Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 28 (2007) 1208–1215
European Journal of Combinatorics
www.elsevier.com/locate/ejc

On some enumerative aspects of generalized associahedra

Christos A. Athanasiadis[1]

Department of Mathematics, University of Crete, 71409 Heraklion, Crete, Greece

Received 15 August 2005; accepted 5 February 2006
Available online 3 March 2006

Abstract

We prove a conjecture of F. Chapoton relating certain enumerative invariants of (a) the cluster complex associated by S. Fomin and A. Zelevinsky with a finite root system and (b) the lattice of noncrossing partitions associated with the corresponding finite real reflection group.

1. The result

Let Φ be a finite root system spanning an n-dimensional Euclidean space V with corresponding finite reflection group W. Let Φ^+ be a positive system for Φ with corresponding simple system Π. The cluster complex $\Delta(\Phi)$ was introduced by Fomin and Zelevinsky within the context of their theory of cluster algebras [10–12]. It is a pure $(n-1)$-dimensional simplicial complex on the vertex set $\Phi^+ \cup (-\Pi)$ which is homeomorphic to a sphere [11]. Although $\Delta(\Phi)$ was initially defined under the assumption that Φ is crystallographic [11], its definition and main combinatorial properties are valid without this restriction [8, Section 5.3] [9]. In the crystallographic case, $\Delta(\Phi)$ was realized explicitly in [7] as the boundary complex of an n-dimensional simplicial convex polytope $P(\Phi)$, known as the simplicial generalized associahedron associated with Φ; see [8] for an expository treatment of cluster complexes and generalized associahedra.

The combinatorics of $\Delta(\Phi)$ is closely related to that of a finite poset $\mathbf{L}_W$, known as the lattice of noncrossing partitions associated with W [3,4] (see Section 2 for definitions). It is known, for

E-mail addresses: caa@math.uoc.gr, caath@math.uoa.gr.

[1] Current address: Department of Mathematics (Division of Algebra-Geometry), University of Athens, Panepistimioupolis, Athens 15784, Greece.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2006.02.002

instance (see [8, Theorem 5.9]), that the h-polynomial of $\Delta(\Phi)$ is equal to the rank generating polynomial of $\mathbf{L}_W$. In particular the number of facets of $\Delta(\Phi)$ is equal to the cardinality of $\mathbf{L}_W$. This number is a Catalan number if Φ has type A_n in the Cartan–Killing classification; in that case $P(\Phi)$ is the polar polytope to the classical n-dimensional associahedron [8, Section 3.1] and $\mathbf{L}_W$ is isomorphic to the lattice of noncrossing partitions of the set $\{1, 2, \ldots, n+1\}$ [8, Section 5.1]. The poset $\mathbf{L}_W$ is a self-dual graded lattice of rank n which plays an important role in the geometric group theory and topology of finite-type Artin groups; see [16] for a related survey article.

The F-triangle for Φ, introduced by Chapoton [6, Section 2], is a refinement of the f-vector of $\Delta(\Phi)$ defined by the generating function

$$F(\Phi) = F(x, y) = \sum_{k=0}^{n} \sum_{\ell=0}^{n} f_{k,\ell}\, x^k y^\ell \tag{1}$$

where $f_{k,\ell}$ is the number of faces of $\Delta(\Phi)$ consisting of k positive roots and ℓ negative simple roots. Clearly $f_{k,\ell} = 0$ unless $k + \ell \leq n$. The M-triangle for W is defined similarly [6, Section 3] as

$$M(W) = M(x, y) = \sum_{a \preceq b} \mu(a, b)\, x^{r(b)} y^{r(a)} \tag{2}$$

where $\preceq$ denotes the order relation of $\mathbf{L}_W$, μ stands for its Möbius function [18, Section 3.6], $r(a)$ is the rank of $a \in \mathbf{L}_W$ and the sum runs over all pairs (a, b) of elements of $\mathbf{L}_W$ with $a \preceq b$. The main objective of this note is to prove the following theorem, the rather surprising statement of which appears as [6, Conjecture 1] and includes many of the known similarities between the enumerative properties of $\Delta(\Phi)$ and $\mathbf{L}_W$ as special cases; see [6, Sections 3.1–3.5].

Theorem 1.1. *The F-triangle for Φ and M-triangle for W are related by the equality*

$$(1-y)^n\, F\left(\frac{x+y}{1-y}, \frac{y}{1-y}\right) = M(-x, -y/x). \tag{3}$$

The proof of Theorem 1.1 (Section 3) relies on two known special cases, one relating the number $f_{n,0}$ of facets of $\Delta(\Phi)$ consisting of only positive roots to the Möbius number of $\mathbf{L}_W$ [6, (23)] and the one already mentioned, relating the h-polynomial of $\Delta(\Phi)$ to the rank generating polynomial of $\mathbf{L}_W$. A case-free proof of the relevant statement was given in [1, Corollary 4.4] in the former case and in [2] in the latter case (see also Remark 9.4 in [17]). To extract the proof of the theorem from the two special cases we utilize the appearance of the cluster complex in the context of the lattice $\mathbf{L}_W$ in the work of Brady and Watt [5]. This connection is briefly outlined in Section 2, where the two special cases are conveniently generalized (Lemmas 2.4 and 2.6). Theorem 1.1 can also be verified with case by case computations; see [6, Sections 4–5], [14, 15] (where a more general version of Chapoton's conjecture is considered) and references given there. The results of [1,2,5], mentioned earlier, and the results of this paper complete a case-free proof of the theorem.

2. Noncrossing partitions and cluster complexes

Throughout this section W is a finite real reflection group of rank n with set of reflections T and Φ is a root system spanning an n-dimensional Euclidean space V with associated reflection group W. We refer the reader to [13,18] for background and any undefined terminology on root systems, finite reflection groups and partially ordered sets.

2.1. The lattice $\mathbf{L}_W$

For $w \in W$ let $r(w) = r_T(w)$ denote the smallest integer k such that w can be written as a product of k reflections in T. Define a partial order $\preceq$ on W by letting

$$u \preceq v \quad \text{if and only if } r(u) + r(u^{-1}v) = r(v),$$

in other words if there exists a shortest factorization of u into reflections in T which is a prefix of such a shortest factorization of v. Since T is invariant under conjugation the function r_T is constant on conjugacy classes of W and we have $u \preceq v$ if and only if $wuw^{-1} \preceq wvw^{-1}$ for $u, v, w \in W$.

Lemma 2.1. *Let a, b, w be elements of W.*

(i) *$a \preceq aw \preceq b$ if and only if $w \preceq a^{-1}b \preceq b$.*
(ii) *$a \preceq aw \preceq b$ if and only if $a \preceq bw^{-1} \preceq b$.*
(iii) *$a \preceq b$ if and only if $a^{-1}b \preceq b$ and, in that case, the interval $[a, b]$ is isomorphic to $[1, a^{-1}b]$.* □

Proof. Consider part (ii) and suppose that $a \preceq aw \preceq b$. From the definition of $\preceq$ we have $r(a) + r(w) = r(aw)$ and $b = awc$ with $r(aw) + r(c) = r(b)$. Let $c' = wcw^{-1}$ and observe that $r(c) = r(c')$. The factorization $b = ac'w$ implies that $r(ac') = r(a) + r(c')$, since $r(ac') \le r(a) + r(c')$ on the one hand and $r(ac') \ge r(b) - r(w) = r(a) + r(c) = r(a) + r(c')$ on the other. It follows that $a \preceq ac' = bw^{-1}$ and $bw^{-1} = ac' \preceq b$, so that $a \preceq bw^{-1} \preceq b$. The converse and part (i) are treated in a similar way. Part (iii) follows from part (i). □

The order $\preceq$ turns W into a graded poset having the identity 1 as its unique minimal element and rank function r_T. For $w \in W$ we denote by $\mathbf{L}_W(w)$ the interval $[1, w]$ in this order. We are primarily interested in the case where w is a Coxeter element γ of W. Since all Coxeter elements of W are conjugate to each other, the isomorphism type of the poset $\mathbf{L}_W(\gamma)$ is independent of γ. This poset is denoted by $\mathbf{L}_W$ when the choice of γ is irrelevant and called the *noncrossing partition lattice* associated with W. If W is reducible, decomposing as a direct product $W_1 \times W_2 \times \cdots \times W_k$ of irreducible parabolic subgroups, then $\mathbf{L}_W$ is isomorphic to the direct product of the posets $\mathbf{L}_{W_i}$.

2.2. The cluster complex

It was shown in [5, Section 8] that the cluster complex $\Delta(\Phi)$ arises naturally in the context of the lattice $\mathbf{L}_W$. We give a brief account of this connection here and refer the reader to [5] for more details. We denote by t_α the reflection in the hyperplane in V orthogonal to α and by N the number of reflections in T. Let Φ^+ be a fixed positive system for Φ and γ be a corresponding bipartite Coxeter element of W, so that $\gamma = \gamma_+\gamma_-$ where

$$\gamma_\pm = \prod_{\alpha \in \Pi_\pm} t_\alpha$$

and $\Pi_+ = \{\alpha_1, \ldots, \alpha_s\}$, $\Pi_- = \{\alpha_{s+1}, \ldots, \alpha_n\}$ are orthogonal sets which form a partition of the simple system Π determined by Φ^+. Assume first that Φ is irreducible. Letting

$\rho_i = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{i-1}}(\alpha_i)$ for $i \geq 1$ (so that $\rho_1 = \alpha_1$), where the α_i are indexed cyclically modulo n, and $\rho_{-i} = \rho_{2N-i}$ for $i \geq 0$ we have

$$\{\rho_1, \rho_2, \ldots, \rho_N\} = \Phi^+,$$
$$\{\rho_{N+i} : 1 \leq i \leq s\} = \{-\rho_1, \ldots, -\rho_s\} = -\Pi_+,$$
$$\{\rho_{-i} : 0 \leq i < n - s\} = \{-\rho_{N-i} : 0 \leq i < n - s\} = -\Pi_-.$$

Define an abstract simplicial complex $\Delta(\gamma)$ on the vertex set $\Phi_{\geq -1} = \Phi^+ \cup (-\Pi) = \{\rho_{-n+s+1}, \ldots, \rho_0, \rho_1, \ldots, \rho_{N+s}\}$ by declaring a set $\sigma = \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$ with $i_1 < i_2 < \cdots < i_k$ to be a face if and only if

$$w_\sigma = t_{\rho_{i_k}} t_{\rho_{i_{k-1}}} \cdots t_{\rho_{i_1}} \tag{4}$$

is an element of $\mathbf{L}_W(\gamma)$ of rank k. If Φ is reducible with irreducible components $\Phi_1, \Phi_2, \ldots, \Phi_m$ then $\gamma = \gamma_1 \gamma_2 \cdots \gamma_m$ where γ_i is a bipartite Coxeter element for the reflection group W_i corresponding to Φ_i and $\mathbf{L}_W(\gamma)$ is isomorphic to the direct product of the posets $\mathbf{L}_{W_i}(\gamma_i)$. We define $\Delta(\gamma)$ as the simplicial join of the complexes $\Delta(\gamma_i)$, so that $\sigma \in \Delta(\gamma)$ if and only if $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_m$ with $\sigma_i \in \Delta(\gamma_i)$ for $1 \leq i \leq m$. In that case we also define

$$w_\sigma = w_{\sigma_1} w_{\sigma_2} \cdots w_{\sigma_m}, \tag{5}$$

where the w_{σ_i} mutually commute, so that w_σ is an element of $\mathbf{L}_W(\gamma)$ of rank equal to the cardinality of σ.

Let $\Delta_+(\gamma)$ and $\Delta_+(\Phi)$ denote the induced subcomplexes of $\Delta(\gamma)$ and $\Delta(\Phi)$, respectively, on the vertex set Φ^+. The following theorem is proved in [5, Section 8] (see also [5, Note 4.2]) in the case of irreducible root systems and extends by definition to the general case.

Theorem 2.2 (*[5]*). *As an abstract simplicial complex $\Delta(\Phi)$ coincides with $\Delta(\gamma)$. In particular, $\Delta_+(\Phi)$ coincides with $\Delta_+(\gamma)$.* □

As a consequence the complexes $\Delta(\gamma)$ and $\Delta_+(\gamma)$ depend only on our fixed choice of Φ^+. Recall (see [3, Proposition 1.6.4]) that any two reflections $t_1, t_2 \in T$ for which $t_1 t_2 \preceq \gamma$ and $t_2 t_1 \preceq \gamma$ are commuting. This implies, in the case of an irreducible root system Φ, that any rearrangement of the product in the right-hand side of (4) which is an element of $\mathbf{L}_W(\gamma)$ of rank k is equal to w_σ. In particular the map $\Delta(\gamma) \mapsto \mathbf{L}_W$ sending σ to w_σ depends only on Φ^+ and γ and not on the specific linear orderings of Π_+ and Π_- used in the definition of $\Delta(\gamma)$. Similar remarks hold for the product in (5) when Φ is reducible. For any $w \in \mathbf{L}_W(\gamma)$ the faces σ of $\Delta_+(\gamma)$ with $w_\sigma = w$ are the facets of a subcomplex $\Delta_+(w)$ of $\Delta_+(\gamma)$ (this corresponds to the complex denoted by $X(w)$ in [5, Section 5]). Clearly the sets of facets of the subcomplexes $\Delta_+(w)$ for $w \in \mathbf{L}_W(\gamma)$ form a partition of the set of faces of $\Delta_+(\gamma)$ into mutually disjoint subsets.

In the next proposition we gather some facts from [11, Section 3], modified according to some observations made in [5, Section 8] (for the non-crystallographic root systems see, for instance, [9]). For the sake of simplicity we assume that Φ is irreducible and let $R : \Phi_{\geq -1} \to \Phi_{\geq -1}$ be the map defined by

$$R(\alpha) = \begin{cases} \gamma^{-1}(\alpha), & \text{if } \alpha \notin \Pi_+ \cup (-\Pi_-) \\ -\alpha, & \text{if } \alpha \in \Pi_+ \cup (-\Pi_-) \end{cases}$$

and let $R(\sigma) = \{R(\alpha) : \alpha \in \sigma\}$ for $\sigma \subseteq \Phi_{\geq -1}$. For $\sigma \subseteq \Pi$ we denote by Φ_σ the standard parabolic root subsystem obtained by intersecting Φ with the linear span of $\Pi \setminus \sigma$, endowed with the induced positive system $\Phi^+_\sigma = \Phi^+ \cap \Phi_\sigma$, and abbreviate Φ_σ as Φ_α when $\sigma = \{\alpha\}$.

Proposition 2.3. *Let Φ be irreducible, $\alpha \in \Pi$ and $\sigma \subseteq \Phi_{\geq -1}$.*

(i) *For $\sigma \in \Delta(\Phi)$ we have $-\alpha \in \sigma$ if and only if $\sigma \setminus \{-\alpha\} \in \Delta(\Phi_\alpha)$.*
(ii) *For any $\beta \in \Phi^+$ there exists i such that $R^i(\beta) \in (-\Pi)$.*
(iii) *$\sigma \in \Delta(\Phi)$ if and only if $R(\sigma) \in \Delta(\Phi)$.* □

2.3. The Möbius function

We write $\mu(w)$ instead of $\mu(\hat{0}, w)$ for the Möbius function between $\hat{0}$ and $w \in P$ of a finite poset P with a unique minimal element $\hat{0}$. Let γ be a bipartite Coxeter element of W, as in Section 2.2. It is known [6, (23)] [1, Corollary 4.4] that the Möbius number $\mu(\gamma)$ of $\mathbf{L}_W = \mathbf{L}_W(\gamma)$ is equal, up to the sign $(-1)^n$, to the number of facets of $\Delta_+(\Phi)$. This fact generalizes as follows.

Lemma 2.4. *For $w \in \mathbf{L}_W(\gamma)$ the number $(-1)^{r(w)}\mu(w)$ is equal to the number of facets of $\Delta_+(w)$.*

Proof. It suffices to treat the case that Φ is irreducible. By [1, Corollary 4.3] $(-1)^{r(w)}\mu(w)$ is equal to the number of factorizations $w = t_{\rho_{i_k}} t_{\rho_{i_{k-1}}} \cdots t_{\rho_{i_1}}$ of length $k = r(w)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. The set of such factorizations is in bijection with the set of faces σ of $\Delta_+(\gamma)$ with $w_\sigma = w$, in other words with the set of facets of $\Delta_+(w)$. □

The next corollary is the specialization $y = 0$ of Theorem 1.1.

Corollary 2.5. *The coefficient of q^k in the characteristic polynomial*

$$\chi(\mathbf{L}_W, q) = \sum_{w \in \mathbf{L}_W} \mu(w)\, q^{r(w)}$$

of $\mathbf{L}_W$ is equal to $(-1)^k$ times the number of faces of $\Delta_+(\Phi)$ of dimension $k-1$.

Proof. This follows from Theorem 2.2, Lemma 2.4 and the fact that the set of faces of $\Delta_+(\gamma)$ of dimension $k-1$ is the disjoint union of the sets of facets of the subcomplexes $\Delta_+(w)$ where w ranges over all elements of $\mathbf{L}_W(\gamma)$ of rank k. □

2.4. Links and h-polynomials

Recall that the h-polynomial of an abstract simplicial complex Δ of dimension $n-1$ is defined as

$$h(\Delta, y) = \sum_{i=0}^{n} f_i(\Delta)\, y^i (1-y)^{n-i}$$

where $f_i(\Delta)$ is the number of faces of Δ of dimension $i-1$. The link of a face σ of Δ is the abstract simplicial complex $\mathrm{lk}_\Delta(\sigma) = \{\tau \setminus \sigma : \sigma \subseteq \tau \in \Delta\}$. It is known [6, Section 3.1] [8, Theorem 5.9] that

$$h(\Delta(\Phi), y) = \sum_{a \in \mathbf{L}_W} y^{r(a)} \tag{6}$$

for any root system Φ (the coefficients of either hand side of (6) are known as the *Narayana numbers* associated with W). This fact generalizes as follows, where γ is as in Section 2.2.

Lemma 2.6. *For any face σ of $\Delta = \Delta(\Phi)$ we have*

$$h(\mathrm{lk}_\Delta(\sigma), y) = \sum_{a \in \mathbf{L}_W(\gamma w_\sigma^{-1})} y^{r(a)}.$$

Proof. We will use induction on the rank of Φ. The proposed equality follows by induction if Φ is reducible, since both hand sides are multiplicative in W, and reduces to (6) if σ is empty. Henceforth we assume that Φ is irreducible and let $\sigma = \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$, in the notation of Section 2.2, with $i_1 < i_2 < \cdots < i_k$ and $k \geq 1$. We distinguish three cases.

Case 1. $\sigma \cap (-\Pi_-) \neq \emptyset$. Then $\rho_{i_1} = -\alpha$ with $\alpha \in \Pi_-$ and hence, by Proposition 2.3(i), we have $\mathrm{lk}_\Delta(\sigma) = \mathrm{lk}_{\Delta'}(\sigma')$ where $\Delta' = \Delta(\Phi_\alpha)$ and $\sigma' = \sigma \setminus \{-\alpha\}$. The induction hypothesis implies that

$$h(\mathrm{lk}_\Delta(\sigma), y) = \sum_{a \in \mathbf{L}_W(\gamma' w_{\sigma'}^{-1})} y^{r(a)} \tag{7}$$

where $\gamma' = \gamma t_\alpha$. Clearly $t_{\rho_{i_k}} \cdots t_{\rho_{i_2}} = w_\sigma t_\alpha$ is a rank $k-1$ element of $\mathbf{L}_W(\gamma')$ and hence $w_{\sigma'} = w_\sigma t_\alpha$. Thus $\gamma' w_{\sigma'}^{-1} = \gamma w_\sigma^{-1}$ and the result follows from (7).

Case 2. $\sigma \cap (-\Pi_+) \neq \emptyset$. Then $\rho_{i_k} = -\alpha$ with $\alpha \in \Pi_+$ and (7) continues to hold with $\gamma' = t_\alpha \gamma$ and $w_{\sigma'} = t_\alpha w_\sigma$, where Δ' and σ' have the same meaning as in the previous case. Since $\gamma' w_{\sigma'}^{-1} = t_\alpha(\gamma w_\sigma^{-1})\, t_\alpha$ is conjugate to γw_σ^{-1}, the poset $\mathbf{L}_W(\gamma' w_{\sigma'}^{-1})$ is isomorphic to $\mathbf{L}_W(\gamma w_\sigma^{-1})$ and the result follows again from (7).

Case 3. $\sigma \cap (-\Pi) = \emptyset$. Let $\ell \geq 1$ be such that $\rho_{i_j} \in \Pi_+$ if and only if $j < \ell$ and let $\sigma' = \{\rho_{i_\ell}, \ldots, \rho_{i_k}\}$. Proposition 2.3(iii) implies that $\mathrm{lk}_\Delta(\sigma)$ is isomorphic to $\mathrm{lk}_\Delta(R(\sigma))$. Since $R(\alpha) = -\alpha \in (-\Pi_+)$ for $\alpha \in \sigma \setminus \sigma' = \sigma \cap \Pi_+$, we have in turn that $\mathrm{lk}_\Delta(R(\sigma)) = \mathrm{lk}_{\Delta'}(R(\sigma'))$ by part (i) of the same proposition, where $\Delta' = \Delta(\Phi_{\sigma \setminus \sigma'})$. Suppose first that $\sigma \cap \Pi_+$ is nonempty, so that $\ell \geq 2$ and $\Phi_{\sigma \setminus \sigma'}$ has smaller rank than Φ. The previous observations and the induction hypothesis imply that

$$h(\mathrm{lk}_\Delta(\sigma), y) = \sum_{a \in \mathbf{L}_W(\gamma' w_{R(\sigma')}^{-1})} y^{r(a)} \tag{8}$$

where $\gamma' = t_{\rho_{i_1}} \cdots t_{\rho_{i_{\ell-1}}} \gamma$. Since $w_\sigma = t_{\rho_{i_k}} \cdots t_{\rho_{i_1}}$ is an element of $\mathbf{L}_W(\gamma)$ of rank k we have that $t_{\rho_{i_k}} \cdots t_{\rho_{i_\ell}}$ is an element of $\mathbf{L}_W(\gamma\, t_{\rho_{i_1}} \cdots t_{\rho_{i_{\ell-1}}})$ of rank $k - \ell + 1$ and hence that $t_{R(\rho_{i_k})} \cdots t_{R(\rho_{i_\ell})} = \gamma^{-1} t_{\rho_{i_k}} \cdots t_{\rho_{i_\ell}} \gamma$ is an element of $\mathbf{L}_W(\gamma')$ of rank $k - \ell + 1$. Therefore $w_{R(\sigma')} = \gamma^{-1} t_{\rho_{i_k}} \cdots t_{\rho_{i_\ell}} \gamma$ and $\gamma' w_{R(\sigma')}^{-1} = w_\sigma^{-1} \gamma$ is conjugate to γw_σ^{-1}, so that the result follows from (8) as in the second case. Finally suppose that $\sigma \cap \Pi_+ = \emptyset$. By the previous argument it suffices to prove that

$$h(\mathrm{lk}_\Delta(R(\sigma)), y) = \sum_{a \in \mathbf{L}_W(\gamma w_{R(\sigma)}^{-1})} y^{r(a)}.$$

In view of Proposition 2.3(ii), applying similarly R sufficiently many times brings us back either to the previous situation or to one of the first two cases. $\square$

3. Proof of Theorem 1.1

Throughout this section γ is a bipartite Coxeter element of W, as in Section 2.2, and $|\sigma|$ denotes the cardinality of a finite set σ.

Proof of Theorem 1.1. To simplify notation let us write Δ, Δ_+ and $\mathbf{L}$ instead of $\Delta(\Phi)$, $\Delta_+(\Phi)$ and $\mathbf{L}_W = \mathbf{L}_W(\gamma)$, respectively, and $a \preceq_{\mathbf{L}} b$ instead of $a \preceq b$ with $a, b \in \mathbf{L}$. From (1) we have

$$\begin{aligned}(1-y)^n F\left(\frac{x+y}{1-y}, \frac{y}{1-y}\right) &= \sum_{k,\ell} f_{k,\ell}(x+y)^k y^\ell (1-y)^{n-k-\ell} \\ &= \sum_{k,\ell,i} f_{k,\ell} \binom{k}{i} x^i y^{k+\ell-i} (1-y)^{n-k-\ell} \\ &= \sum_{\tau \in \Delta} \sum_{\substack{\sigma \in \Delta_+ \\ \sigma \subseteq \tau}} x^{|\sigma|} y^{|\tau|-|\sigma|} (1-y)^{n-|\tau|} \\ &= \sum_{\sigma \in \Delta_+} x^{|\sigma|} \sum_{\substack{\tau \in \Delta \\ \sigma \subseteq \tau}} y^{|\tau|-|\sigma|} (1-y)^{n-|\tau|} \\ &= \sum_{\sigma \in \Delta_+} x^{|\sigma|} \sum_{\tau' \in \mathrm{lk}_\Delta(\sigma)} y^{|\tau'|} (1-y)^{n-|\sigma|-|\tau'|}\end{aligned}$$

and hence that

$$(1-y)^n\, F\left(\frac{x+y}{1-y}, \frac{y}{1-y}\right) = \sum_{\sigma \in \Delta_+} x^{|\sigma|}\, h(\mathrm{lk}_\Delta(\sigma), y). \tag{9}$$

Similarly, using (2) and observing from Lemma 2.1(iii) that $\mu(a,b) = \mu(w)$, where $w = a^{-1}b$, we have

$$\begin{aligned}M(-x, -y/x) &= \sum_{a \preceq_{\mathbf{L}} b} \mu(a,b)(-x)^{r(b)-r(a)} y^{r(a)} \\ &= \sum_{a \preceq_{\mathbf{L}} aw} \mu(w)(-x)^{r(w)} y^{r(a)} \\ &= \sum_{w \in \mathbf{L}} (-x)^{r(w)} \mu(w) \sum_{a \preceq_{\mathbf{L}} \gamma w^{-1}} y^{r(a)}.\end{aligned}$$

Observe that we have used Lemma 2.1(ii) to conclude that for $a, w \in \mathbf{L}$ we have $a \preceq aw \in \mathbf{L}$ if and only if $a \preceq \gamma w^{-1} \in \mathbf{L}$. From the last expression and Lemma 2.4 we have

$$M(-x, -y/x) = \sum_{w \in \mathbf{L}} x^{r(w)} \sum_{\substack{\sigma \in \Delta_+ \\ w_\sigma = w}} \sum_{a \preceq_{\mathbf{L}} \gamma w^{-1}} y^{r(a)}$$

or, equivalently,

$$M(-x, -y/x) = \sum_{\sigma \in \Delta_+} x^{|\sigma|} \sum_{a \preceq_{\mathbf{L}} \gamma w_\sigma^{-1}} y^{r(a)}. \tag{10}$$

In view of Lemma 2.6, the result follows by comparing (9) and (10). □

References

[1] C.A. Athanasiadis, T. Brady, C. Watt, Shellability of noncrossing partition lattices, preprint, 10 pages, Proc. Amer. Math. Soc. (in press).

[2] C.A. Athanasiadis, T. Brady, J. McCammond, C. Watt, h-Vectors of generalized associahedra and noncrossing partitions, preprint, 2006, 20 pages.
[3] D. Bessis, The dual braid monoid, Ann. Sci. Ecole Norm. Sup. 36 (2003) 647–683.
[4] T. Brady, C. Watt, $K(\pi, 1)$'s for Artin groups of finite type, in: Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa 2000), Geom. Dedicata, 94, 2002, pp. 225–250.
[5] T. Brady, C. Watt, Lattices in finite real reflection groups, preprint, 29 pages, Trans. Amer. Math. Soc. (in press).
[6] F. Chapoton, Enumerative properties of generalized associahedra, Sém. Lothar. Combin. 51 (2004) 16. Article B51b, (electronic).
[7] F. Chapoton, S. Fomin, A.V. Zelevinsky, Polytopal realizations of generalized associahedra, Canad. Math. Bull. 45 (2002) 537–566.
[8] S. Fomin, N. Reading, Root systems and generalized associahedra, in: Geometric Combinatorics, IAS/Park City Math. Ser. (in press).
[9] S. Fomin, N. Reading, Generalized cluster complexes and Coxeter combinatorics, Int. Math. Res. Not. 44 (2005) 2709–2757.
[10] S. Fomin, A.V. Zelevinsky, Cluster algebras I: Foundations, J. Amer. Math. Soc. 15 (2002) 497–529.
[11] S. Fomin, A.V. Zelevinsky, Y-systems and generalized associahedra, Ann. of Math. 158 (2003) 977–1018.
[12] S. Fomin, A.V. Zelevinsky, Cluster algebras II: finite type classification, Invent. Math. 154 (2003) 63–121.
[13] J.E. Humphreys, Reflection groups and Coxeter groups, in: Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, England, 1990.
[14] C. Krattenthaler, The F-triangle of the generalized cluster complex, in: Topics in Discrete Mathematics, M. Klazar (Ed.), Nesetril Festschrift, Springer-Verlag, Berlin, New York (in press).
[15] C. Krattenthaler, The M-triangle of generalized noncrossing partitions for the types E_7 and E_8, preprint, 2006, 34 pages.
[16] J. McCammond, An introduction to Garside structures, preprint, 2004, 28 pages.
[17] N. Reading, Clusters, Coxeter sortable elements and noncrossing partitions, preprint, 28 pages, ArXiV preprint math.CO/0507186, Trans. Amer. Math. Soc. (in press).
[18] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge University Press, Cambridge, 1997.