Europ. J. Combinatorics (1998) **19**, 7–18

On Free Deformations of the Braid Arrangement†

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We classify the hyperplane arrangements between the cones of the braid arrangement and the Shi arrangement of type A_{n-1} which are free, in the sense of Terao. We also prove that the cones of the extended Shi arrangements of type A_{n-1} are free, verifying part of a conjecture of Edelman and Reiner.

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1. INTRODUCTION

There has been considerable interest in the past in analyzing specific families of hyperplane arrangements from the perspective of freeness. Examples of such families have primarily included classes of subarrangements of Coxeter arrangements. The subarrangements of the braid arrangement $\mathcal{A}_n$, the Weyl arrangement of type A_{n-1}, are known as the *graphical arrangements*. They correspond naturally to graphs on n vertices. It follows mainly from the work of Stanley [14] and is recorded in [5, §3] that free graphical arrangements correspond to chordal graphs. Certain classes of arrangements between the root systems A_{n-1} and B_n were studied by Józefiak and Sagan [9]. These arrangements can also be related to graphs. Edelman and Reiner [5] gave a complete classification of the free arrangements in this case and showed that they correspond to threshold graphs. In a more recent work [6] these authors classified free arrangements which arise as discriminantal arrangements of two-dimensional zonotopes with integer side lengths.

We will be concerned with *deformations* of $\mathcal{A}_n$. The combinatorics of such arrangements was first studied in a systematic way by Stanley and collaborators [15]. They are the affine arrangements which have each of their hyperplanes parallel to one of the hyperplanes $x_i - x_j = 0$ of $\mathcal{A}_n$. A central role in what follows will be played by the *Shi arrangement* of type A_{n-1}, introduced by J.-Y. Shi in [13]. It is the arrangement of affine hyperplanes in $\mathbb{R}^n$ of the form

$$x_i - x_j = 0, 1 \qquad \text{for } 1 \le i < j \le n$$

and will be denoted by $\hat{\mathcal{A}}_n$. The characteristic polynomial of $\hat{\mathcal{A}}_n$ was shown to factor completely over the integers as $q(q-n)^{n-1}$ by Headley [7, 8]. This agrees with Shi's result [13] that it divides $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions. A simple counting proof of Headley's result, based on the 'finite field method', was given in [1, 2]. Freeness of the associated homogenized linear arrangement, or cone, was conjectured in [6].

Our motivation comes primarily from [1, 2]. In this work the characteristic polynomials of large classes of deformations of Coxeter arrangements were shown to factor completely over the nonnegative integers and the question of freeness of their cones was naturally raised [1, §7, 2, §8.4]. Our objective is to answer this question in some cases of interest and indicate the importance of the finite field method as a technique to detect freeness.

Specifically, we will prove that the cone of $\hat{\mathcal{A}}_n$ is inductively free. This gives another explanation to the results of Shi and Headley from the point of view of freeness, as well as simple inductive proofs. In fact we will classify all free hyperplane arrangements between the cones of $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$. These arrangements again correspond naturally to graphs on n vertices. We will show that the free ones correspond to a family of graphs which is quite

†The present research was supported by a postdoctoral fellowship in the framework of the EC programme Human Capital and Mobility, EC Network DIMANET, Contract no. ERBCHRXCT-94 0429.

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simple to define and has already appeared in [1, 2]. These free arrangements are as good in providing counterexamples to Orlik's conjecture as the arrangements considered in [5]. We include similar results for other deformations of $\mathcal{A}_n$.

It follows, at least in the cases we can work out, that complete factorization of the characteristic polynomial for deformations of $\mathcal{A}_n$ is a very strong indication of freeness. Also, in these cases, all free arrangements turn out to be inductively free.

2. Background and Motivation

We assume familiarity with basic material about hyperplane arrangements and their freeness. Here we will only recall some crucial definitions, facts and conventions about arrangements. We will also review some results from [1, 2] to motivate the rest. We refer the reader to the book by Orlik and Terao [10] for an extensive treatment of the theory of hyperplane arrangements, or to one of the papers [5, §2, 12, §3] for the highlights of the parts of the theory that we will need.

We will only be concerned with real hyperplane arrangements. Thus a *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine subspaces of codimension one in some real finite dimensional vector space, which is usually the euclidean space $\mathbb{R}^n$. The arrangement $\mathcal{A}$ is called *central* if all of its hyperplanes are linear subspaces, i.e. they pass through the origin. Freeness was defined by Terao for central hyperplane arrangements [10, 17, Ch. 4]. The standard way to pass from any hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ to a central one is to construct the *cone* $\boldsymbol{c}\mathcal{A}$. This is the arrangement in $\mathbb{R}^{n+1}$ obtained by homogenizing each hyperplane

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + d = 0$$

of $\mathcal{A}$ to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + dx_0 = 0$$

and adding the hyperplane $x_0 = 0$. Here x_0 is the new coordinate attached to $\mathbb{R}^n$. The cone $\boldsymbol{c}\hat{\mathcal{A}}_n$ of the Shi arrangement, for example, has hyperplanes

$$\begin{aligned} x_i - x_j &= 0 \qquad \text{for } 1 \le i < j \le n, \\ x_i - x_j - x_0 &= 0 \qquad \text{for } 1 \le i < j \le n, \\ x_0 &= 0. \end{aligned}$$

The Factorization Theorem of Terao [10, 18, Thm 4.137] states that the characteristic polynomial $\chi(\mathcal{A}, q)$ of $\mathcal{A}$ [10, §2.3] factors completely over the nonnegative integers for any free arrangement $\mathcal{A}$. Its roots are the *exponents* of $\mathcal{A}$. The operation of taking the cone has a very simple effect on the characteristic polynomial of $\mathcal{A}$ [10, Prop. 2.51], namely

$$\chi(\boldsymbol{c}\mathcal{A}, q) = (q-1)\chi(\mathcal{A}, q). \tag{1}$$

In particular it preserves the property of complete factorization of the characteristic polynomial. Hence the result of Headley about $\chi(\hat{\mathcal{A}}_n, q)$, mentioned in the introduction, naturally raises the question of freeness for $\boldsymbol{c}\hat{\mathcal{A}}_n$.

The *inductively free* arrangements are the free arrangements that usually come up in examples. They form a subclass of the class of free arrangements by the Addition Theorem [10, Thm 4.50] and provide a standard tool for proving freeness. For this reason we include a definition. Let $\mathcal{A}$ be any hyperplane arrangement and let $H \in \mathcal{A}$ be a distinguished hyperplane. The corresponding *deleted arrangement* is

$$\mathcal{A}' = \mathcal{A} - \{H\}.$$

The *restricted arrangement* to H has H as its ambient space and is given by

$$\mathcal{A}'' = \{H' \cap H \mid H' \in \mathcal{A}'\}.$$

The triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a *triple of arrangements*. For the following definition and lemma let $\exp \mathcal{A}$ be the multiset of roots of $\chi(\mathcal{A}, q)$. When no assumption on $\mathcal{A}$ is made, these roots are complex numbers which are not necessarily integers. The class of inductively free arrangements $\mathcal{IF}$ is the smallest class of central hyperplane arrangements which satisfies the following two conditions:

(1) The empty arrangement in $\mathbb{R}^n$ is in $\mathcal{IF}$ for all $n \geq 0$.
(2) If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements with $\mathcal{A}', \mathcal{A}'' \in \mathcal{IF}$ and $\exp \mathcal{A}'' \subseteq \exp \mathcal{A}'$ then $\mathcal{A} \in \mathcal{IF}$.

The Factorization Theorem follows easily for inductively free arrangements from the elementary Deletion-Restriction Theorem [10, Cor. 2.57], which states that $\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q)$. Specifically, we will need the following corollary of the Deletion-Restriction Theorem, which we state as a lemma.

LEMMA 2.1. *If* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *is a triple of arrangements and* $\exp \mathcal{A}' = \{e_1, e_2, \ldots, e_n\}$, $\exp \mathcal{A}'' = \{e_1, e_2, \ldots, e_{n-1}\}$, *then*

$$\exp \mathcal{A} = \{e_1, e_2, \ldots, e_{n-1}, e_n + 1\}.$$

It follows that for an inductively free arrangement $\mathcal{A}$, the elements of $\exp \mathcal{A}$, that is the roots of $\chi(\mathcal{A}, q)$, are all nonnegative integers. Note that the Deletion-Restriction Theorem is valid for general hyperplane arrangements, as opposed to central ones, and that the definition of inductive freeness still makes sense if we remove the requirement that the arrangements are central.

CONVENTION. From now on we will use this more general notion of inductive freeness. Thus an inductively free arrangement $\mathcal{A}$ need not be central. We refer again to the roots of $\chi(\mathcal{A}, q)$, which are nonnegative integers, as the exponents of $\mathcal{A}$. Note that if $\mathcal{A}$ is inductively free then so is the central arrangement $\boldsymbol{c}\mathcal{A}$. In particular, by [10, Thm 4.50], $\boldsymbol{c}\mathcal{A}$ is free with an extra exponent equal to 1.

We say that two hyperplane arrangements in $\mathbb{R}^n$ are *affinely equivalent* if there is an invertible affine endomorphism of $\mathbb{R}^n$ that maps the hyperplanes of one onto the hyperplanes of the other. Inductive freeness is a *combinatorial property* and hence it is preserved under affine equivalence.

Headley's result was generalized in several ways in [1, 2]. We recall one such generalization next. For the purposes of some further generalizations, we find it convenient to think of a simple graph S on the vertex set $[n] = \{1, 2, \ldots, n\}$ as a directed graph. Each edge ij is directed as (j, i), i.e. from j to i, if $i < j$. In other words, S is a subset of the set

$$E_n = \{(j, i) \mid 1 \leq i < j \leq n\},$$

which is the edge set of the complete graph. Note that the arrangements between $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$ correspond to simple graphs on the vertex set $[n]$. More precisely, each such arrangement is of the form

$$\begin{aligned} x_i - x_j &= 0 \qquad \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 \qquad \text{for } (j, i) \in S \end{aligned}$$

for some $S \subseteq E_n$. We denote this arrangement by $\hat{\mathcal{A}}_{n,S}$. The Shi arrangment $\hat{\mathcal{A}}_n$ corresponds to the complete graph $S = E_n$ and the braid arrangement $\mathcal{A}_n$ to the empty graph. The following theorem produces a family of arrangements between $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$ whose characteristic polynomials have nonnegative integers as roots.

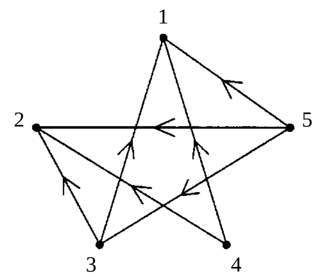

FIGURE 1. An example with $n = 5$.

THEOREM 2.2 ([1, THM 3.4, 2, THM 6.2.2]). *Suppose that the graph $S \subseteq E_n$ has the following property: if $1 \le i < j < k \le n$ and $(j, i) \in S$ then $(k, i) \in S$. Then*

$$\chi(\hat{\mathcal{A}}_{n,S}, q) = q \prod_{1<j\le n} (q - c_j),$$

where $c_j = n + a_j - j + 1$ and $a_j = \#\{i < j \mid (j, i) \in S\}$ is the outdegree of j in S, for $1 < j \le n$.

Figure 1 shows a graph satisfying the condition in Theorem 2.2. For this graph we have $a_2 = 0$, $a_3 = a_4 = 2$, $a_5 = 3$, so $c_2 = 4$, $c_3 = 5$, $c_4 = c_5 = 4$ and the corresponding characteristic polynomial is $q(q - 4)^3(q - 5)$.

3. INDUCTIVE FREENESS

In this section we prove inductive freeness of the Shi arrangement, as well as the extended Shi arrangements and the ones mentioned in Theorem 2.2. In particular, their cones are free. We will see in the next section that, up to a suitable permutation of the coordinates, the arrangements in Theorem 2.2 are all the arrangements between $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n$ with free cones.

We first give a detailed proof of inductive freeness of $\hat{\mathcal{A}}_n$ as a prototype for the proofs in the more general cases. We need to find a family of arrangements larger than the class of Shi arrangements which will contain enough deletions and restrictions to guarantee inductive freeness of all arrangements in the family. Our task becomes easier by the fact that such a family is included in the family already produced by the finite field method in [2, Thm 6.2.10].

NOTATION. In what follows we write $\{a_1^{m_1}, a_2^{m_2}, \dots, a_r^{m_r}\}$ for a multiset, where m_1, $m_2, \dots, m_r$ denote multiplicities.

THEOREM 3.1. *For any integers $m \ge 0$ and $2 \le k \le n + 1$, the arrangement*

$$\begin{aligned} x_1 - x_j &= 0, 1, \dots, m \qquad \text{for } 2 \le j < k, \\ x_1 - x_j &= 0, 1, \dots, m + 1 \qquad \text{for } k \le j \le n, \\ x_i - x_j &= 0, 1 \qquad \text{for } 2 \le i < j \le n \end{aligned} \tag{2}$$

is inductively free with multiset of exponents $\{0^1, (n + m - a)^{k-2}, (n + m)^{n-k+1}\}$.

PROOF. Figure 2 illustrates (2) with a graph. The edges (j, i) of the graph are labeled with sets of integers s such that the hyperplanes $x_i - x_j = s$ are in (2). We proceed by double induction on n and $n - k$. The result is clear for $n = 2$, so pick an $n \ge 3$. The idea of the proof is to remove the hyperplanes involving x_1 one by one, until none remains, in such a

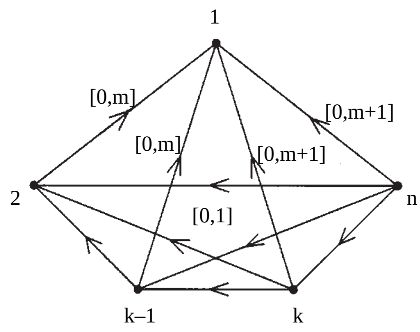

FIGURE 2. An edge labeled graph.

way that the deletions and restrictions at each step are arrangements to which the induction hypothesis applies.

First we treat the case $m = 0$ and $k = n + 1$. The arrangement in question is

$$\begin{aligned} x_1 - x_j &= 0 \qquad \text{for } 2 \le j \le n, \\ x_i - x_j &= 0, 1 \qquad \text{for } 2 \le i < j \le n. \end{aligned} \tag{3}$$

Note that the arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i - x_j = 0, 1 \qquad \text{for } 2 \le i < j \le n \tag{4}$$

is inductively free with multiset of exponents $\{0^2, (n-1)^{n-2}\}$, since $\hat{\mathcal{A}}_{n-1}$ is inductively free by induction with multiset of exponents $\{0^1, (n-1)^{n-2}\}$. By Lemma 2.1, adding r of the hyperplanes $x_1 - x_j = 0$ in any order produces an inductively free arrangement whose multiset of exponents is $\{0^1, r^1, (n-1)^{n-2}\}$. Indeed, the restriction to the last hyperplane added at each step is affinely equivalent to (4). The case $r = n-1$ gives the desired result for (3).

We can now assume $2 \le k \le n$, since the arrangement (2) having parameters $m \ge 1$ and $k = n+1$ coincides with (2) having parameters $m-1$ and $k = 2$. Consider the hyperplane H of (2) with equation $x_1 - x_k = m + 1$. Deletion of this hyperplane produces an arrangement which is of the same form as (2), with k replaced by $k+1$. Hence, by induction, it is inductively free with multiset of exponents $\{0^1, (n+m-1)^{k-1}, (n+m)^{n-k}\}$. Restriction to H produces again an arrangement of the same form, with n replaced by $n-1$ and m replaced by $m+1$. To see this just set $x_k = x_1 - m - 1$ in the equations involving x_k. The equation $x_k - x_n = 1$, for example, becomes $x_1 - x_n = m + 2$. Again by induction, the restricted arrangement is inductively free with multiset of exponents $\{0^1, (n+m-1)^{k-2}, (n+m)^{n-k}\}$. These exponents are contained in the multiset of exponents of the deleted arrangement. It follows from the definition of inductive freeness and Lemma 2.1 that (2) is inductively free as well, with the claimed exponents. □

For $m = 0$, $k = 2$ we get the following corollary.

COROLLARY 3.2. *The Shi arrangement $\hat{\mathcal{A}}_n$ is inductively free with multiset of exponents $\{0^1, n^{n-1}\}$. In particular, $\chi(\hat{\mathcal{A}}_n, q) = q(q-n)^{n-1}$.*

Let $a \ge 1$ be an integer. As in [2], we denote the *extended Shi arrangement* corresponding to a by $\hat{\mathcal{A}}_n^{[-a+1,a]}$. It has hyperplanes

$$x_i - x_j = -a+1, -a+2, \ldots, a \qquad \text{for } 1 \le i < j \le n$$

and reduces to $\hat{\mathcal{A}}_n$ for $a = 1$. It was conjectured by Edelman and Reiner that the cone $\boldsymbol{c}\hat{\mathcal{A}}_n^{[-a+1,a]}$ is free with multiset of exponents $\{0^1, 1^1, (an)^{n-1}\}$. This is a special case for the irreducible crystallographic root system A_{n-1} of one half of Conjecture 3.3 in [6]. The weaker statement $\chi(\hat{\mathcal{A}}_n^{[-a+1,a]}, q) = q(q-an)^{n-1}$ can easily be derived using the finite field method [2, Cor. 7.1.2]. For additional work on these arrangements see [11, 16]. Theorem 3.1 can be extended to the following result.

THEOREM 3.3. *Fix an integer $a \geq 1$. For integers m, k as in Theorem 3.1, the arrangement*

$$\begin{aligned} x_1 - x_j &= -a+1, \dots, m \qquad \text{for } 2 \leq j < k, \\ x_1 - x_j &= -a+1, \dots, m+1 \qquad \text{for } k \leq j \leq n, \\ x_i - x_j &= -a+1, \dots, a \qquad \text{for } 2 \leq i < j \leq n \end{aligned} \tag{5}$$

is inductively free with multiset of exponents $\{0^1, (an+m-a)^{k-2}, (an+m-a+1)^{n-k+1}\}$.

PROOF. We use again double induction on n and $n-k$ and distinguish two cases. Suppose first that $k = n+1$ and $m = 0$. The arrangement in question is affinely equivalent to

$$\begin{aligned} x_1 - x_j &= 0, \dots, a-1 \qquad \text{for } 2 \leq j \leq n, \\ x_i - x_j &= -a+1, \dots, a \qquad \text{for } 2 \leq i < j \leq n \end{aligned} \tag{6}$$

by the transformation which sends $x_1 \to x_1 - a + 1$ and fixes the other coordinates. The inductive freeness of (6) follows, as in the proof of Theorem 3.1, by adding in any order the first set of hyperplanes to

$$x_i - x_j = -a+1, \dots, a \qquad \text{for } 2 \leq i < j \leq n.$$

The restricted arrangement at each step is affinely equivalent to $\hat{\mathcal{A}}_{n-1}^{[-a+1,a]}$, for which induction applies. After adding r hyperplanes of the form $x_1 - x_j = s$, the multiset of exponents of the intermediate arrangement is $\{0^1, r^1, (an-a)^{n-2}\}$. The case $r = an - a$ gives the exponents of (6).

As in the proof of Theorem 3.1 we can now assume that $2 \leq k \leq n$. We delete and restrict to the hyperplane $x_1 - x_k = m+1$. The deleted arrangement has the form (5) with k replaced by $k+1$ and so does the restricted arrangement, with n replaced by $n-1$ and m replaced by $m+a$, once we substitute $x_k = x_1 - m - 1$. Induction and Lemma 2.1 apply and give the result for (5). □

The extended Shi arrangements correspond to the case $m = a-1$, $k = 2$.

COROLLARY 3.4. *The extended Shi arrangement $\hat{\mathcal{A}}_n^{[-a+1,a]}$ is inductively free with multiset of exponents* $\{0^1, (an)^{n-1}\}$. □

We now come back to the arrangements $\hat{\mathcal{A}}_{n,S}$ of Theorem 2.2. We need to extend the notation used there for more general deformations of $\mathcal{A}_n$. Suppose that $\mathcal{A}$ contains $\mathcal{A}_n$ and that it has hyperplanes of the form $x_i - x_j = s$, where $s \in \mathbb{Z}$ and $1 \leq i < j \leq n$. For $1 < j \leq n$ let a_j be the number of hyperplanes $x_i - x_j = s$ of $\mathcal{A}$ with $i < j$ and $s \neq 0$. Also let

$$c_j = n + a_j - j + 1.$$

We call these numbers the a and c *parameters* of $\mathcal{A}$ respectively. Note that this notation agrees with the one in Theorem 2.2. A generalization of Theorem 2.2 [2, Thm 6.2.10] produced a large class of deformations of $\mathcal{A}_n$, in the form described above, whose characteristic polynomials factor completely over the nonnegative integers. Thus, under assumptions, the characteristic polynomial of $\mathcal{A}$ equals $q(q-c_2)\cdots(q-c_n)$. Note that the exponents proposed for the arrangements in Theorem 3.1 are exactly the corresponding c parameters.

THEOREM 3.5. *Let T be a graph on the vertex set $[2, n]$, i.e. $(j, i) \in T$ implies $2 \le i < j \le n$. Suppose that T satisfies the condition in Theorem 2.2: if $2 \le i < j < k \le n$ and $(j, i) \in T$ then $(k, i) \in T$. Let m, k be integers as in Theorem 3.1. The arrangement*

$$\begin{aligned} &x_1 - x_j = 0, 1, \dots, m \quad \text{for } 2 \le j < k, \\ &x_1 - x_j = 0, 1, \dots, m+1 \quad \text{for } k \le j \le n, \\ &x_i - x_j = 0 \quad \text{for } 2 \le i < j \le n, \\ &x_i - x_j = 1 \quad \text{for } (j, i) \in T \end{aligned} \tag{7}$$

is inductively free with exponents $0, c_2, \dots, c_n$, where c_j, $2 \le j \le n$ are its c parameters.

PROOF. We use the same induction argument. The result follows easily in the case $k = n+1$, $m = 0$ by adding the hyperplanes $x_1 - x_j = 0$ in any order to

$$\begin{aligned} &x_i - x_j = 0 \quad \text{for } 2 \le i < j \le n, \\ &x_i - x_j = 1 \quad \text{for } (j, i) \in T, \end{aligned}$$

to which induction applies. Otherwise we delete and restrict to the hyperplane $x_1 - x_k = m + 1$. Let $\{a_j\}$ and $\{c_j\}$ be the a and c parameters of (7). Deletion simply reduces a_k, and hence c_k, by one. The restricted arrangement is affinely equivalent to an arrangement of the form (7), obtained by substituting $x_k = x_1 - m - 1$ in the equations involving x_k. Induction and Lemma 2.1 complete the proof, once we check that the c parameters of the restricted arrangement are $c_2, \dots, c_{k-1}, c_{k+1}, \dots, c_n$. We omit the details which are straightforward. □

The case $m = 0$ gives the result promised at the end of Section 2.

COROLLARY 3.6. *Under the assumptions and notation of Theorem 2.2, the arrangement $\hat{\mathcal{A}}_{n,S}$ is inductively free with exponents $0, c_2, \dots, c_n$.*

4. FREE ARRANGEMENTS BETWEEN $\boldsymbol{c}\mathcal{A}_n$ AND $\boldsymbol{c}\hat{\mathcal{A}}_n$

In this section we show that the cones of the arrangements of Theorem 2.2 are essentially the only free arrangements between $\boldsymbol{c}\mathcal{A}_n$ and $\boldsymbol{c}\hat{\mathcal{A}}_n$.

We first recall another fundamental result in the theory of free arrangements, the Localization Theorem [10, Thm 4.37]. Let X be an element of the intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$ [10, §2.1]. The *localization* $\mathcal{A}_X$ is the subarrangement of $\mathcal{A}$:

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

The Localization Theorem asserts that any localization of a free arrangement is free. It can easily provide obstructions to freeness [19] and is therefore quite useful in classifying free arrangements [5, 6]. Our main result can be stated as follows.

THEOREM 4.1. *Let $S \subseteq E_n$. The following are equivalent:*

(i) $\hat{\mathcal{A}}_{n,S}$ is inductively free.
(ii) $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$ is free.
(iii) S does not contain any of the two directed graphs in Figure 3 as induced subgraphs.
(iv) There is a permutation $w = w_1 w_2 \dots w_n$ of $[n]$ such that

$$w^{-1} \cdot S = \{(j, i) \mid (w_j, w_i) \in S\}$$

is contained in E_n and satisfies the condition in Theorem 2.2.

FIGURE 3. Obstructions to freeness.

PROOF. The implication (i) $\Rightarrow$ (ii) is clear and (iv) $\Rightarrow$ (i) follows immediately from Corollary 3.6. We show the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

Suppose that (ii) holds. For $U \subseteq [n]$ let S_U be the induced subgraph of S on U. Note that the subspace X_U defined by the equations $x_0 = 0$, $x_i = x_j$ for $i, j \in U$ is in the intersection lattice $L_{\boldsymbol{c}\hat{\mathcal{A}}_{n,S}}$ and that the localization of $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$ on X_U is affinely equivalent to $\boldsymbol{c}\hat{\mathcal{A}}_{k,T}$, where $k = \#U$ and T is isomorphic to S_U. By the Localization Theorem [10, 4.37] these localizations are free. Hence to prove (iii) it suffices to check that the arrangements $\boldsymbol{c}\hat{\mathcal{A}}_{3,S_1}$ and $\boldsymbol{c}\hat{\mathcal{A}}_{4,S_2}$ are not free, where S_1 is the path $\{(3, 2), (2, 1)\}$ and $S_4 = \{(2, 1), (4, 3)\}$. It follows from [2, Thm 7.1.5] (see also [1, Thm 5.6]) and can easily be checked otherwise that

$$\chi(\hat{\mathcal{A}}_{3,S_1}, q) = q(q^2 - 5q + 7)$$

and

$$\chi(\hat{\mathcal{A}}_{4,S_2}, q) = q(q - 3)(q^2 - 5q + 7).$$

The Factorization Theorem [10, Thm 4.137] and (1) imply that $\boldsymbol{c}\hat{\mathcal{A}}_{3,S_1}$ and $\boldsymbol{c}\hat{\mathcal{A}}_{4,S_2}$ are not free.

Finally suppose that (iii) holds. Equivalently, we require the following two conditions:

(I) For distinct indices i, j, k with $1 \le i < j < k \le n$, $(k, j) \in S$ and $(j, i) \in S$ imply $(k, i) \in S$.

(II) For $1 \le i < j \le n$, $1 \le k < l \le n$ and i, j, k, l distinct, $(j, i) \in S$ and $(l, k) \in S$ imply $(l, i) \in S$ or $(j, k) \in S$ or both.

We denote by $\mathrm{out}(w)$ the outdegree of a vertex w of S and let $w_1, w_2, \ldots, w_n$ be any linear ordering of the vertices $1, 2, \ldots, n$ of S which satisfies $\mathrm{out}(w_i) \le \mathrm{out}(w_j)$ for $i < j$. First note that, by (I), $(w_j, w_i) \in S$ implies $\mathrm{out}(w_i) < \mathrm{out}(w_j)$ and hence $i < j$. This means that $w^{-1} \cdot S \subseteq E_n$, as claimed. To prove (iv) it remains to check the condition in Theorem 2.2. Let $1 \le i < j < l \le n$ with $(w_j, w_i) \in S$. We want to show that $(w_l, w_i) \in S$, so suppose the contrary. By (II), whenever $(w_l, w_k) \in S$ we have $(w_j, w_k) \in S$. Note also that, by (I), (w_l, w_j) is not in S. It follows that $\mathrm{out}(w_j) > \mathrm{out}(w_l)$, contradicting the fact that $j < l$. □

In contrast to the situation in [5], very few of the arrangements $\boldsymbol{c}\hat{\mathcal{A}}_{3,S}$ of Theorem 4.1 are supersolvable. For the sake of completeness we give next a precise result. We recall that the Localization Theorem remains true if freeness is replaced by supersolvability [14, Prop. 3.2] and refer the reader to [14] and [10, §2.1] for the relevant background.

THEOREM 4.2. *Let $S \subseteq E_n$. The arrangement $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$ is supersolvable if and only if all the edges in S have the same terminal vertex or they all have the same initial vertex.*

PROOF. Suppose that all the edges in S have the same terminal vertex, say

$$S = \{(2, 1), (3, 1), \ldots, (k, 1)\}$$

for some $k \le n$. It is easy to check that the subspace defined by the equations $x_0 = 0$, $x_1 = x_2 = \cdots = x_{n-1}$ is a modular coatom of the intersection lattice $L_{\boldsymbol{c}\hat{\mathcal{A}}_{n,S}}$. The arrangement

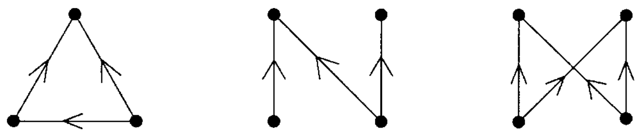

FIGURE 4. Obstructions to supersolvability.

$\boldsymbol{c}\hat{\mathcal{A}}_{n-1,S_{[n-1]}}$, defined by the induced subgraph $S_{[n-1]}$, has the form assumed for $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$, so induction produces the desired maximal chain of modular elements. The second case can be reduced to the first one by applying the linear transformation which sends $x_i \to -x_i$ for $1 \le i \le n$ and fixes x_0.

Now suppose that $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$ is supersolvable. Since supersolvable arrangements are inductively free [10, Thm 4.58] S satisfies the equivalent conditions of Theorem 4.1, in particular condition (iii). To show that S has the form proposed in the theorem it suffices to show that none of the two directed graphs in Figure 3 can occur as a subgraph of S. Equivalently, it suffices to show that none of the three directed graphs in Figure 4 can occur as an *induced* subgraph of S. By the Localization Theorem for supersolvability [14, Prop. 3.2] it suffices to check that the arrangements $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$, defined by these three directed graphs, are not supersolvable. The argument given in [5, Lemma 4.5(c)] applies here as well. If $\mathcal{A}$ is supersolvable with largest exponent b, then it contains a modular coatom having at least $\#\mathcal{A} - b$ hyperplanes. However, one can easily check that in all three cases, any subarrangement with $\#\mathcal{A} - b$ hyperplanes has full rank. □

REMARK. Curiously, the same directed graphs as in the previous two theorems have appeared in recent work by G. D. Bailey [3] and were shown to correspond to the free and supersolvable arrangements, respectively, in a *different* class. This class consists of certain discriminantal arrangements of zonotopes. There seems to be no obvious connection between the two types of result.

We now note that the family of free arrangements in Theorem 4.1 contains simple counterexamples to Orlik's conjecture [10, p. 10, 155], which stated that the restriction of a free arrangement to any of its hyperplanes is free. This was first disproved by Edelman and Reiner [4]. The same authors provided infinitely many counterexamples in [5], including one of dimension 4 with 10 hyperplanes. A counterexample contained in the family of Theorem 4.1 is provided by $\boldsymbol{c}\hat{\mathcal{A}}_{4,S_3}$, where S_3 is shown in Figure 5 and corresponds to the middle graph in Figure 4. This arrangement is free by Corollary 3.6 and has rank 4 and 10 hyperplanes. The restriction of $\boldsymbol{c}\hat{\mathcal{A}}_{4,S_3}$ to the hyperplane $x_2 = x_4$ is affinely equivalent to $\boldsymbol{c}\hat{\mathcal{A}}_{3,S_1}$, corresponding to the first forbidden graph of Figure 3, and hence is not free. As Reiner has pointed out, $\boldsymbol{c}\hat{\mathcal{A}}_{4,S_3}$ is projectively equivalent to the minimum-dimensional counterexample given in [5].

Clearly, any arrangement $\boldsymbol{c}\hat{\mathcal{A}}_{n,S}$ such that S contains an isomorphic copy of S_3 as an induced subgraph is a counterexample to Orlik's conjecture.

A result similar to Theorem 4.1 can be obtained in the same way for the arrangements between $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{A}}_n^{[-1,1]}$, i.e. the one with hyperplanes

$$x_i - x_j = -1, 0, 1 \qquad \text{for } 1 \le i < j \le n.$$

Such an arrangement is of the form

$$\begin{aligned} x_i - x_j &= 0 \qquad \text{for } 1 \le i < j \le n, \\ x_i - x_j &= 1 \qquad \text{for } (j, i) \in G, \end{aligned} \tag{8}$$

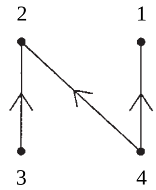

FIGURE 5. The graph S_3.

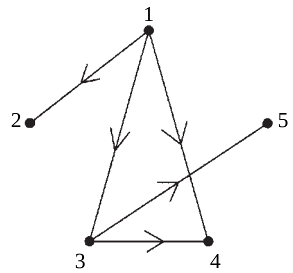

FIGURE 6. A directed graph satisfying (∗).

where $E_n \subseteq G \subseteq \mathcal{E}_n$ and

$$\mathcal{E}_n = \{(i, j) \mid i \neq j, 1 \leq i, j \leq n\}$$

is the edge set of the complete directed graph on the vertex set $[n]$. We denote this arrangement by $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$. The role of the arrangements of Theorem 2.2 will be played by the ones for which G satisfies the following condition:

$$\text{If } 1 \leq i < j < k \leq n \text{ and } (i, k) \in G \text{ then } (i, j) \in G. \qquad (*)$$

Figure 6 shows the edges not in E_n of such a G, for $n = 5$. The arrangement $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$ is also determined by $\mathcal{E}_n - G$. We denote by S this set with the orientation of each edge reversed, so that $S \subseteq E_n$. Clearly, G satisfies (∗) if and only if S satisfies the condition in Theorem 2.2. In the case of Figure 6, for example, we have $S = \{(5, 1), (3, 2), (4, 2), (5, 2), (5, 4)\}$. We state the analogue of Theorem 4.1 and give an outline of the proof. The arrangements $\hat{\mathcal{A}}_n^{[-1,1]}$ were shown to be inductively free by Edelman and Reiner (see the proof of Theorem 3.2 in [6]).

THEOREM 4.3. *Let $E_n \subseteq G \subseteq \mathcal{E}_n$. With the notation above, the following are equivalent:*

(i) $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$ is inductively free.
(ii) $\mathbf{c}\hat{\mathcal{A}}_{n,G}^{[-1,1]}$ is free.
(iii) S does not contain any of the two directed graphs in Figure 3 as induced subgraphs.
(iv) There is a permutation $w = w_1 w_2 \dots w_n$ of $[n]$ such that

$$w^{-1} \cdot S = \{(j, i) \mid (w_j, w_i) \in S\}$$

is contained in E_n and satisfies the condition in Theorem 2.2.

PROOF. The implication (i) ⇒ (ii) is again clear. The implication (iv) ⇒ (i) follows from the inductive freeness of the class of arrangements

$$\begin{aligned} x_1 - x_j &= 0, 1, \dots, m+1 \qquad \text{for } 2 \le j \le k, \\ x_1 - x_j &= 0, 1, \dots, m \qquad \text{for } k < j \le n, \\ x_i - x_j &= 0, 1 \qquad \text{for } 2 \le i < j \le n, \\ x_j - x_i &= 1 \qquad \text{for } (i, j) \in T, 2 \le i < j \le n, \end{aligned}$$

where T satisfies $(*)$. The exponents are the c parameters, together with 0. The proof is as in Theorem 3.5, except that now we remove the hyperplanes $x_1 - x_j = s$ in the order that decreases j (and s). The implication (ii) $\Rightarrow$ (iii) follows again from the Localization Theorem [10, 4.37] by computing explicitly the characteristic polynomial of $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$ when S is one of the two directed graphs of Figure 3 as $q(q^2 - 7q + 13)$ and $q(q-5)(q^2 - 11q + 31)$ respectively. The implication (iii) $\Rightarrow$ (iv) coincides with the corresponding implication in Theorem 4.1. □

5. Other Classes of Deformations

Classes of arrangements that correspond to pairs of graphs seem to be more complicated to analyze from the point of view of freeness. This is the case, for example, with the class of all subarrangements of the Coxeter arrangement of type B_n, as remarked in [5]. We have no obvious suggestion for what all subarrangements of $\hat{\mathcal{A}}_n$ with free cones should look like.

The case of the arrangements between $\mathcal{A}_n$ and $\hat{\mathcal{A}}_n^{[-1,1]}$ seems to deserve special mention. Such an arrangement can be modeled by a directed graph $G \subseteq \mathcal{E}_n$ and has the form (8), so we can still denote it by $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$. The following result, which extends Theorem 2.2 and is a special case of the more general [2, Thm 6.2.10], suggests an explicit answer for this case.

Theorem 5.1 ([1, Thm 3.9, 2, Thm 6.2.7]). *Suppose that the set $G \subseteq \mathcal{E}_n$ has the following properties:*

(i) If $i, j < k$, $i \neq j$ and $(i, j) \in G$, then $(i, k) \in G$ or $(k, j) \in G$ or both.
(ii) If $i, j < k$, $i \neq j$ and $(i, k) \in G$, $(k, j) \in G$, then $(i, j) \in G$.

Then the characteristic polynomial of $\hat{\mathcal{A}}_{n,G}^{[-1,1]}$ factors as in Theorem 2.2, where c_j are the corresponding c parameters.

Note that if $G \subseteq E_n$, the conditions in the previous theorem reduce to the one in Theorem 2.2, while if $E_n \subseteq G$ they reduce to $(*)$. The arguments of Section 3 do not trivially extend to show inductive freeness of the arrangements in Theorem 5.1.

The fact that all deformations of $\mathcal{A}_n$ with free cones that we have encountered in the previous sections turned out to be inductively free suggests that this might be a property of general deformations of $\mathcal{A}_n$, or of deformations of any Weyl arrangement.

Acknowledgements

I feel indebted to the Department of Mathematics at the Technical University of Berlin, and especially to Günter Ziegler, for their hospitality and support which made this research possible. The question of freeness of the Shi arrangement of type A_{n-1} was first suggested to me by Richard Stanley.

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Received 28 August 1996 and accepted in revised form 17 March 1997

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