

THE GENERALIZED BAUES PROBLEM FOR CYCLIC POLYTOPES II

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ABSTRACT. Given an affine surjection of polytopes $\pi : P \rightarrow Q$, the Generalized Baues Problem asks whether the poset of all proper polyhedral subdivisions of Q which are induced by the map π has the homotopy type of a sphere. We extend earlier work of the last two authors on subdivisions of cyclic polytopes to give an affirmative answer to the problem for the natural surjections between cyclic polytopes $\pi : C(n, d') \rightarrow C(n, d)$ for all $1 \leq d < d' < n$.

1. INTRODUCTION

The Generalized Baues Problem, posed by Billera, Kapranov and Sturmfels [4], is a question in combinatorial geometry and topology, motivated by the theory of fiber polytopes [5] [18, Lecture 9]. Given an affine surjection of polytopes $\pi : P \rightarrow Q$, the problem asks to determine whether the *Baues poset* $\omega(P \xrightarrow{\pi} Q)$ of all proper polyhedral subdivisions of Q which are induced in a certain way by the map π , endowed with a standard topology [6], has the homotopy type of a sphere of dimension $\dim(P) - \dim(Q) - 1$. We refer to [11] for a concise introduction and [15] for a recent survey.

Although the Generalized Baues Problem is known to have a negative answer in general [14], various special cases have remained of interest in the literature; see [15, Section 4]. One such relates to subdivisions of cyclic polytopes. Another is the case where P is a simplex, in which $\omega(P \xrightarrow{\pi} Q)$ is the poset of *all* proper polyhedral subdivisions of Q and is simply denoted $\omega(Q)$. In [9] an affirmative answer to the problem was given in the case of the poset of all subdivisions of cyclic polytopes of dimension at most 3. This was recently improved in [13] to all dimensions, as follows.

Theorem 1.1. ([13, Theorem 1.1]) *For all $1 \leq d < n$, the Baues poset $\omega(C(n, d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n, d)$ is homotopy equivalent to an $(n - d - 2)$ -sphere.*

For $1 \leq d < d' < n$, one can consider the natural projections $\pi : C(n, d') \rightarrow C(n, d)$ between cyclic polytopes [1]. The Baues poset $\omega(C(n, d))$ in Theorem 1.1 is the Baues poset of the projection π for $d' = n - 1$. In this paper we use the “sliding” technique of [13] to give an affirmative answer to the Generalized Baues Problem for π for all d, d' and n .

Theorem 1.2. *For $1 \leq d < d' < n$, the Baues poset $\omega(C(n, d') \xrightarrow{\pi} C(n, d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n, d)$ which are induced by π is homotopy equivalent to a $(d' - d - 1)$ -sphere.*

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Theorem 1.2 was conjectured by Reiner [15] on the basis of the following special cases:

- $d = 2, d' = n - 2$ [1, Corollary 6.3],
- $d' = n - 1$ (Theorem 1.1),
- $d = 2, n < 2d' + 2, d' \geq 9$ [16, Corollary 15].

Other previously known special cases are those of $d = 1$ and $d' - d \leq 2$, which follow from more general results of [4] and [14], respectively: for any polytope projection $\pi : P \rightarrow Q$, the poset $\omega(P \xrightarrow{\pi} Q)$ of all proper π -induced subdivisions of Q is homotopy equivalent to a sphere whenever $\dim(Q) = 1$ or $\dim(P) - \dim(Q) \leq 2$.

Our argument is a modification of the one used in [13, Section 4] to prove Theorem 1.1 and therefore relies heavily on the constructions of [13]. In the next section we review some basic definitions and facts. In Section 3 we give a sketch of the proof of Theorem 1.2, thereby recalling those constructions from [13] that will be essential here. Section 4 contains the remaining details, which amount to proving that two certain posets of subdivisions are contractible.

2. PRELIMINARIES

2.1. Polyhedral subdivisions. By a point configuration \mathcal{A} in \mathbb{R}^d we mean a finite labeled subset of \mathbb{R}^d . We allow \mathcal{A} to have repeated points which are distinguished by their labels. The convex hull $\text{conv}(\mathcal{A})$ of \mathcal{A} is a polytope.

A face of a subconfiguration $\sigma \subseteq \mathcal{A}$ is a subconfiguration $F^\omega \subseteq \sigma$ consisting of all points on which some linear functional $\omega \in (\mathbb{R}^d)^*$ takes its minimum over σ .

We say that two subconfigurations σ_1 and σ_2 of \mathcal{A} *intersect properly* if the following two conditions are satisfied:

- $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 ;
- $\text{conv}(\sigma_1) \cap \text{conv}(\sigma_2) = \text{conv}(\sigma_1 \cap \sigma_2)$.

A subconfiguration of \mathcal{A} is said to be *full-dimensional*, or *spanning*, if it affinely spans \mathbb{R}^d . In that case we call it a *cell*. Following [3] and [10, Section 7.2] we say that a collection S of cells of \mathcal{A} is a (*polyhedral*) *subdivision* of \mathcal{A} if the elements of S intersect pairwise properly and cover $\text{conv}(\mathcal{A})$ in the sense that

$$\cup_{\sigma \in S} \text{conv}(\sigma) = \text{conv}(\mathcal{A}).$$

Cells that share a common facet are *adjacent*. The set of subdivisions of \mathcal{A} is partially ordered by the *refinement* relation

$$S_1 \leq S_2 \quad : \iff \quad \forall \sigma_1 \in S_1, \exists \sigma_2 \in S_2 : \sigma_1 \subset \sigma_2.$$

The poset of subdivisions of \mathcal{A} has a unique maximal element which is the trivial subdivision $\{\mathcal{A}\}$. The minimal elements are the subdivisions all of whose cells are affinely independent, which are called *triangulations* of \mathcal{A} . We call subdivisions of a polytope Q the subdivisions of its vertex set.

2.2. Induced subdivisions. Now let $P \subset \mathbb{R}^p$ be a polytope, and let $\pi : \mathbb{R}^p \rightarrow \mathbb{R}^d$ be a linear projection map. We can consider the point configuration $\mathcal{A} = \pi(\text{vert}(P))$ arising from the projection of the vertex set of P . An element in \mathcal{A} is labeled by the vertex of P of which it is considered to be the image. In other words, π induces a bijection between the vertex set of P and \mathcal{A} , even if different vertices of P have the same projection.

A subdivision S of \mathcal{A} is said to be π -induced if every cell of S is the projection of the vertex set of a face of P . If P is a simplex then all subdivisions of \mathcal{A} are π -induced. This concept of π -induced subdivisions was introduced in [5].

A π -induced subdivision S contains the same information as the collection of faces of P whose vertex sets are in S . In this sense one can say that a π -induced subdivision of \mathcal{A} is a polyhedral subdivision whose cells are projections of faces of P (this statement is not accurate; see [11, 14, 18] for an accurate definition of π -induced subdivisions in terms of faces of P).

The poset of π -induced subdivisions excluding the trivial one is denoted by $\omega(P \xrightarrow{\pi} \pi(P))$. Its minimal elements are the subdivisions for which every cell comes from a $\dim(\mathcal{A})$ -dimensional face of P . They are called *tight* π -induced subdivisions.

In [4] it was conjectured that the Baues poset $\omega(P \xrightarrow{\pi} \pi(P))$ is homotopy equivalent to a sphere of dimension $p - d - 1$. Evidence for this were the cases $p - d = 1$ (trivial) and $d = 1$ (proved in [4]) together with the fact that $\omega(P \xrightarrow{\pi} \pi(P))$ always contains a subposet *homeomorphic* to a sphere of dimension $p - d - 1$ (the poset of *coherent* π -induced subdivisions [5]). The conjecture was known as the *generalized Baues conjecture* since the case $d = 1$ had been conjectured by J. Baues in a different form, until it was disproved in [14]. Still, several cases remain of interest. Theorem 1.1 is the case where π is the natural projection from a simplex to a cyclic polytope and our Theorem 1.2 is the case where π is the natural projection between two cyclic polytopes. Other cases where the statement is known to be true are when $p - d = 2$ [14] and when P is a simplex and $d = 2$ [8].

See [5, 15, 18] for more information on π -induced subdivisions and the Baues problem.

2.3. Poset topology. When referring to the topology of a finite poset we mean the topology of its *order complex*, i.e., the simplicial complex of chains in the poset [6]. For a poset P and $x \in P$ we denote by $P_{\leq x}$ the set $\{y \in P : y \leq x\}$. We will use the following tool from [2] to relate the homotopy type of two posets. A proof is given in [17, Section 3].

Lemma 2.1. (Babson) *Let $f : \omega \rightarrow \omega'$ be an order preserving map of posets. If*

- (i) $f^{-1}(y)$ is contractible for every $y \in \omega'$ and
- (ii) $\omega_{\leq x} \cap f^{-1}(y)$ is contractible for every $x \in \omega$ and $y \in \omega'$ with $f(x) > y$

then f induces a homotopy equivalence.

2.4. Cyclic polytopes. The cyclic polytope $C(n, d)$ is the convex hull of any n points on the moment curve $\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\}$ in \mathbb{R}^d . We consider it as the point configuration consisting of these n points, which are the vertices for $d \geq 2$. Hence, all the notions for induced subdivisions make sense for cyclic polytopes. Also, we extend the usual definition by the trivial case of $d = 0$: the cyclic polytope $C(n, 0)$ is just the set of n copies of the only point in \mathbb{R}^0 . The cyclic polytope $C(n, 1)$ consists of n distinct points in the real line \mathbb{R} .

As usual, we label the vertices of $C(n, d)$ with the numbers $1, \dots, n$, in the order they appear along the moment curve and refer to faces of $C(n, d)$ by the index sets of their vertices, i.e. as subsets of $[n] := \{1, 2, \dots, n\}$.

The face lattice of $C(n, d)$ is known to be independent of the choice of points on the curve and is characterized by Gale's evenness criterion, which is as follows (see also [18, p. 14] or [1, Theorem 5.2]). For a subset $F \subset [n]$ with complement

$[n] \setminus F = \{a_1, a_2, \dots, a_k\}$, we divide F in its *initial interval* $\{1, \dots, a_1 - 1\}$, its *final interval* $\{a_k + 1, \dots, n\}$ and its *interior intervals* $\{a_i + 1, \dots, a_{i+1} - 1\}$, $i = 1, \dots, k - 1$. The initial and final intervals may be empty. An interval is called *odd* if it has an odd number of elements and *even* otherwise. Then, F is a face of $C(n, d)$ if and only if the cardinality of F plus the number of odd interior intervals does not exceed d . Two obvious consequences of this description are that cyclic polytopes are simplicial and that faces of $C(n, d)$ are also faces of $C(n, d')$ for $d' > d$.

Moreover, if d is the smallest integer for which F is a face of $C(n, d)$, then F is an *upper face* of $C(n, d)$ (meaning that its normal cone contains only vectors with last coordinate positive) if the final interval in F is odd and F is a *lower face* (meaning that its normal cone contains only vectors with last coordinate negative) if the final interval in F is even (or empty).

2.5. The canonical projections between cyclic polytopes. For a fixed pair of dimensions $d' > d$ we will be interested in the surjection $\pi : C(n, d') \rightarrow C(n, d)$, induced by the map $\pi : \mathbb{R}^{d'} \rightarrow \mathbb{R}^d$ which forgets the last $d' - d$ coordinates. The fiber polytopes for this family of surjections were studied in [1]. The associated Baues posets were studied in the special case $d = 2$ in [16]. For the ease of notation, we will write $\omega_{d'}(C(n, d))$ for the Baues poset $\omega(C(n, d') \xrightarrow{\pi} C(n, d))$. This poset is also independent of the choice of points used to define $C(n, d')$. Note that the Baues poset $\omega_{d'}(C(n, 0)) = \omega(C(n, d') \xrightarrow{\pi} C(n, 0))$ is isomorphic to the poset of proper faces of $C(n, d')$ for all $d' > 0$, hence homeomorphic to a $(d' - 1)$ -sphere.

3. STRUCTURE OF THE PROOF

The idea for proving Theorem 1.2 is as follows. Let us fix the dimensions $2 \leq d < d'$ and then use induction on the number n of vertices. The result is already known in the cases $d = 0, 1$. The base case $n = d' + 1$ for the induction is provided by Theorem 1.1. For the inductive step, we will use the same approach as in [13]: via the *deletion operation* of vertex n from a subdivision of $C(n, d)$, we will define a map between the posets $\omega_{d'}(C(n, d))$ and $\omega_{d'}(C(n - 1, d))$ and will prove it to be a homotopy equivalence. This deletion operation is a generalization of the deletion operation on triangulations of $C(n, d)$ from [12].

For two collections S and T of finite pointsets in \mathbb{R}^d we define

$$\begin{aligned} \text{spanning}(S) &:= \{ \sigma \in S : \sigma \text{ is spanning} \} \\ \text{ast}_S(i) &:= \{ \sigma \in S : i \notin \sigma \} \\ \text{lk}_S(i) &:= \{ \sigma - \{i\} : \sigma \in S, i \in \sigma \} \\ S * T &:= \{ \sigma \cup \tau : \sigma \in S, \tau \in T \}. \end{aligned}$$

As was discussed in [13, Section 4], if S is a subdivision of $C(n, d)$ then $\text{lk}_S(n)$ is a subdivision of $C(n - 1, d - 1)$. Moreover, Gale's evenness criterion easily implies that if S is in $\omega_{d'}(C(n, d))$ then $\text{lk}_S(n)$ is in $\omega_{d-1}(C(n - 1, d - 1))$.

Definition 3.1. ([13]) *Given a subdivision S of $C(n, d)$, the deletion $S \setminus n$ is*

$$S \setminus n := \text{spanning}(\{ \sigma \setminus n : \sigma \in S \}),$$

where

$$\sigma \setminus n := \begin{cases} (\sigma - \{n\}) \cup \{n-1\}, & \text{if } n \in \sigma, \\ \sigma, & \text{otherwise.} \end{cases}$$

Equivalently,

$$S \setminus n := \text{ast}_S(n) \cup \text{spanning}(\text{lk}_S(n) * \{n-1\}).$$

Using the idea of “sliding” vertex n to $n-1$, it is proved in [13, Theorem 3.2] that $S \setminus n$ is a subdivision of $C(n-1, d)$. The deletion of n defines a map between the posets $\omega_{d'}(C(n, d))$ and $\omega_{d'}(C(n-1, d))$:

Proposition 3.2. *Let $n \geq d' + 2$. The deletion map $\Pi_{d'} : \omega_{d'}(C(n, d)) \rightarrow \omega_{d'}(C(n-1, d))$*

$$\Pi_{d'}(S) = S \setminus n$$

between the Baues posets of proper π -induced subdivisions is well-defined and order preserving.

Proof. In order to see that $\Pi_{d'}$ is well-defined we just need to check that if σ is a proper face of $C(n, d')$ then $\sigma \setminus n$, introduced in Definition 3.1, is a proper face of $C(n-1, d')$. It follows easily from Gale’s evenness criterion that $\sigma \setminus n$ is a face of $C(n-1, d')$. Moreover, since σ is proper and $C(n, d')$ is simplicial, σ has at most $d' \leq n-2$ vertices. Thus $\sigma \setminus n$ has at most $n-2$ vertices and is a proper face of $C(n-1, d')$.

That $\Pi_{d'}$ is order preserving follows trivially from the fact that if $\sigma \subset \sigma'$ then $\sigma \setminus n \subset \sigma' \setminus n$. \square

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In order to apply Lemma 2.1 to the map $\Pi_{d'}$ we need to understand its fibers. The following concept of subdivisions of $C(n, d)$ induced by a certain subdivision \bar{S} of $C(n, d+1)$ will be crucial for this.

Let \bar{S} be a subdivision of the cyclic polytope $C(n, d+1)$ and S a subdivision of $C(n, d)$. Following [13], we say that S is induced by \bar{S} if every cell $\sigma \in S$ is a face (not necessarily proper) of a cell $\sigma' \in \bar{S}$. We can think of S as a cellular section of the natural projection $C(n, d+1) \rightarrow C(n, d)$ which uses only cells in \bar{S} or their faces. Observe that for every cell σ'' of \bar{S} we can tell whether σ'' is above, on or below (the section corresponding to) a subdivision S induced by \bar{S} . We will denote by $\text{above}(S, \bar{S})$ and $\text{below}(S, \bar{S})$ the set of cells of \bar{S} which lie above and below S , respectively.

We denote by $\omega(\bar{S})$ the poset of all subdivisions of $C(n, d)$ which are induced by \bar{S} , partially ordered by refinement, so that $\omega(\bar{S})$ is a subposet $\omega(C(n, d))$.

From the definition of the deletion $S \setminus n$ it follows trivially that $\text{lk}_S(\{n, n-1\}) := \text{lk}_{\text{lk}_S(n)}(n-1) \subset \text{lk}_{S \setminus n}(n-1)$ for any $S \in \omega(C(n, d))$. Let $T \in \omega_{d'}(C(n-1, d))$ and let $S \in \omega_{d'}(C(n, d))$ be such that $S \setminus n = T$, i.e., $S \in \Pi_{d'}^{-1}(T)$. Then the subdivision $\text{lk}_S(\{n, n-1\})$ of $C(n-2, d-2)$ is induced by the subdivision $\text{lk}_T(n-1) \in \omega(C(n-2, d-1))$. In other words, we have a map $\Pi_{d'}^{-1}(T) \rightarrow \omega(\text{lk}_T(n-1))$ defined by $S \mapsto \text{lk}_S(\{n, n-1\})$. The following much stronger statement follows from [13, Lemma 4.7].

Lemma 3.3. *Let $2 \leq d < d' \leq n-2$ and consider the deletion map $\Pi_{d'} : \omega_{d'}(C(n, d)) \rightarrow \omega_{d'}(C(n-1, d))$. Let $T \in \omega_{d'}(C(n-1, d))$ and $\bar{S} = \text{lk}_T(n-1) \in \omega(C(n-2, d-1))$. Then:*

- (1) The map $\omega(C(n, d)) \rightarrow \omega(C(n-2, d-2))$ given by $S \mapsto \text{lk}_S(\{n-1, n\})$ restricts to a poset isomorphism between $\Pi_{d'}^{-1}(T)$ and a subposet $\omega_{d'}(\bar{S})$ of $\omega(\bar{S})$.
- (2) The inverse map $\tau : \omega_{d'}(\bar{S}) \rightarrow \Pi_{d'}^{-1}(T)$ is given by

$$\begin{aligned} \tau(S) := & \{ \sigma \in T : n-1 \notin \sigma \} \\ & \cup \{ \sigma \cup \{n\} : \sigma \in \bar{S}, \sigma \text{ is below } S \} \\ & \cup \{ \sigma \cup \{n-1\} : \sigma \in \bar{S}, \sigma \text{ is above } S \} \\ & \cup \{ \sigma \cup \{n, n-1\} : \sigma \in S \}. \end{aligned}$$

Moreover,

$$\omega_{d'}(\bar{S}) = \{ S \in \omega(\bar{S}) : \tau(S) \in \omega_{d'}(C(n, d)) \}.$$

- (3) Let $T' \in \omega_{d'}(C(n, d))$ be such that $T' \setminus n$ is coarser than T and let $S_0 = \text{lk}_{T'}(\{n, n-1\}) \in \omega(C(n-2, d-2))$. Then, the previous isomorphism restricts to an isomorphism between $\omega_{d'}(C(n, d))_{\leq T'} \cap \Pi_{d'}^{-1}(T)$ and

$$\omega_{d'}(\bar{S})_{\leq S_0} := \{ S \in \omega_{d'}(\bar{S}) : S \text{ refines } S_0 \} = \omega_{d'}(\bar{S}) \cap \omega(C(n-2, d-2))_{\leq S_0}.$$

By Lemma 2.1 applied to the map $\Pi_{d'}$ introduced in Proposition 3.2, Lemma 3.3 implies that in order to prove Theorem 1.2 we just need to show that, under the assumptions of the lemma, both $\omega_{d'}(\bar{S})$ and $\omega_{d'}(\bar{S})_{\leq S_0}$ are contractible. We will do this in the next section, following the ideas of [13].

4. THE DETAILS

Throughout this section we assume that the hypotheses of Lemma 3.3 hold and we fix an element $T \in \omega_{d'}(C(n-1, d))$ and an element $T' \in \omega_{d'}(C(n, d))$ such that T refines $T' \setminus n$. We also let $\bar{S} = \text{lk}_T(n-1)$ and $S_0 = \text{lk}_{T'}(\{n, n-1\})$. Our task is to prove that both $\omega_{d'}(\bar{S})$ and $\omega_{d'}(\bar{S})_{\leq S_0}$ are contractible. The proof for $\omega_{d'}(\bar{S})_{\leq S_0}$ is easier and we do it in the following proposition. The proof for $\omega_{d'}(\bar{S})$ occupies the rest of this section.

Proposition 4.1. *Under the assumptions of part 3 of Lemma 3.3, let $\omega(\bar{S})_{\leq S_0} := \omega(\bar{S}) \cap \omega(C(n-2, d-2))_{\leq S_0}$. Then:*

- (1) $\omega_{d'}(\bar{S})_{\leq S_0} = \omega(\bar{S})_{\leq S_0}$ and hence
- (2) $\omega_{d'}(\bar{S})_{\leq S_0}$ is contractible.

Proof. The second statement follows from [13, Corollary 4.6], where $\omega(\bar{S})_{\leq S_0}$ is proved to be contractible.

For the first statement, let $T' \in \omega_{d'}(C(n, d))$ be such that $T' \setminus n$ is coarser than T and let $S_0 = \text{lk}_{T'}(\{n-1, n\}) \in \omega(C(n-2, d-2))$. Observe that S_0 might not be in $\omega(\bar{S})$ but it is in $\omega(\bar{S}')$, where $\bar{S}' := \text{lk}_{T' \setminus n}(n-1)$ is coarser than \bar{S} . By parts 1 and 2 of Lemma 3.3 we have that S_0 is in $\omega_{d'}(\bar{S}')$.

Let $S \in \omega(\bar{S})$ be a refinement of S_0 . We will prove that $\tau(S)$ is in $\omega(C(n, d'))$, i.e. $S \in \omega_{d'}(\bar{S})$. Thus $S \in \omega_{d'}(\bar{S})_{\leq S_0}$. For the proof we only use the fact that $S_0 \in \omega_{d'}(\bar{S}')$, that S refines S_0 and that \bar{S} refines \bar{S}' .

Let $\sigma \in \text{above}(S, \bar{S})$ and choose $\sigma' \in \bar{S}'$ such that $\sigma \subset \sigma'$. Since S refines S_0 , either $\sigma' \in \text{above}(S_0, \bar{S}')$ or $\sigma' \in S_0$. In both cases $\sigma' \cup \{n-1\}$, and hence $\sigma \cup \{n-1\}$, is a face of $C(n, d')$. In the same way, if $\sigma \in \text{below}(S, \bar{S})$ then $\sigma \cup \{n\}$ is a face of $C(n, d')$.

Finally, if $\sigma \in S$, then there is $\sigma' \in S_0$ with $\sigma \subset \sigma'$ and since $\sigma' \cup \{n, n-1\}$ is a face of $C(n, d')$, $\sigma \cup \{n, n-1\}$ is a face too. \square

We are now concerned with the poset $\omega(\bar{S}) \subset \omega(C(n-2, d-2))$ and its subposet

$$\omega_{d'}(\bar{S}) = \{S \in \omega(\bar{S}) : \tau(S) \in \omega_{d'}(C(n, d))\}.$$

Our goal is to prove that $\omega_{d'}(\bar{S})$ is contractible. Actually, we will never use the fact that \bar{S} is a link of a subdivision $T \in C(n-1, d)$ but only that, since $T \in \omega_{d'}(C(n-1, d))$, its link \bar{S} is in $\omega_{d'-1}(C(n-2, d-1))$. In other words, we will prove the following result.

Theorem 4.2. *Let \bar{S} be a proper subdivision of $C(n-2, d-1)$, induced by $C(n-2, d'-1)$, and let $\omega_{d'}(\bar{S}) \subset \omega(\bar{S})$ be the poset of sections S of \bar{S} which have the properties:*

- (1) *For any $\sigma \in \text{above}(S, \bar{S})$, $\sigma \cup \{n-1\}$ is a face of $C(n, d')$.*
- (2) *For any $\sigma \in \text{below}(S, \bar{S})$, $\sigma \cup \{n\}$ is a face of $C(n, d')$.*
- (3) *For any $\sigma \in S$, $\sigma \cup \{n, n-1\}$ is a face of $C(n, d')$.*

Then $\omega_{d'}(\bar{S})$ is contractible.

Let us recall the following technical property of subdivisions of cyclic polytopes, proved and called *stackability* in [13]. Let S be a subdivision of a cyclic polytope $C(n, d)$, for arbitrary $n > d$. For any two cells $\sigma_1, \sigma_2 \in S$ which share a facet, their common facet is an upper facet of one of σ_1, σ_2 and a lower facet of the other. If it is a lower facet of σ_2 and an upper facet of σ_1 we say that “ σ_2 is above σ_1 ” and “ σ_1 below σ_2 ”.

Lemma 4.3. ([13, Lemma 2.16]) *The relation “ σ_1 is below σ_2 ” just defined has no cycles. Hence, its transitive closure is a partial order on the collection of all cells of S .*

In the sequel we denote by $<_{st}$ this partial order on the cells of the subdivision $\bar{S} \in \omega_{d'-1}(C(n-2, d-1))$.

Lemma 4.4. *Let $\bar{S} \in \omega_{d'-1}(C(n-2, d-1))$. Let $\sigma \subset [n-2]$ be a face of $C(n-2, d'-1)$ (not necessarily a cell of \bar{S}) and let σ_+ and σ_- be cells in \bar{S} such that $\sigma_- <_{st} \sigma_+$. Then:*

- (1) *At least one of $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ is a proper face of $C(n, d')$.*
- (2) *If $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are both proper faces of $C(n, d')$, then so is $\sigma \cup \{n, n-1\}$.*
- (3) *If $\sigma_+ \cup \{n\}$ and $\sigma_- \cup \{n-1\}$ are both proper faces of $C(n, d')$ then so is either $\sigma_- \cup \{n\}$ or $\sigma_+ \cup \{n-1\}$.*

Proof. 1. If σ is a face of $C(n-2, d'-2)$ then $\sigma \cup \{n, n-1\}$ is a face of $C(n, d')$. Hence, both $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are faces of $C(n, d')$ as well.

If σ is not a face of $C(n-2, d'-2)$ then σ is either an upper or a lower face of $C(n-2, d'-1)$. In the first case $\sigma \cup \{n-1\}$ is a face of $C(n, d')$ and in the second case $\sigma \cup \{n\}$ is a face of $C(n, d')$, as follows easily from Gale’s evenness criterion.

2. We will show that $\sigma \cup \{n, n-1\}$ has at least one interior component of odd length less than either $\sigma \cup \{n\}$ or $\sigma \cup \{n-1\}$. Taking m to be the maximum element in $[n-2] \setminus \sigma$, we observe that this is the case for $\sigma \cup \{n-1\}$ if $n-m$ is even and for $\sigma \cup \{n\}$ if $n-m$ is odd.

3. Let m_+ (respectively m_-) be the maximum in $[n-2] \setminus \sigma_+$ (respectively $[n-2] \setminus \sigma_-$). We will prove that either $\{m_-+1, \dots, n-2\}$ has an even number of elements (and then $\sigma_- \cup \{n\}$ is a face of $C(n, d')$), or $\{m_++1, \dots, n-2\}$ has an odd number of elements (and then $\sigma_+ \cup \{n-1\}$ is a face of $C(n, d')$).

For this let $\sigma_- = \sigma_0 <_{st} \sigma_1 <_{st} \dots <_{st} \sigma_k = \sigma_+$ be a chain of cells of \overline{S} such that every two consecutive ones share a facet. Let m be the maximum integer in $[n-2] \setminus \bigcap_{i=0}^k \sigma_i$. We will consider separately the following three cases: (i) $m \notin \bigcup_{i=0}^k \sigma_i$, (ii) $m \in \bigcup_{i=0}^k \sigma_i$ and $n - m$ is even and (iii) $m \in \bigcup_{i=0}^k \sigma_i$ and $n - m$ is odd.

- (i) If $m \notin \bigcup_{i=0}^k \sigma_i$ then $m = m_+ = m_-$. Obviously, $\{m + 1, \dots, n - 2\}$ has either an even or an odd number of elements.
- (ii) If $n - m$ is even then the common interval $\{m + 1, \dots, n - 2\}$ to all the σ_i has an even number of elements. This implies that if $m \in \sigma_i$ for an $i \leq k - 1$ then $m \in \sigma_{i+1}$ too. Indeed, the common facet τ between σ_i and σ_{i+1} is an upper facet of σ_i and, hence, its last interval has odd length. So it is impossible to have $m \in \sigma_i \setminus \tau$ and $\{m + 1, \dots, n - 2\} \subset \tau$. In particular, m cannot be in $\sigma_- = \sigma_0$ because then it would be in $\bigcap_{i=0}^k \sigma_i$. Hence, $m = m_-$ and $\{m_- + 1, \dots, n - 2\}$ has an even number of elements.
- (iii) This case is analogous: If $n - m$ is odd then the common interval $\{m + 1, \dots, n - 2\}$ to all the σ_i has an odd number of elements. This implies that if $m \in \sigma_i$ for an $i \geq 1$ then $m \in \sigma_{i-1}$ too. Indeed, the common facet τ between σ_i and σ_{i-1} is a lower facet of σ_i and, hence, its last interval has even length. So it is impossible to have $m \in \sigma_i \setminus \tau$ and $\{m + 1, \dots, n - 2\} \subset \tau$. In particular, m cannot be in $\sigma_+ = \sigma_k$ because then it would be in $\bigcap_{i=0}^k \sigma_i$. Hence, $m = m_+$ and $\{m_+ + 1, \dots, n - 2\}$ has an odd number of elements.

□

Our next goal is to prove that $\omega_{d'}(\overline{S})$ is not empty, and hence that the map $\Pi_{d'}$ is surjective. Clearly, $\omega(\overline{S})$ is not empty so we are interested in which elements $S \in \omega(\overline{S})$ lie also in $\omega_{d'}(\overline{S}) \subset \omega(\overline{S})$.

Lemma 4.5. *Let $S \in \omega(\overline{S})$. Then $S \in \omega_{d'}(\overline{S})$ if and only if for any cell σ of \overline{S} we have:*

- if $\sigma \in S \cup \text{above}(S, \overline{S})$ then $\sigma \cup \{n - 1\}$ is a face of $C(n, d')$,*
- if $\sigma \in S \cup \text{below}(S, \overline{S})$ then $\sigma \cup \{n\}$ is a face of $C(n, d')$.*

Proof. Necessity of the conditions is obvious by the definition of $\omega_{d'}(\overline{S})$ in Lemma 3.3. Sufficiency is not obvious since a cell σ of S might not be a (spanning) cell of \overline{S} . We need to prove under the conditions in the statement that for such a cell σ , $\sigma \cup \{n, n - 1\}$ is a face of $C(n, d')$.

Let σ be in $S \setminus \overline{S}$. Then σ is a simplex of dimension $d - 2$ in $C(n - 2, d - 1)$ and there is a cell $\sigma_+ \in \overline{S}$ (respectively σ_-) of which σ is a lower (respectively upper) facet unless σ is an upper (respectively lower) facet of $C(n - 2, d - 1)$. We will prove that $\sigma \cup \{n\}$ and $\sigma \cup \{n - 1\}$ are faces of $C(n, d')$. Then by part 2 of Lemma 4.4 we conclude that $\sigma \cup \{n, n - 1\}$ is a face of $C(n, d')$.

If σ is an upper (respectively lower) facet of $C(n - 2, d - 1)$ then $\sigma \cup \{n - 1\}$ (respectively $\sigma \cup \{n\}$) is a lower (respectively upper) facet of $C(n, d)$, hence a face of $C(n, d')$. If σ is an upper (respectively lower) facet of $\sigma_- \in \overline{S}$, (respectively of σ_+) then σ_- is below S (respectively σ_+ is above S) and by hypothesis $\sigma_- \cup \{n\}$ (respectively $\sigma_+ \cup \{n - 1\}$) is a face of $C(n, d')$. Thus $\sigma \cup \{n\}$ (respectively $\sigma \cup \{n - 1\}$) is also a face.

□

For the sequel, for $\bar{S} \in \omega_{d'-1}(C(n-2, d-1))$, let us define the following collections, which depend on d' :

$$\begin{aligned} \text{above}(\bar{S}) &= \{\sigma \in \bar{S} : \forall \sigma' \in \bar{S} \text{ with } \sigma \leq_{st} \sigma', \sigma' \cup \{n-1\} \text{ is a face of } C(n, d')\}, \\ \text{below}(\bar{S}) &= \{\sigma \in \bar{S} : \forall \sigma' \in \bar{S} \text{ with } \sigma' \leq_{st} \sigma, \sigma' \cup \{n\} \text{ is a face of } C(n, d')\}. \end{aligned}$$

By definition, $\text{below}(\bar{S})$ and $\text{above}(\bar{S})$ are a lower and an upper ideal respectively in $<_{st}$. This implies that the upper envelope S_{up} of $\text{below}(\bar{S})$ and the lower envelope S_{down} of $\text{above}(\bar{S})$ are valid sections in $\omega(\bar{S})$. We show that they are also in $\omega_{d'}(\bar{S})$. Observe that $\text{above}(\bar{S}) = \text{above}(S_{down}, \bar{S})$ and $\text{below}(\bar{S}) = \text{below}(S_{up}, \bar{S})$.

Lemma 4.6. *We have:*

- (1) $\text{below}(\bar{S}) \cup \text{above}(\bar{S}) = \bar{S}$.
- (2) Let $S \in \omega(\bar{S})$. Then $S \in \omega_{d'}(\bar{S})$ if and only if

$$\begin{aligned} \text{above}(S, \bar{S}) \cup \text{below}(\bar{S}) &= \bar{S}, \\ \text{below}(S, \bar{S}) \cup \text{above}(\bar{S}) &= \bar{S}. \end{aligned}$$
- (3) In particular, S_{up} and S_{down} are in $\omega_{d'}(\bar{S})$.

Proof. 1. Let $\sigma \in \bar{S}$ and suppose $\sigma \notin \text{below}(\bar{S})$. By definition of $\text{below}(\bar{S})$ this means that there is a $\sigma' \leq_{st} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. Since σ' is a face of $C(n-2, d'-1)$, parts 1 and 3 of Lemma 4.4 imply, respectively, that $\sigma' \cup \{n-1\}$ and any $\sigma'' \cup \{n-1\}$ with $\sigma'' \geq_{st} \sigma'$ are faces of $C(n, d')$. In particular, $\sigma \in \text{above}(\bar{S})$.

2. We first prove the necessity of the conditions. If $\sigma \in \bar{S} \setminus (\text{above}(S, \bar{S}) \cup \text{below}(\bar{S}))$ then $\sigma \in S \cup \text{below}(S, \bar{S})$ and there is a $\sigma' <_{st} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. This σ' will also be in $S \cup \text{below}(S, \bar{S})$ and hence $S \notin \omega_{d'}(\bar{S})$ by Lemma 4.5. The case of a $\sigma \in \bar{S} \setminus (\text{below}(S, \bar{S}) \cup \text{above}(\bar{S}))$ is analogous.

For the sufficiency, let $S \in \omega(\bar{S})$ be such that $\text{above}(S, \bar{S}) \cup \text{below}(\bar{S}) = \bar{S}$ and $\text{below}(S, \bar{S}) \cup \text{above}(\bar{S}) = \bar{S}$. We will prove that $S \in \omega_{d'}(\bar{S})$ using Lemma 4.5. Let $\sigma \in \bar{S}$ and suppose that $\sigma \in S \cup \text{above}(S, \bar{S})$. This is equivalent to $\sigma \notin \text{below}(S, \bar{S})$ and hence $\sigma \in \text{above}(\bar{S})$. Hence, $\sigma \cup \{n-1\}$ is a face of $C(n, d')$. In the same way, if $\sigma \in S \cup \text{below}(S, \bar{S})$ we prove that $\sigma \cup \{n\}$ is a face of $C(n, d')$.

3. From the definition of S_{up} , it follows that S_{up} does not contain any cell of \bar{S} (i.e. $\text{above}(S_{up}, \bar{S}) \cup \text{below}(S_{up}, \bar{S}) = \bar{S}$) and also that $\text{below}(S_{up}, \bar{S}) = \text{below}(\bar{S})$. Putting these two facts together and using part 1, we conclude that S_{up} satisfies the conditions of part 2. The same holds for S_{down} . \square

Remark 4.7. The last result can be interpreted using the following poset structure on the collection of subdivisions induced by \bar{S} .

Definition 4.8. Let $\text{St}(\bar{S})$ be the set of subdivisions of $C(n, d)$ induced by \bar{S} , partially ordered by $S_1 \leq S_2$ if and only if S_1 lies below S_2 as a cellular section of the natural projection $C(n, d+1) \rightarrow C(n, d)$ or, equivalently, if $\text{above}(S_2, \bar{S}) \subset \text{above}(S_1, \bar{S})$ and $\text{below}(S_1, \bar{S}) \subset \text{below}(S_2, \bar{S})$.

Let $\text{St}_{d'}(\bar{S})$ be the induced subset of $\text{St}(\bar{S})$ on the subset $\omega_{d'}(\bar{S})$. We call $\text{St}(\bar{S})$ and $\text{St}_{d'}(\bar{S})$ the Stasheff orders on $\omega(\bar{S})$ and $\omega_{d'}(\bar{S})$.

The above definition reminds of the second higher Stasheff-Tamari order on the set of all triangulations of a cyclic polytope and its characterization as closed sets in

dimensions 2 and 3 [7]. In this context the structure is well-behaved in all dimensions.

Using the Stasheff order, Lemma 4.6 can be rewritten as follows.

Lemma 4.9. *An element S of $\omega(\overline{S})$ is in $\omega_{d'}(\overline{S})$ if and only if $S_{down} \leq_{st} S \leq_{st} S_{up}$. Thus, $St_{d'}(\overline{S})$ is a nonempty interval in $St(\overline{S})$.*

It is also easy to see that $\omega_{d'}(\overline{S})$ is a lattice, where for every $S_1, S_2 \in \omega_{d'}(\overline{S})$ the join $S_1 \vee S_2$ and the meet $S_1 \wedge S_2$ are the elements satisfying

$$\begin{aligned} \text{above}(S_1 \vee S_2, \overline{S}) &:= \text{above}(S_1, \overline{S}) \cap \text{above}(S_2, \overline{S}), \\ \text{below}(S_1 \vee S_2, \overline{S}) &:= \text{below}(S_1, \overline{S}) \cup \text{below}(S_2, \overline{S}), \\ \text{above}(S_1 \wedge S_2, \overline{S}) &:= \text{above}(S_1, \overline{S}) \cup \text{above}(S_2, \overline{S}), \\ \text{below}(S_1 \wedge S_2, \overline{S}) &:= \text{below}(S_1, \overline{S}) \cap \text{below}(S_2, \overline{S}). \end{aligned}$$

In a sense, $S_1 \vee S_2$ and $S_1 \wedge S_2$ are the common upper and lower envelopes of S_1 and S_2 , except that if a cell σ is in $S_1 \cap S_2$ then σ (instead of its upper or lower envelope) is also in $S_1 \vee S_2$ and $S_1 \wedge S_2$.

In what follows we argue that the proof of [13, Theorem 4.5], showing that $\omega(\overline{S})$ is contractible, can be modified to prove that $\omega_{d'}(\overline{S})$ is contractible. The original proof is based on a total ordering of the cells of \overline{S} compatible with the partial order $<_{st}$. Here we also want our total order to behave nicely with respect to $\text{above}(\overline{S})$ and $\text{below}(\overline{S})$.

Lemma 4.10. *There is a total order, i.e. a numbering $\overline{S} = \{\sigma_1, \dots, \sigma_k\}$, of the cells of \overline{S} such that for every $i, j \in \{1, \dots, k\}$ we have:*

- (1) *If $\sigma_i <_{st} \sigma_j$ then $i < j$.*
- (2) *If $\sigma_i \in \text{below}(\overline{S})$ and $\sigma_j \notin \text{below}(\overline{S})$ then $i < j$.*
- (3) *If $\sigma_i \in \text{above}(\overline{S})$ and $\sigma_j \notin \text{above}(\overline{S})$ then $i > j$.*

Proof. We first order the cells not in $\text{above}(\overline{S})$ with the numbers from 1 to k_1 , in a way compatible with the partial order $<_{st}$. Then we order the cells in $\text{above}(\overline{S}) \cap \text{below}(\overline{S})$ with numbers $k_1 + 1, \dots, k_2$ and then those not in $\text{below}(\overline{S})$ with $k_2 + 1, \dots, k$, both times again in a way compatible with $<_{st}$. This can be done since $<_{st}$ is a partial order.

The order so obtained satisfies conditions 2 and 3 by construction and it also satisfies condition 1 since $\text{below}(\overline{S})$ is a lower ideal in $<_{st}$ (so that if $\sigma_i <_{st} \sigma_j$, it is impossible that $\sigma_j \in \text{below}(\overline{S})$ and $\sigma_i \notin \text{below}(\overline{S})$) and $\text{above}(\overline{S})$ is an upper ideal in $<_{st}$ (so that if $\sigma_i <_{st} \sigma_j$, it is impossible that $\sigma_i \in \text{above}(\overline{S})$ and $\sigma_j \notin \text{above}(\overline{S})$). \square

The proof of the following proposition follows closely the one of [13, Theorem 4.5] but we include it for the sake of completeness. It establishes Theorem 4.2 and finishes the proof of Theorem 1.2.

Proposition 4.11. *The subposet $\omega_{d'}(\overline{S})$ of $\omega(C(n-2, d-2))$ is contractible.*

Proof. Let the cells of \overline{S} be totally ordered as in Lemma 4.10, so that $\{\sigma_1, \dots, \sigma_{k_1}\} = \text{below}(\overline{S}) \setminus \text{above}(\overline{S})$, $\{\sigma_{k_1+1}, \dots, \sigma_{k_2}\} = \text{below}(\overline{S}) \cap \text{above}(\overline{S})$ and $\{\sigma_{k_2+1}, \dots, \sigma_k\} = \text{above}(\overline{S}) \setminus \text{below}(\overline{S})$.

For any $S \in \omega(\bar{S})$ we call *height* of S the maximum index i of a cell σ_i on or below S . For each $i = 0, \dots, k$ we denote by $\omega_{d'}(\bar{S}; i)$ the subposet of $\omega_{d'}(\bar{S})$ consisting of the subdivisions of height at most i .

By definition, S_{down} has height k_1 and S_{up} has height k_2 . Moreover, by Lemma 4.6, $\omega_{d'}(\bar{S}) = \omega_{d'}(\bar{S}; k_2)$ and $\omega(\bar{S}; k_1)$ has only the element S_{down} . We will prove that $\omega_{d'}(\bar{S}; i)$ and $\omega_{d'}(\bar{S}; i - 1)$ are homotopically equivalent for every $i = k_1 + 1, \dots, k_2$.

Consider first the following situation. Let $S \in \omega_{d'}(\bar{S})$ with $\sigma_i \in S$. We can get two new elements S_{σ_i+} and S_{σ_i-} of $\omega_{d'}(\bar{S})$ by substituting σ_i in S for its upper and lower envelope, respectively.

We now construct the homotopy equivalence $f_i : \omega_{d'}(\bar{S}; i) \rightarrow \omega_{d'}(\bar{S}; i - 1)$. We define f_i to be the identity on those $S \in \omega_{d'}(\bar{S}; i)$ with height at most $i - 1$. If S has height i then either S contains σ_i , in which case we take $f_i(S) = S_{\sigma_i-}$, or S contains the upper envelope of σ_i . In this case $S = T_{\sigma_i+}$ for some $T \in \omega_{d'}(\bar{S})$. We then define $f_i(T_{\sigma_i+}) = T_{\sigma_i-}$. In this way, the inverse image of an element $S \in \omega_{d'}(\bar{S}; i - 1)$ is given as follows:

- (i) It is S itself, if S does not contain the lower envelope of σ_i .
- (ii) If S contains the lower envelope of σ_i , then $S = T_{\sigma_i-}$ for some $T \in \omega_{d'}(\bar{S}; i)$ and $f^{-1}(S) = f^{-1}(T_{\sigma_i-}) = \{T, T_{\sigma_i-}, T_{\sigma_i+}\}$.

Define the following order-preserving map:

$$g_i : \begin{cases} \omega(\bar{S}; i - 1) & \rightarrow \omega(\bar{S}; i), \\ S & \mapsto \begin{cases} S & \text{in case (i),} \\ T & \text{in case (ii).} \end{cases} \end{cases}$$

Then $f_i \circ g_i = \text{id}_{\omega_{d'}(\bar{S}; i-1)}$ and $g_i \circ f_i \geq \text{id}_{\omega_{d'}(\bar{S}; i)}$, which means that f_i and g_i are homotopy inverses to each other by Quillen's order homotopy theorem [6, 10.11]. Thus, $\omega_{d'}(\bar{S}; i)$ is homotopy equivalent to $\omega_{d'}(\bar{S}; i - 1)$. \square

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