hence the set $\{\epsilon_1, \ldots, \epsilon_k\}$ is the vertex set of some facet of $X(w)$. Note that Lemma 3.2 continues to hold for this reordered set, since its statement does not depend at all on the ordering of the ϵ_i. Given $1 \leq i \leq k$, we have $\mu(\epsilon_i) \cdot \tau = 0$ for any $\tau \in \Phi^+(w)$ satisfying $\tau < \epsilon_i$ and, by our convention on the ϵ's, we have $\mu(\epsilon_j) \cdot \tau = 0$ for $j \geq i$ as well. This forces such a root τ into a linear subspace of dimension $i - 1$, namely, the intersection of $\mathcal{M}(w)$ with the hyperplanes $\mu(\epsilon_j)^\perp$ for $i \leq j \leq k$. Thus, for any facet F of $X(w)$, the ith vertex of F cannot precede ϵ_i in the order (2.7). ■

Corollary 3.7. Let $\tau \in \Phi^+(w)$. Then $w^{-1}(\tau)$ is a negative root if and only if τ is a vertex of the first facet of $X(w)$ in the lexicographic order. □

Proof. Combine Proposition 3.6 with the last statement of Lemma 3.3. ■

Remark 3.8. This reordering of orthogonal roots to get the ordered vertex set of the first facet is sometimes necessary even for the case $w = \gamma$. Example 6.14 of [6] illustrates this for A_3.

3.2 The lexicographically last facet

We define the roots

$$\zeta_i = R(\delta_k) \cdots R(\delta_{i+1})\delta_i \tag{3.5}$$

for $1 \leq i \leq k$, so that $\zeta_k = \delta_k$. As with $\{\epsilon_1, \ldots, \epsilon_k\}$ in Section 3.1 we see that $\{\zeta_1, \ldots, \zeta_k\}$ is a linearly independent subset of $\Phi^+(w)$. From (3.5) we have

$$w = R(\delta_1) \cdots R(\delta_{i-1})R(\delta_{i+1}) \cdots R(\delta_k)R(\zeta_i) \tag{3.6}$$

which, combined with (3.3), gives $R(\epsilon_i)w = wR(\zeta_i)$ or $R(\zeta_i) = w^{-1}R(\epsilon_i)w$ and hence $\zeta_i = \pm w^{-1}(\epsilon_i)$. Since $w^{-1}(\epsilon_i)$ is a negative root by Lemma 3.3, we deduce that $\zeta_i = -w^{-1}(\epsilon_i)$.

Lemma 3.9. It holds that

$$\mu(\delta_i) \cdot \zeta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.7}$$

In particular, $\mu(\tau) \cdot \zeta_i \geq 0$ for all i and for all $\tau \in \Phi^+(w)$. □

Proof. Using Lemma 2.4(i) we get

$$\mu(\delta_i) \cdot \zeta_j = -\mu(\delta_i) \cdot w^{-1}(\epsilon_j) = -w(\mu(\delta_i)) \cdot \epsilon_j = \mu(\epsilon_j) \cdot \delta_i, \tag{3.8}$$

so the first statement follows from Lemma 3.1. The second statement follows as in the proof of Lemma 3.1. ■

The proofs of Lemmas 3.10, 3.11, and 3.12 and of Corollary 3.13 below are completely analogous to those of corresponding statements of Section 3.1 and are omitted.

Lemma 3.10. If $\tau \in \Phi^+(w)$ satisfies $\tau > \zeta_i$, then $\mu(\tau) \cdot \zeta_i = 0$ and $\zeta_i \cdot \tau \geq 0$. □

Lemma 3.11. Let $\tau \in \Phi^+(w)$. Given $1 \leq i \leq k$, $\tau \in \{\zeta_i, \zeta_{i+1}, \ldots, \zeta_k\}$ if and only if $R(\delta_i)R(\delta_{i+1}) \cdots R(\delta_k)\tau$ is a negative root. In particular, $\tau \in \{\zeta_1, \ldots, \zeta_k\}$ if and only if $w(\tau)$ is a negative root. □

Lemma 3.12. If $\tau \in \Phi^+(w)$, $\tau > \zeta_i$ and $\tau \notin \{\zeta_{i+1}, \ldots, \zeta_k\}$, then $\zeta_i \cdot \tau = 0$. □

Corollary 3.13. If $i < j$ and $\zeta_i > \zeta_j$, then $\zeta_i \cdot \zeta_j = 0$. □

Proposition 3.14. The set $\{\zeta_1, \ldots, \zeta_k\}$ is the vertex set of the last facet of $X(w)$ in the lexicographic order. □

Proof. As in the proof of Proposition 3.6 we have $R(\zeta_k) \cdots R(\zeta_1) = w$ and we may relabel, so that $\zeta_1 < \cdots < \zeta_k$. Hence the set $\{\zeta_1, \ldots, \zeta_k\}$ is the vertex set of some facet of $X(w)$. Given $1 \leq i \leq k$, Lemma 3.10 implies that for $1 \leq j \leq i$ and any $\tau \in \Phi^+(w)$ with $\tau > \zeta_i$ we have $\mu(\tau) \cdot \zeta_j = 0$. Since μ is an invertible linear transformation, it follows as in the proof of Proposition 3.6 that any such root τ lies in a linear subspace of dimension $k - i$ and hence the ith vertex of any facet F of $X(w)$ cannot succeed ζ_i in the order (2.7). ■

Corollary 3.15. Let $\tau \in \Phi^+(w)$. Then $w(\tau)$ is a negative root if and only if τ is a vertex of the last facet of $X(w)$ in the lexicographic order. □

Proof. Combine Proposition 3.14 with the last statement of Lemma 3.11. ■

Remark 3.16. In the special case $w = \gamma$, the vertices of the last facet of $X(\gamma)$ are the last n positive roots by Remark 2.8.

Remark 3.17. In the sequel we will denote by $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_k\}$ the vertex sets of the first and last facet of $X(w)$, respectively, in the lexicographic order, reordered, so that $\epsilon_1 < \cdots < \epsilon_k$ and $\zeta_1 < \cdots < \zeta_k$. Lemmas 3.2, 3.4, 3.10, and 3.12, as well as the last statements of Lemmas 3.1 and 3.9, continue to hold. Indeed, except for Lemmas 3.4 and 3.12, the statements do not depend at all on the ordering of the ϵ_i and the ζ_i. For Lemma 3.4, it suffices to note that if $\epsilon_1 < \cdots < \epsilon_k$ and $\tau \in \Phi^+(w)$ satisfies $\tau < \epsilon_i$ and $\tau \notin \{\epsilon_1, \ldots, \epsilon_{i-1}\}$, then $\tau \notin \{\epsilon_1, \ldots, \epsilon_k\}$, so the hypotheses of the original version of the lemma continue to hold. Lemma 3.12 is similar.

Remark 3.18. It follows from Proposition 3.14 that, for $1 < i \leq k$, the set $\{\zeta_i, \ldots, \zeta_k\}$ is the vertex set of the last facet of $X(wR(\zeta_1) \cdots R(\zeta_{i-1}))$. Similarly, it follows from Proposition 3.6 that, for $1 \leq i < k$, the set $\{\epsilon_1, \ldots, \epsilon_i\}$ is the vertex set of the first facet of $X(R(\epsilon_{i+1}) \cdots R(\epsilon_k)w)$.

4 Vertex type

In this section we define the notion of vertex type which will be used to define the bijection ϕ, mentioned in Section 1. We let F denote a face of $\Delta(\gamma)$ with ordered vertex set $\{\tau_1, \tau_2, \ldots, \tau_k\}$, let $w = R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_1) \preceq \gamma$, and let $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_k\}$ be the reordered vertex sets of the first and last facets of $X(w)$, respectively, in the lexicographic order.

Definition 4.1. For $1 \leq i \leq k$, set

$$\begin{aligned} u_i(F) &= R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_i), \\ v_i(F) &= R(\tau_1)R(\tau_2) \cdots R(\tau_i) \end{aligned} \tag{4.1}$$

and say that τ_i is a left vertex in F if $u_i(F)\tau_i$ is a positive root. Otherwise say that τ_i is a right vertex in F.

Lemma 4.2. Any vertex of F which is a negative root is a left vertex in F. □

Proof. Suppose that $\tau_i \in -\Pi$ and recall that the first $n - s$ vertices of $\Delta(\gamma)$ in the total order (2.7) are in $-\Pi_2$ while the last s are in $-\Pi_1$. If $\tau_i \in -\Pi_2$, then $\tau_1, \ldots, \tau_{i-1} \in -\Pi_2$ as well. In particular, τ_i is orthogonal to each of $\tau_1, \ldots, \tau_{i-1}$ and hence $u_i(F)\tau_i = w(\tau_i)$. Since $-\tau_i \in \Pi_2 \subseteq \Omega$, Corollary 2.9(iii) implies that $w(\tau_i)$ is a positive root, so that τ_i is a left vertex in F. Similarly, if $\tau_i \in -\Pi_1$, then $\tau_{i+1}, \ldots, \tau_n \in -\Pi_1$. In particular, τ_i is orthogonal to each of $\tau_{i+1}, \ldots, \tau_k$ and hence $u_i(F)\tau_i = R(\tau_i)\tau_i = -\tau_i$ is a positive root, so that τ_i is a left vertex in F. ■

Lemma 4.3. The vertex τ_i is a right vertex in F if and only if τ_i is a vertex of the last facet of $X(u_i(F))$ in the lexicographic order. Moreover, if F is positive, then τ_i is the first vertex of this facet. □

Proof. Since any right vertex in F must be a positive root by Lemma 4.2, the first statement is a direct consequence of the definition and Corollary 3.15. The second statement is obvious since $\{\tau_i, \ldots, \tau_k\}$ is the ordered vertex set of a facet of $X(u_i(F))$. ■

Let F be positive, so that F is a facet of $X(w)$. We will describe the wall of F opposite (i.e., not containing) the vertex τ_i. To simplify notation, define the roots $\eta_i = u_i(F)\tau_i$ and $\theta_i = v_i(F)\tau_i$. Then

$$\begin{aligned} R(\eta_i)R(\tau_k) \cdots R(\tau_{i+1})R(\tau_{i-1}) \cdots R(\tau_1) &= w \preceq \gamma, \\ R(\tau_k) \cdots R(\tau_{i+1})R(\tau_{i-1}) \cdots R(\tau_1)R(\theta_i) &= w \preceq \gamma. \end{aligned} \tag{4.2}$$

Note that $\eta_i, \theta_i \in \Phi(w)$ for all i, although they are not necessarily positive roots. As in the proof of Lemma 3.1, we find that

$$\mu(\eta_i) \cdot \tau_j = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{4.3}$$

$$\mu(\tau_j) \cdot \theta_i = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.4}$$

As a result, the wall of F opposite τ_i is the intersection of $S^{n-1} \cap \mathcal{M}(w)$ with $\mu(\eta_i)^\perp$, the linear hyperplane orthogonal to $\mu(\eta_i)$. It follows by linearity from (4.3) and (4.4) that $\mu(\eta_i) \cdot x = \mu(x) \cdot \theta_i$ for all $x \in \mathcal{M}(w)$. Note that since $X(w)$ is homeomorphic to a ball, every codimension one face is contained in either exactly one or exactly two facets of $X(w)$ and the first case occurs if and only if this face is contained in the boundary of $X(w)$.

Remark 4.4. This discussion leads to a simple interpretation of the notion of vertex type when F is a facet of $\Delta(\gamma)$. Let τ_i' be the positive root parallel to η_i, so that $\tau_i' = \eta_i$ in the case of a left vertex while $\tau_i' = -\eta_i$ in the case of a right vertex. In the case $k = n$, (4.3) allows us to define the positive side of the hyperplane containing $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$ to be the set of vectors x satisfying $\mu(\tau_i') \cdot x \geq 0$. With this definition, we see that right vertices of a facet F are precisely those τ_i which lie on the positive side of the hyperplane containing the other $n-1$ vertices of F.

Proposition 4.5. Suppose that F is positive.

(i) The root τ_i is a left vertex in F if and only if there exists a vertex $\tau > \tau_i$ of $X(w)$ such that $(V \backslash \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Moreover, such a vertex τ is unique.

(ii) The root $\nu_i(F)\tau_i$ is positive if and only if there exists a vertex $\tau < \tau_i$ of $X(w)$ such that $(V \backslash \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Moreover, such a vertex τ is unique. □

Proof. The uniqueness of τ in both parts follows from the discussion preceding the statement of the proposition.

(i) Suppose τ_i is a left vertex in F, so that $\eta_i = u_i(F)\tau_i$ is a positive root. Since $\mu(\eta_i) \cdot \tau_i < 0$ by (4.3) and $\mu(\eta_i) \cdot \eta_i = 1$, we see that $\mu(\eta_i)^\perp$ separates the positive roots τ_i and η_i. It follows that the wall of F opposite τ_i does not lie in the boundary of $X(w)$ and, as a result, there exists a facet of $X(w)$ other than F with vertex set $(V \backslash \{\tau_i\}) \cup \{\tau\}$. Since $\mu(\eta_i)^\perp$ separates τ from τ_i, we must have $\mu(\eta_i) \cdot \tau > 0$. In view of Lemma 2.6, this inequality combined with $\mu(\eta_i) \cdot \tau_i < 0$ gives $\tau > \tau_i$. For the converse, suppose that $(V \backslash \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$ for some $\tau > \tau_i$. Then $\mu(\eta_i) \cdot \tau_i < 0$ and $\mu(\eta_i) \cdot \tau > 0$. Hence Lemma 2.6 implies that η_i is a positive root, meaning that τ_i is a left vertex in F.

(ii) Suppose that $\theta_i = \nu_i(F)\tau_i$ is a positive root. Since $\mu(\eta_i) \cdot \tau_i < 0$ by (4.3) and $\mu(\eta_i) \cdot \theta_i = \mu(\theta_i) \cdot \theta_i = 1$, we see that $\mu(\eta_i)^\perp$ separates the positive roots τ_i and θ_i. As in part (i), it follows that the wall of F opposite τ_i does not lie in the boundary of $X(w)$ and, as a result, there exists a facet of $X(w)$ other than F with vertex set $(V \backslash \{\tau_i\}) \cup \{\tau\}$. Since $\mu(\eta_i)^\perp$ separates τ from τ_i, we must have $\mu(\eta_i) \cdot \tau > 0$. Thus $\mu(\tau) \cdot \theta_i = \mu(\eta_i) \cdot \tau > 0$. In view of Lemma 2.6, this inequality combined with $\mu(\tau_i) \cdot \theta_i = \mu(\eta_i) \cdot \tau_i < 0$ gives $\tau < \tau_i$. The proof of the converse proceeds as in part (i). ■

Corollary 4.6. If F is a positive face and τ_i is a left vertex in F, then $\nu_i(F)\tau_i$ is a negative root. □

Proof. By part (i) of Proposition 4.5, there exists a vertex $\tau > \tau_i$ of $X(w)$ such that $(V \backslash \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Since $X(w)$ is a manifold, there cannot be a vertex $\tau' < \tau_i$ of $X(w)$ such that $(V \backslash \{\tau_i\}) \cup \{\tau'\}$ is the vertex set of a facet of $X(w)$. Thus the root $\nu_i(F)\tau_i$ cannot be positive by part (ii) of Proposition 4.5. ■

Corollary 4.7. If $\tau \in \Phi^+(w)$ satisfies $\tau \leq \zeta_1$, then there exists a facet of $X(w)$ having τ as its smallest vertex. In particular, τ precedes all vertices of the last facet of $X(wR(\tau))$ in the lexicographic order. □

Proof. The statement is obvious in case $\tau = \zeta_1$, so suppose $\tau < \zeta_1$. Let F be any facet of $X(w)$ having τ as a vertex and let $\{\tau_1, \tau_2, \ldots, \tau_k\}$ be the ordered vertex set of F. If $\tau = \tau_1$, there is nothing to prove. Otherwise, since $\tau_1 < \zeta_1$, the vertex τ_1 is a left vertex in F by Corollary 3.15. Proposition 4.5(i) implies that there exists a vertex τ' of $X(w)$ with $\tau_1 < \tau'$ such that $(V \backslash \{\tau_1\}) \cup \{\tau'\}$ is the vertex set of a facet F' of $X(w)$. Clearly τ is a vertex of F' and the smallest vertex of F' succeeds that of F in the order (2.7). Therefore, applying the same argument to F' repeatedly, if necessary, we can find a facet of $X(w)$ having τ as its smallest vertex. ■

The following technical fact will be used in Section 6. It demonstrates that for a root τ with $R(\tau) \preceq w \preceq \gamma$ the last facet of $X(w)$ may share some vertices with the last facet of $X(wR(\tau))$.

Proposition 4.8. If $\tau \in \Phi^+(w)$ satisfies

$$\zeta_1 < \cdots < \zeta_{i-1} < \tau < \zeta_i < \cdots < \zeta_k \tag{4.5}$$

for some $1 \le i \le k$, then

(i) τ is orthogonal to $\zeta_1, \dots, \zeta_{i-1}$,

(ii) $R(\tau) \preceq wR(\zeta_1) \cdots R(\zeta_{i-1})$,

(iii) the last facet of $X(wR(\tau))$ in the lexicographic order has ordered vertex set

$$\{\zeta_1, \dots, \zeta_{i-1}, \zeta'_{i+1}, \dots, \zeta'_k\}, \tag{4.6}$$

where $\{\zeta'_{i+1}, \dots, \zeta'_k\}$ is the ordered vertex set of the lexicographically last facet of $X(wR(\zeta_1) \cdots R(\zeta_{i-1})R(\tau))$. Moreover $\tau < \zeta'_{i+1}$. □

Proof. Part (i) follows from **Lemma 3.12**. In view of **Lemma 2.5** and from the fact that $R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq \gamma$ has length $i-1$, we can deduce from **Lemma 3.10** that $R(\tau)R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq \gamma$ has length i. Since $R(\tau) \preceq w$ and $R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq w$ by the assumptions, it follows from **Lemma 2.1**(iv) that

$$R(\tau)R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq w, \tag{4.7}$$

which proves (ii). Let $w' = wR(\zeta_1) \cdots R(\zeta_{i-1}) = R(\zeta_k) \cdots R(\zeta_i)$ and observe that $\{\zeta_i, \dots, \zeta_k\}$ is the ordered vertex set of the last facet of $X(w')$ in the lexicographic order by **Remark 3.18**. Since $R(\tau) \preceq w'$ and $\tau < \zeta_i$, **Corollary 4.7** implies that $\tau < \zeta'_{i+1}$ and, in particular, $\zeta_{i-1} < \zeta'_{i+1}$. Since $R(\tau)$ commutes with $R(\zeta_j)$ for $1 \le j \le i-1$ by (i), we have

$$R(\zeta'_k) \cdots R(\zeta'_{i+1}) = w'R(\tau) = wR(\tau)R(\zeta_1) \cdots R(\zeta_{i-1}) \tag{4.8}$$

and hence the set $\{\zeta_1, \dots, \zeta_{i-1}, \zeta'_{i+1}, \dots, \zeta'_k\}$ is the ordered vertex set of a facet of $X(wR(\tau))$. To complete the proof of (iii) it suffices to show that ζ_j is the jth vertex of the last facet of $X(wR(\tau))$ in the lexicographic order for $1 \le j < i$. This follows from the claim that any positive root $\tau' > \zeta_j$ in $\mathcal{M}(wR(\tau))$ lies in the $(k-j-1)$-dimensional space $\mathcal{M}(wR(\zeta_1) \cdots R(\zeta_j)R(\tau))$. Indeed, we have $R(\tau') \preceq wR(\tau)$, $R(\zeta_j) \cdots R(\zeta_1) \preceq wR(\tau)$ and, by **Lemmas 2.5** and **3.10**, $R(\tau')R(\zeta_j) \cdots R(\zeta_1) \preceq \gamma$. From **Lemma 2.1**(iv) we deduce that $R(\tau')R(\zeta_j) \cdots R(\zeta_1) \preceq wR(\tau)$. ■

5 The map ϕ

Definition 5.1. For a facet F of $\Delta(\gamma)$ with ordered vertex set $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$, define

$$\phi(F) = c(F, \sigma_n)c(F, \sigma_{n-1}) \cdots c(F, \sigma_1), \tag{5.1}$$

where

$$c(F, \sigma_i) = \begin{cases} R(\sigma_i) & \text{if } \sigma_i \text{ is a right vertex in } F, \\ I & \text{otherwise.} \end{cases} \tag{5.2}$$

In Figure 2.1, the right vertices of each facet are indicated with a small circle and the value of ϕ on each facet can be deduced. The vertex types can be verified using Definition 4.1, but the calculations are simplified greatly by using Lemmas 4.3 and 6.3 and Theorem 5.2.

Clearly $\phi(F) \in \mathbf{L}(\gamma)$ and thus ϕ is a map from the set of facets of $\Delta(\gamma)$ to $\mathbf{L}(\gamma)$ such that, for any facet F of $\Delta(\gamma)$, the rank of $\phi(F)$ in $\mathbf{L}(\gamma)$ is equal to the number of right vertices of F. It will be shown in Section 6 that ϕ is a bijection. Part of the injectivity of ϕ will be proved in this section. The next theorem gives a fundamental property of ϕ.

Theorem 5.2. Let F be a positive facet of $\Delta(\gamma)$ and let $w = \phi(F)$.

(i) The set of right vertices in F is equal to the vertex set of the last facet of $X(w)$ in the lexicographic order.

(ii) The set of left vertices in F is equal to the vertex set of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order. □

For the proof of the theorem, we need the following lemma.

Lemma 5.3. Suppose that F is a positive face of $\Delta(\gamma)$ with ordered vertex set $\{\tau_1, \tau_2, \ldots, \tau_k\}$. If τ_i is a right vertex and τ_j is a left vertex in F for some $i < j$, then $\tau_i \cdot \tau_j = 0$. □

Proof. Proceeding by induction on $j - i$, we may assume that (i) the result holds for the face with ordered vertex set $\{\tau_i, \ldots, \tau_{j-1}\}$ and (ii) the result holds for the face with ordered vertex set $\{\tau_{i+1}, \ldots, \tau_j\}$. Furthermore, we may also assume that all roots in $\{\tau_{i+1}, \ldots, \tau_{j-1}\}$ are right vertices in F. (If not, then replace F by the face F' obtained from F by removing all roots in $\{\tau_{i+1}, \ldots, \tau_{j-1}\}$ which are left vertices in F and observe that, in view of our assumption (i) made earlier, τ_i and τ_j are still right and left vertices in F', resp.) Under these assumptions, let $w = u_i(F)$ and let $\{\zeta_1, \zeta_2, \ldots, \zeta_r\}$ be the ordered vertex set of the last facet of $X(w)$ in the lexicographic order, where $r = k - i + 1$. We have $\tau_i = \zeta_1$ by Lemma 4.3 and

$$R(\tau_j) \preceq u_{i+1}(F) = wR(\tau_i) = R(\zeta_r)R(\zeta_{r-1}) \cdots R(\zeta_2). \tag{5.3}$$

By Remark 3.18, the last facet of $X(u_{i+1}(F))$ in the lexicographic order has vertex set $\{\zeta_2, \ldots, \zeta_r\}$. By our assumptions τ_j is orthogonal to all roots in $\{\tau_{i+1}, \ldots, \tau_{j-1}\}$ and hence $u_{i+1}(F)\tau_j = u_j(F)\tau_j$. Since τ_j is a left vertex in F we conclude that $u_{i+1}(F)\tau_j$ is a positive root. It follows from Lemma 3.11 that $\tau_j \notin \{\zeta_2, \ldots, \zeta_r\}$. Since $\tau_j \in \mathcal{M}(w)$ and $\tau_j > \zeta_1 = \tau_i$, Lemma 3.12 applies to give $\tau_i \cdot \tau_j = 0$. ■

Proof of Theorem 5.2. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the ordered vertex set of F and let $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$ and $\{\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_\ell}\}$ be the ordered sets of right and left vertices in F, respectively, so that $w = R(\sigma_{i_k}) \cdots R(\sigma_{i_1})$. To prove (i) it suffices to show that σ_{i_p} is the first vertex of the last facet of $X(R(\sigma_{i_k}) \cdots R(\sigma_{i_p}))$ in the lexicographic order for $1 \leq p \leq k$. In view of Corollary 3.15, this is equivalent to the statement that the root $R(\sigma_{i_k}) \cdots R(\sigma_{i_p})\sigma_{i_p}$ is negative. The last statement holds since $R(\sigma_{i_k}) \cdots R(\sigma_{i_p})\sigma_{i_p} = u_{i_p}(F)\sigma_{i_p}$ by Lemma 5.3 and σ_{i_p} is a right vertex in F.

Similarly, by Lemma 5.3, we have $w^{-1}\gamma = R(\sigma_{j_\ell}) \cdots R(\sigma_{j_1})$. To prove (ii) it suffices to show that σ_{j_q} is the last vertex of the lexicographically first facet of $X(R(\sigma_{j_q}) \cdots R(\sigma_{j_1}))$ for $1 \leq q \leq \ell$. By Corollary 3.7 this is equivalent to the statement that $R(\sigma_{j_1}) \cdots R(\sigma_{j_q})\sigma_{j_q}$ is a negative root. Since $R(\sigma_{j_1}) \cdots R(\sigma_{j_q})\sigma_{j_q} = v_{j_q}(F)\sigma_{j_q}$ by Lemma 5.3, this follows from Corollary 4.6. ■

Corollary 5.4. The restriction of the map ϕ to the set of positive facets of $\Delta(\gamma)$ is injective. □

Proof. If F is a positive facet of $\Delta(\gamma)$ with $w = \phi(F)$, then parts (i) and (ii) of Theorem 5.2 imply that the vertices of F are determined by w and hence F is the unique positive facet whose image under ϕ is w. ■

The image of the facets of $X(\gamma)$ can be characterized in terms of nonperipheral elements. Recall from Section 2.4 that $w \in L(\gamma)$ is peripheral if and only if $R(\rho) \preceq w^{-1}\gamma$ for some $\rho \in \Omega$.

Proposition 5.5. If F is a facet of $\Delta(\gamma)$, then the element $\phi(F)$ is nonperipheral if and only if F is positive. □

Proof. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the ordered vertex set of F. Assume that σ_i is a negative simple root for some $1 \leq i \leq n$, say $\sigma_i = -\alpha_j \in -\Pi_1$ with $1 \leq j \leq s$ (the case $\sigma_i \in -\Pi_2$ is similar). As in the proof of Lemma 4.2, $R(\sigma_i)$ commutes with each of $R(\sigma_{i+1}), \ldots, R(\sigma_n)$. By the same lemma, σ_i is a left vertex in F and hence

$$\phi(F) \preceq R(\sigma_n) \cdots R(\sigma_{i+1}) R(\sigma_{i-1}) \cdots R(\sigma_1) = R(\sigma_i)\gamma = R(\alpha_j)\gamma, \tag{5.4}$$

which implies that $\phi(F)$ is peripheral.

The converse is proven by contradiction. Let F be positive and assume that $\phi(F)$ is peripheral. Set $w = \phi(F)$, $v = w^{-1}\gamma$, and $r = \ell(v)$. Let $\{\delta'_1, \ldots, \delta'_r\}$ be the ordered simple system for $\Phi^+(\nu)$ and let $\{\epsilon'_1, \ldots, \epsilon'_r\}$ be the set of vertices of the first facet of $X(\nu)$, defined as in (3.1). Recall from [6, Theorem 5.1] that δ'_r is the largest root in $\Phi^+(\nu)$ with respect to the order (2.7). Since w is peripheral, the set Ω intersects $\Phi^+(\nu)$ by Proposition 2.11 and hence $\delta'_r \in \Omega$. Since ϵ'_r is a vertex of the first facet of $X(\nu)$ in the lexicographic order, ϵ'_r is a left vertex in F by Theorem 5.2(ii), say $\epsilon'_r = \sigma_i$. Let $u_i = u_i(F) = R(\sigma_n) \cdots R(\sigma_i)$ and $\nu_i = \nu_i(F) = R(\sigma_1) \cdots R(\sigma_i)$. By Corollary 3.5 and Proposition 3.6, the root σ_i is orthogonal to all later left vertices in F (if any). It follows from this fact and Lemma 5.3 that $\nu_i(\sigma_i) = \nu^{-1}(\sigma_i)$. However $\nu^{-1}(\sigma_i) = \nu^{-1}(\epsilon'_r) = -\delta'_r \in -\Omega$ and hence $\gamma(\nu_i(\sigma_i)) \in -\gamma(\Omega)$ is a positive root by Remark 2.8. On the other hand, $\gamma(\nu_i(\sigma_i)) = -u_i(\sigma_i)$ since $\gamma = u_i R(\sigma_i) \nu_i^{-1}$. But $-u_i(\sigma_i)$ (and hence $\gamma(\nu_i(\sigma_i))$) is a negative root since σ_i is a left vertex in F. This is the required contradiction. ■

6 Bijectivity of ϕ

The first step in establishing bijectivity of ϕ is to show that ϕ maps the set of facets of $X(\gamma)$ surjectively onto the set of nonperipheral elements of $\mathbf{L}(\gamma)$.

Lemma 6.1. If $w \in \mathbf{L}(\gamma)$ is nonperipheral, then there exists a facet F of $X(\gamma)$ such that the left vertices in F are precisely the vertices of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order. In particular, $\phi(F) = w$. □

Proof. Let $\{\tau_1, \ldots, \tau_k\}$ be the ordered vertex set of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order. For $0 \le i \le k$, we define

$$\nu_i = R(\tau_i) \cdots R(\tau_1), \qquad w_i = \gamma \nu_i^{-1}, \tag{6.1}$$

so that $\nu_0 = I$, $w_0 = \gamma$, $\nu_k = w^{-1}\gamma$, and $w_k = w$. Note that $\nu_i = R(\tau_i)\nu_{i-1}$ and $w_{i-1} = w_i R(\tau_i)$ for $1 \le i \le k$. We first claim that $w_{i-1}(\tau_i)$ is a positive root for $1 \le i \le k$. Since τ_i is a vertex of the first facet of $X(\nu_i)$ in the lexicographic order, the root $\nu_i^{-1}(\tau_i)$ is negative by Corollary 3.7. Therefore $-\nu_i^{-1}(\tau_i)$ is a positive root, clearly in $\mathcal{M}(\nu_i)$. Since w, and hence w_i, is nonperipheral, we know from Proposition 2.11 that this root cannot be in the set Ω of the last n positive roots. Hence $\gamma(-\nu_i^{-1}(\tau_i))$ must be a positive root. This proves the claim since $w_{i-1}(\tau_i) = -w_i(\tau_i) = -\gamma(\nu_i^{-1}(\tau_i))$.

We will show by induction that for each $0 \le i \le k$ there exists a facet F_i of $X(\gamma)$ such that the ordered set of left vertices in F_i is equal to $\{\tau_1, \ldots, \tau_i\}$. For $i = 0$, Lemma 4.3 implies that the last facet F_0 of $X(\gamma)$ in the lexicographic order, having Ω as its vertex set, has the desired property.

For the inductive step, let $1 \leq i \leq k$ and assume that $X(\gamma)$ has a facet F_{i-1} whose ordered set of left vertices is $\{\tau_1, \ldots, \tau_{i-1}\}$. Let $\{\zeta_i, \ldots, \zeta_n\}$ be the ordered set of right vertices in F_{i-1}. Since $\gamma = w_{i-1} v_{i-1}$, we know from Lemma 5.3 and Theorem 5.2(ii) that $\phi(F_{i-1}) = w_{i-1}$ and that $\{\zeta_i, \ldots, \zeta_n\}$ is the vertex set of the last facet of $X(w_{i-1})$ in the lexicographic order. From $w_{i-1} = w_i R(\tau_i)$ we get $R(\tau_i) \preceq w_{i-1}$. Furthermore, $\tau_i \notin \{\zeta_i, \ldots, \zeta_n\}$ by Corollary 3.15, since $w_{i-1}(\tau_i)$ is a positive root by the earlier claim. Thus $\zeta_i < \cdots < \zeta_{j-1} < \tau_i < \zeta_j < \cdots < \zeta_n$ for some j. Therefore Proposition 4.8 applies to w_{i-1} and τ_i to give the following:

(i) τ_i is orthogonal to each of $\zeta_i, \ldots, \zeta_{j-1}$,

(ii) $R(\tau_i) \preceq w_{i-1} R(\zeta_i) \cdots R(\zeta_{j-1})$,

(iii) the ordered vertex set of G_i, the last facet of $X(w_{i-1} R(\tau_i))$ in the lexicographic order, is $\{\zeta_i, \ldots, \zeta_{j-1}, \zeta'_{j+1}, \ldots, \zeta'_n\}$,

(iv) $\tau_i < \zeta'_{j+1}$,

where $w' = w_{i-1} R(\zeta_i) \cdots R(\zeta_{j-1}) R(\tau_i)$ and $\{\zeta'_{j+1}, \ldots, \zeta'_n\}$ is the ordered vertex set of the last facet of $X(w')$ in the lexicographic order.

The union of $\{\tau_1, \ldots, \tau_{i-1}, \tau_i\}$ with the vertex set of G_i is the vertex set of a facet F_i of $X(\gamma)$, since the roots in

$$V = \{\tau_1, \ldots, \tau_{i-1}\} \cup \{\zeta_i, \ldots, \zeta_{j-1}\} \tag{6.2}$$

can be ordered as they were in F_{i-1}, while τ_i can be positioned between ζ_{j-1} and ζ'_{j+1} by (iv). Since the roots in V are the first $j-1$ roots in both F_{i-1} and F_i and are ordered in precisely the same way, it follows that τ_r remains a left vertex in F_i for all $1 \leq r \leq i-1$. By (i) and (iii), ζ_t remains a right vertex in F_i for all $i \leq t \leq j-1$ and each ζ'_t for $j < t \leq n$ is a right vertex in F_i. Finally, using (i), we have that $w' R(\tau_i)(\tau_i) = w_{i-1} R(\zeta_i) \cdots R(\zeta_{j-1})(\tau_i) = w_{i-1}(\tau_i)$ is a positive root by our claim, so that τ_i is a negative vertex in F_i, as desired. Thus $\phi(F_i) = w_i$ and the induction is complete. ■

Remark 6.2. The proof of this lemma shows, independently of ϕ, that if $w \in \mathbf{L}_W$ is nonperipheral, then the union of the vertices of the last facet of the subcomplex $X(w)$ and the vertices of the first facet of the subcomplex $X(w^{-1}\gamma)$ is the vertex set of a facet F of $X(\gamma)$. This surprisingly precise description of the combinatorial closeness of the disjoint subcomplexes $X(w)$ and $X(w^{-1}\gamma)$ of $X(\gamma)$ suggests that our definition of ϕ is canonical.

Let us call an element of $\mathbf{L}(\gamma)$ of the form

$$\gamma' = R(\alpha_{i_1}) \cdots R(\alpha_{i_t}), \quad \text{with } 1 \leq i_1 < \cdots < i_t \leq n, \tag{6.3}$$

a *standard parabolic Coxeter element* of W (with respect to the ordered simple system Π). Note that the intersection of the moved spaces of two standard parabolic Coxeter elements is again the moved space of some standard parabolic Coxeter element. It follows from **Lemma 2.1**(i) that, given $w \preceq \gamma$, there is a minimum (with respect to $\preceq$) standard parabolic Coxeter element γ' satisfying $w \preceq \gamma'$. Clearly w is nonperipheral with respect to γ' and γ' is the unique standard parabolic Coxeter element with this property.

Lemma 6.3. If F is a face of $\Delta(\gamma)$ and F' is the face of F obtained by removing a set of negative simple roots from the vertex set of F, then the sets of right vertices of F and F' coincide. □

Proof. It suffices to consider the case where F' is obtained from F by removing a single negative simple root. Let $V = \{\tau_1, \ldots, \tau_k\}$ be the ordered vertex set of F. By definition of the order (**2.7**) there exist integers i and j with $0 \leq i < j \leq k$ such that

$$\{\tau_1, \ldots, \tau_i\} \subseteq -\Pi_2, \qquad \{\tau_{i+1}, \ldots, \tau_j\} \subseteq \Phi^+, \qquad \{\tau_{j+1}, \ldots, \tau_k\} \subseteq -\Pi_1. \tag{6.4}$$

In view of **Lemma 4.2** it suffices to show that for $i+1 \leq p \leq j$, the type of τ_p is unchanged if a negative simple root is removed from V. Since the type of τ_p is determined by $\{\tau_p, \tau_{p+1}, \ldots, \tau_k\}$, we need only consider the removal of a negative simple root in the set $\{\tau_{j+1}, \ldots, \tau_k\}$. Suppose that τ_q is such a negative simple root and that the roots $u_p(F)(\tau_p)$ and $u'_p(F)(\tau_p)$ have different signs, where

$$u_p(F) = R(\tau_k) \cdots R(\tau_p), \qquad u'_p(F) = R(\tau_k) \cdots R(\tau_{q+1}) R(\tau_{q-1}) \cdots R(\tau_p). \tag{6.5}$$

Since Π_1 is an orthonormal set, we have $u'_p(F) = R(\tau_q) u_p(F)$ whence, since τ_q is a negative simple root, we conclude that $u'_p(F)(\tau_p) = \pm\tau_q$. However this forces a linear dependence on the set $\{\tau_p, \ldots, \tau_k\}$, giving a contradiction. ■

Theorem 6.4. The map ϕ is a bijection from the set of facets of $\Delta(\gamma)$ to $\mathbf{L}(\gamma)$. □

Proof. To prove surjectivity of ϕ let $w \preceq \gamma$ and choose the unique standard parabolic Coxeter element γ' of the form (**6.3**) with respect to which $w \preceq \gamma'$ is nonperipheral. Let ϕ' be the map of **Definition 5.1** corresponding to γ'. By **Lemma 6.1** we can find a facet F' of $X(\gamma')$ such that $\phi'(F') = w$. Extend F' to a facet F of $\Delta(\gamma)$ by adding the negative simple roots not present in (**6.3**) and note that $\phi(F) = w$ by **Lemma 6.3**.

To prove injectivity of ϕ let $w \preceq \gamma$. We need to show that there is at most one facet F of $\Delta(\gamma)$ with $\phi(F) = w$. Let $\{\tau_1, \ldots, \tau_k\}$ be the ordered set of positive vertices of such a facet F, which determines a face $\widetilde{F}$ of F. Denote by γ' and $\widetilde{\gamma}$ the unique standard parabolic Coxeter elements with respect to which $w \preceq \gamma'$ and $R(\tau_k) \cdots R(\tau_1) \preceq \widetilde{\gamma}$ are nonperipheral, respectively. By Lemma 6.3 we have $\widetilde{\phi}(\widetilde{F}) = w$, where $\widetilde{\phi}$ is the map of Definition 5.1 corresponding to $\widetilde{\gamma}$. Proposition 5.5 implies that w is nonperipheral with respect to $\widetilde{\gamma}$ and hence we must have $\widetilde{\gamma} = \gamma'$. Thus the set of negative vertices of F is equal to the negative of the set of simple roots not appearing in the expression (6.3) for γ' and, as a result, this set of negative vertices is uniquely determined by w. Finally, since $\widetilde{\phi}(\widetilde{F}) = w$, $\widetilde{F}$ is uniquely determined by w by Corollary 5.4. ■

Corollary 6.5. The number of facets of $\Delta(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$. □

Corollary 6.6. The map ϕ restricts to a bijection from the set of facets of $X(\gamma)$ to the set of nonperipheral elements of $\mathbf{L}(\gamma)$. □

Proof. Combine Theorem 6.4 with Proposition 5.5. ■

In particular, the number of facets of $X(\gamma)$, and hence of $\Delta_+(\Phi)$, is equal to the number of nonperipheral elements of $\mathbf{L}(\gamma)$. This fact was found independently by Reading [15, Corollary 9.2]. The number of facets of $\Delta_+(\Phi)$ (positive clusters) is given by a product formula similar to (1.1); see [12, equation (3.8)].

7 Shellings and h-vectors

In this section we describe an explicit shelling of $\Delta(\gamma)$ and use it to prove Theorem 1.1. We consider the reverse of the lexicographic ordering for the various complexes under consideration instead of the lexicographic ordering itself only because this makes some of the statements technically easier to prove.

Theorem 7.1. The reverse of the lexicographic ordering on the facets of $X(w)$ is a shelling of $X(w)$ for any $w \in \mathbf{L}(\gamma)$. □

Proof. Let F and F' be two facets of $X(w)$ with vertex sets V and V', respectively, such that F' succeeds F in the lexicographic order. We need to show that there exists $\rho \in V \backslash V'$ such that $V \backslash \{\rho\}$ is contained in the vertex set of a facet of $X(w)$ which succeeds F in the lexicographic order. We proceed by induction on the length k of w. The statement is trivial for $k \leq 1$, so suppose $k \geq 2$. Let τ and τ' be the smallest elements of V and V', respectively, so that $\tau \leq \tau'$. If $\tau < \tau'$, then τ is not a vertex of the last facet of $X(w)$ in the lexicographic order and hence $w(\tau)$ is a positive root by Corollary 3.15. This means that τ is a left vertex in F. Clearly $\tau \notin F'$ and the result follows in this case from Proposition 4.5(i) with $\rho = \tau$. Suppose now that $\tau = \tau'$. Then $V \backslash \{\tau\}$ and $V' \backslash \{\tau\}$ are the vertex sets of facets G and G', respectively, of $X(wR(\tau))$ such that G' succeeds G in the lexicographic order. By induction G precedes a facet of $X(wR(\tau))$ with vertex set of the form $(V\backslash\{\tau,\rho\}) \cup \{\rho'\}$ for some $\rho \in V\backslash V'$, so that necessarily $\rho < \rho'$. It follows that $(V\backslash\{\rho\})\cup\{\rho'\}$ is the vertex set of a facet of $X(w)$ which succeeds F. This completes the induction. ■

Let $n(F)$ denote the number of vertices of a face F of $\Delta(\gamma)$ and F_+ (resp., F_-) denote the face of F whose vertices are the positive (resp., negative) vertices of F. Define the order $\trianglelefteq$ on the set of facets of $\Delta(\gamma)$ as follows. For two such facets F and F', we have $F' \triangleleft F$ if and only if either $n(F'_+) > n(F_+)$ or $n(F'_+) = n(F_+)$ and F_+ precedes F'_+ in the lexicographic order. This order is a total order on the set of facets of $\Delta(\gamma)$ since such a facet F is determined by F_+.

Lemma 7.2. Let V be the vertex set of a facet F of $\Delta(\gamma)$, let $\tau \in V$, and let τ' be the unique vertex of $\Delta(\gamma)$ other than τ such that $(V\backslash\{\tau\}) \cup \{\tau'\}$ is the vertex set of a facet of $\Delta(\gamma)$.

(i) If τ is a negative root, then τ' is a positive root.

(ii) Let τ be positive. Then τ is a left vertex in F if and only if τ' is positive and $\tau < \tau'$. □

Proof. (i) Observe that if τ is a negative root, then the wall of F opposite τ contains all other negative simple roots.

(ii) Let V_+ denote the set of positive vertices of F and let $w \preceq \gamma$ be such that F_+ is a facet of $X(w)$. By Lemma 6.3 we have that τ is a left vertex in F if and only if it is a left vertex in F_+. By Proposition 4.5(i) this happens if and only if there exists a positive root $\sigma > \tau$ such that $(V_+\backslash\{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $X(w)$. To complete the proof observe that for σ positive, $(V_+\backslash\{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $X(w)$ if and only if $(V\backslash\{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $\Delta(\gamma)$. ■

Theorem 7.3. (i) The order $\trianglelefteq$ is a shelling order for $\Delta(\gamma)$.

(ii) The restriction set of a facet F of $\Delta(\gamma)$ with respect to this shelling is equal to the set of left vertices in F. □

Proof. (i) Let F and F' be two facets of $\Delta(\gamma)$ with vertex sets V and V', respectively, such that $F' \triangleleft F$. It suffices to show that there exists $\rho \in V\backslash V'$ such that $V\backslash\{\rho\}$ is contained in the vertex set of a facet of $\Delta(\gamma)$ which precedes F in the order $\trianglelefteq$. If $F'_- = F_-$, then the statement follows from Theorem 7.1. Otherwise one can choose ρ to be any negative root in $V\backslash V'$ since then, by Lemma 7.2 (i), $V\backslash\{\rho\}$ is contained in the vertex set of a facet of $\Delta(\gamma)$ having one more positive vertex than F.

(ii) This is an immediate consequence of the definition of the restriction set and of Lemmas 4.2 and 7.2. ■

Corollary 7.4. (i) The entry $h_i(\Delta(\gamma))$ of the h-vector of $\Delta(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank i.

(ii) The entry $h_i(X(\gamma))$ of the h-vector of $X(\gamma)$ is equal to the number of nonperipheral elements of $\mathbf{L}(\gamma)$ of rank $n-i$. □

Proof. (i) It follows from Theorem 7.3 and (2.13) that $h_i(\Delta(\gamma))$ is equal to the number of facets of $\Delta(\gamma)$ with i left vertices or, equivalently, to the number of facets of $\Delta(\gamma)$ with $n-i$ right vertices. By Theorem 6.4 this is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank $n-i$. Statement (i) follows since $\mathbf{L}(\gamma)$ is self-dual.

(ii) The facets of $X(\gamma)$ come first in the shelling of Theorem 7.3 and hence $h_i(X(\gamma))$ is equal to the number of facets of $X(\gamma)$ with i left vertices or, equivalently, to the number of facets of $X(\gamma)$ with $n-i$ right vertices. This is equal to the number of nonperipheral elements of $\mathbf{L}(\gamma)$ of rank $n-i$ by Corollary 6.6. ■

Part (ii) of the previous corollary gives a combinatorial interpretation of the entries of the h-vector of $X(\gamma)$, and hence of $\Delta_+(\Phi)$. Several different combinatorial interpretations to these numbers appear in [2]. The following statement, which is a direct consequence of Proposition 2.11 and part (ii) of Corollary 7.4, provides a similar interpretation.

Corollary 7.5. The entry $h_i(X(\gamma))$ of the h-vector of $X(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank i which are not preceded by any of the reflections corresponding to the last n positive roots. □

Remark 7.6. The nonperipheral elements of $\mathbf{L}(\gamma)$ of rank one are the reflections in W which do not lie in any proper standard parabolic subgroup of W. The number f_W of such reflections was studied by Chapoton [8] and can be expressed in terms of the exponents and Coxeter number of W [8, Proposition 1.1]. Hence, in the special case $i = n-1$, part (ii) of Corollary 7.4 provides a conceptual explanation of the equality $h_{n-1}(\Delta_+(\Phi)) = f_W$, which can be checked case by case on the basis of the data provided in [2, Section 6] and [8, Section 1].

Proof of Theorem 1.1. Let Φ and W be as in the statement of the theorem. If $\Phi = \Phi_1 \times \Phi_2 \times \cdots \times \Phi_m$ is the decomposition of Φ into irreducible components and $W = W_1 \times W_2 \times \cdots \times W_m$ is the corresponding decomposition of W, then $\Delta(\Phi)$ is the simplicial join of the $\Delta(\Phi_i)$ and $\mathbf{L}_W$ is the direct product of the $\mathbf{L}_{W_i}$. Hence the h-polynomial of $\Delta(\Phi)$ (the polynomial with coefficients the entries of the h-vector) and the rank generating polynomial of $\mathbf{L}_W$ are multiplicative with respect to these decompositions. Therefore we may assume that Φ (equivalently, W) is irreducible. Since the combinatorial structure of

$\Delta(\Phi)$ is unaffected by rescaling of roots, we may assume further that the elements of Φ are of unit length. Under these assumptions the result follows from Corollary 7.4(i) and Theorem 2.12(i). ■

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