

Characteristic Polynomials of Subspace Arrangements and Finite Fields*

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Let \mathcal{A} be any subspace arrangement in \mathbb{R}^n defined over the integers and let \mathbb{F}_q denote the finite field with q elements. Let q be a large prime. We prove that the characteristic polynomial $\chi(\mathcal{A}, q)$ of \mathcal{A} counts the number of points in \mathbb{F}_q^n that do not lie in any of the subspaces of \mathcal{A} , viewed as subsets of \mathbb{F}_q^n . This observation, which generalizes a theorem of Blass and Sagan about subarrangements of the \mathcal{B}_n arrangement, reduces the computation of $\chi(\mathcal{A}, q)$ to a counting problem and provides an explanation for the wealth of combinatorial results discovered in the theory of hyperplane arrangements in recent years. The basic idea has its origins in the work of Crapo and Rota (1970). We find new classes of hyperplane arrangements whose characteristic polynomials have simple form and very often factor completely over the nonnegative integers. © 1996 Academic Press, Inc.

1. INTRODUCTION

The present work was motivated by Sagan's expository paper [20]. In [20] the author surveys three methods that have been used in the past to show that the characteristic polynomials of certain graded lattices factor completely over the nonnegative integers. The first method considers the characteristic polynomial of certain subspace arrangements. The author gives the most general class of real subspace arrangements known for which a combinatorial interpretation of the characteristic polynomial exists.

Our objective in the present paper is to give a much more general theorem which provides such a combinatorial interpretation that is missing from [20]. The key idea to the problem is quite old. It is contained in a theorem of Crapo and Rota, related to the famous critical problem. We quote from [7, Sect. 16], where V_n stands for a vector space of dimension

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n over the finite field with q elements and S is a set of points in V_n not including the origin.

THEOREM 1. *The number of linearly ordered sequences (L_1, L_2, \dots, L_k) of k linear functionals in V_n which distinguish the set S is given by $p(q^k)$, where $p(v)$ is the characteristic polynomial of the geometric lattice spanned by the set S .*

The sequence (L_1, L_2, \dots, L_k) is said to distinguish the set S if for each $s \in S$ there exists at least one i for which $L_i(s) \neq 0$. Specialized to $k = 1$, this theorem expresses $p(q)$ as the number of linear functionals L satisfying $L(s) \neq 0$ for all $s \in S$. Thinking dually, we can replace each point of S with the hyperplane H_s which is “orthogonal” to s and passes through the origin. The geometric lattice spanned by the set S is the intersection lattice of the resulting central hyperplane arrangement. The previous statement is equivalent to saying that $p(q)$ counts the number of points in V_n , not in any of the hyperplanes H_s . This is, in a special case, the theorem we will present.

Unfortunately, the Crapo–Rota theorem was overlooked for quite a long time in the later development of the theory of hyperplane arrangements. For example, some of its immediate enumerative consequences for real central hyperplane arrangements were only derived much later and independently by Zaslavsky [35]. The purpose of this paper is twofold. In the first place we state the theorem more generally for affine subspace arrangements, which have appeared in the meantime. Secondly, we show that even when restricted to hyperplane arrangements, this theorem provides a powerful enumerative tool which simplifies and extends enormously much of the work done in the past decade on the combinatorics of special classes of real hyperplane arrangements. As far as we know, this tool has not been of use in the past although it is stated for hyperplane arrangements over finite fields in the standard reference [16]. We are able to transfer the Crapo–Rota idea to the real case simply because real arrangements defined by linear equations with integer coefficients can be reduced to arrangements over finite fields having the same intersection semilattice.

Overview. In the rest of this section we introduce basic background and terminology about the modern theory of subspace arrangements. In Section 2 we motivate and give the main theorem of the present paper about the characteristic polynomial of a rational subspace arrangement, as well as a few examples. Sections 3–5 contain applications of this theorem to various rational hyperplane arrangements. In Section 3 we provide simple proofs for the formulas giving the characteristic polynomials of the Shi arrangements and several generalizations. In Section 4 we give a simple formula for the characteristic polynomial of the Linial arrangement, thus providing another proof of a theorem of Postnikov recently conjectured by Stanley,

for its number of regions. In Section 5 we give further applications to related arrangements. In Section 6 we give a two variable generalization of the main theorem and, as an application, we compute the face numbers of the Shi arrangement of type A. We close with some directions for further research.

Background. A *subspace arrangement* \mathcal{A} in \mathbb{R}^n is a finite collection of proper affine subspaces of \mathbb{R}^n . If all the affine subspaces in \mathcal{A} are hyperplanes, i.e. they have dimension $n - 1$, then \mathcal{A} is called a *hyperplane arrangement*. The theory of hyperplane arrangements has deep connections with areas of mathematics other than combinatorics, see for example [16]. A nice exposition for the more modern theory of subspace arrangements can be found in [2].

Here we will be concerned with subspace arrangements \mathcal{A} in \mathbb{R}^n that can be defined over the integers. This just means that every affine subspace of \mathcal{A} is an intersection of hyperplanes of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = d, \tag{1}$$

where the a_i 's and d are integers. We will also call these arrangements *rational* for obvious reasons. A subspace arrangement \mathcal{A} is called *central* if all its subspaces are linear subspaces, i.e. they pass through the origin. We call \mathcal{A} *centered* if its subspaces have nonvoid intersection and *centerless* otherwise. We will mostly focus on centerless hyperplane arrangements. Classical examples of rational hyperplane arrangements in \mathbb{R}^n are the arrangements $\mathcal{A}_n, \mathcal{B}_n, \mathcal{D}_n$, defined as

$$\begin{aligned} \mathcal{A}_n &= \{x_i = x_j \mid 1 \leq i < j \leq n\}, \\ \mathcal{D}_n &= \mathcal{A}_n \cup \{x_i = -x_j \mid 1 \leq i < j \leq n\}, \\ \mathcal{B}_n &= \mathcal{D}_n \cup \{x_i = 0 \mid 1 \leq i \leq n\}. \end{aligned}$$

They are the arrangements of reflecting hyperplanes corresponding to the finite Coxeter groups of type A_{n-1}, B_n and D_n respectively. A subspace arrangement that has received a lot of attention recently (see for example [2, Sect. 3] and [5, Sect. 6]) is the *k-equal* arrangement

$$\mathcal{A}_{n,k} = \{x_{i_1} = x_{i_2} = \dots = x_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}.$$

A fundamental combinatorial object associated to \mathcal{A} is its intersection semilattice $L_{\mathcal{A}}$. It consists of all subspaces of \mathbb{R}^n that can be written as the intersection of some of the subspaces of \mathcal{A} . The partial order on $L_{\mathcal{A}}$ is reverse inclusion. Thus the empty intersection, which is \mathbb{R}^n itself, is the unique minimal element $\hat{0}$ of $L_{\mathcal{A}}$. The poset $L_{\mathcal{A}}$ is a meet-semilattice, i.e.,

any two elements have a greatest lower bound (meet). If \mathcal{A} is centered, the intersection of its subspaces is the unique maximal element and $L_{\mathcal{A}}$ is actually a lattice. For example, with the notation introduced above, $L_{\mathcal{A}_n}$ is the partition lattice Π_n and $L_{\mathcal{A}_{n,k}}$ is the lattice of set partitions of an n -element set with no blocks of size $2, \dots, k-1$, ordered by refinement. For basic facts and terminology about posets and lattices we refer the reader to [26, Ch. 3]. If \mathcal{A} and \mathcal{B} are subspace arrangements in \mathbb{R}^n , we say that \mathcal{A} is embedded in \mathcal{B} if every subspace of \mathcal{A} is the intersection of some of the subspaces of \mathcal{B} , i.e. $\mathcal{A} \subseteq L_{\mathcal{B}}$.

Our main object of study is the *characteristic polynomial* $\chi(\mathcal{A}, q)$, defined by

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{\dim x},$$

where μ stands for the Möbius function of $L_{\mathcal{A}}$ [26, Ch. 3]. The characteristic polynomial plays an important role in the combinatorial and topological aspects of the theory of arrangements. For the arrangements \mathcal{A}_n , \mathcal{B}_n and \mathcal{D}_n it is given by the well known formulas

$$\chi(\mathcal{A}_n, q) = q(q-1)(q-2) \cdots (q-n+1),$$

$$\chi(\mathcal{B}_n, q) = (q-1)(q-3) \cdots (q-2n+1),$$

$$\chi(\mathcal{D}_n, q) = (q-1)(q-3) \cdots (q-2n+3)(q-n+1).$$

When $\chi(\mathcal{A}, q)$ has q as a factor, we will use the notation

$$\tilde{\chi}(\mathcal{A}, q) = \frac{1}{q} \chi(\mathcal{A}, q).$$

For example, $\tilde{\chi}(\mathcal{A}_n, q) = (q-1)(q-2) \cdots (q-n+1)$. This is the characteristic polynomial of \mathcal{A}_n , restricted in the hyperplane $x_1 + x_2 + \cdots + x_n = 0$.

We write $\bigcup \mathcal{A}$ for the set theoretic union of the elements of the collection \mathcal{A} . Let \mathcal{A} be a hyperplane arrangement in \mathbb{R}^n . We denote by $r(\mathcal{A})$ the number of regions of \mathcal{A} , that is the number of connected components of the space $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$. Similarly, we denote by $b(\mathcal{A})$ the number of *bounded* regions of \mathcal{A} . A good reason to study the characteristic polynomial for hyperplane arrangements is the following theorem, discovered by Zaslavsky [33, Sect. 2].

THEOREM 1.1. *For any hyperplane arrangement \mathcal{A} in \mathbb{R}^n we have*

$$r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1) = \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)|$$

and

$$b(\mathcal{A}) = (-1)^{\rho(L_{\mathcal{A}})} \chi(\mathcal{A}, 1) = \left| \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \right|,$$

where $\rho(L_{\mathcal{A}})$ denotes the rank (one less than the number of levels) of the intersection semilattice $L_{\mathcal{A}}$.

For example, it is easy to see that $r(\mathcal{A}_n) = n!$, as predicted by Theorem 1.1 and the formula for $\chi(\mathcal{A}_n, q)$. We will give some nontrivial applications of Zaslavsky’s theorem in the following sections.

We note here that the numbers $r(\mathcal{A})$ and $b(\mathcal{A})$ also give information about the topology of the complexified hyperplane arrangement $\mathcal{A}^{\mathbb{C}}$. Indeed, let $M_{\mathcal{A}^{\mathbb{C}}} = \mathbb{C}^n - \bigcup \mathcal{A}$ be the complement in \mathbb{C}^n of the union of the hyperplanes of \mathcal{A} , now viewed as subsets of \mathbb{C}^n . Let $\beta^i(M_{\mathcal{A}^{\mathbb{C}}})$ be the Betti numbers of $M_{\mathcal{A}^{\mathbb{C}}}$, i.e the ranks of the singular cohomology groups $H^i(M_{\mathcal{A}^{\mathbb{C}}})$. It follows from the work of Orlik and Solomon [15] and Theorem 1.1 that

$$r(\mathcal{A}) = \sum_{i \geq 0} \beta^i(M_{\mathcal{A}^{\mathbb{C}}})$$

and

$$b(\mathcal{A}) = \left| \sum_{i \geq 0} (-1)^i \beta^i(M_{\mathcal{A}^{\mathbb{C}}}) \right|.$$

We refer the reader to [2, Sect. 1] for more details. A similar result, asserting that $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$ has Euler characteristic $(-1)^n \chi(\mathcal{A}, -1)$ holds for any real subspace arrangement \mathcal{A} [2, Thm. 7.3.1].

2. RATIONAL ARRANGEMENTS

At this point we briefly review previous work on the combinatorics of the characteristic polynomial of rational subspace arrangements. As noted before, it is unfortunate that this work proceeded independently of the “finite field” point of view of Crapo and Rota.

We commented in the previous section that there are nice product formulas for the characteristic polynomial of the Coxeter arrangements \mathcal{A}_n , \mathcal{B}_n , \mathcal{D}_n . In general, the characteristic polynomial of an arrangement seems to factor over the nonnegative integers much more often than random polynomials do. It was thus natural to ask if there exists a combinatorial interpretation of the characteristic polynomial that can explain this

factorization phenomenon. Results in this direction constitute the first method described in [20]. The other two methods are the theory of free hyperplane arrangements [16, Ch. 4] [31] and Stanley's factorization theorem for supersolvable lattices [24, Thm. 4.1] and its generalizations. Stanley's method is also combinatorial.

The first combinatorial interpretation known was for the class of hyperplane arrangements contained in \mathcal{A}_n . Such an arrangement \mathcal{G} is called *graphical* because its set of hyperplanes $x_i = x_j$ may be identified with the set of edges ij of a graph G with vertices $1, 2, \dots, n$. The characteristic polynomial of such an arrangement is the chromatic polynomial of the corresponding graph.

Zaslavsky generalized this fact with his theory of signed graph coloring [34, 35, 36]. He proved that the characteristic polynomial of any hyperplane arrangement $\mathcal{A} \subseteq \mathcal{B}_n$ is the "signed chromatic polynomial" of a certain "signed graph" associated to \mathcal{A} . Another interpretation as the chromatic polynomial of a certain hypergraph is implicit in [27, Thm. 3.4]. This theorem applies to subspace arrangements embedded in \mathcal{A}_n . It gives a simple proof of a theorem, first obtained by Björner and Lovász [4], which computes the characteristic polynomial of the k -equal arrangement in an exponential generating function form.

Blass and Sagan [6, Thm. 2.1], [20, Thm. 2.2] generalized both results by giving a combinatorial interpretation of $\chi(\mathcal{A}, q)$ for any subspace arrangement \mathcal{A} embedded in \mathcal{B}_n . They proved their result by interpreting the quantity $t^{\dim x}$ as the cardinality of a set and using Möbius inversion. Below we denote by $[-s, s]$ the set of integers $\{-s, -s+1, \dots, s\}$, for any non-negative integer s . We also use the notation $\#S$ for the cardinality of the finite set S , to avoid confusion with the absolute value symbol.

THEOREM 2.1 [6, Thm. 2.1; 20, Thm. 2.2]. *If \mathcal{A} is any subspace arrangement embedded in \mathcal{B}_n , then for any $q = 2s + 1$,*

$$\chi(\mathcal{A}, q) = \# \left([-s, s]^n - \bigcup \mathcal{A} \right).$$

In [6] the authors comment that this theorem is the *only* combinatorial interpretation known for the characteristic polynomial of a class of subspace (as opposed to hyperplane) arrangements. We show below that the ideas of Crapo and Rota extend naturally and give a similar simple combinatorial interpretation for the whole class of rational subspace arrangements.

To motivate our own reasoning, we consider the arrangement

$$\mathcal{A}'_n = \mathcal{A}_n \cup \{x_1 - x_n = 1\},$$

obtained from \mathcal{A}_n by adding the hyperplane $x_1 - x_n = 1$. This is a reflecting hyperplane, one corresponding to the highest root $e_1 - e_n$, of the infinite arrangement associated to the affine Weyl group of type A_{n-1} . A few computations suffice to conjecture that

$$\chi(\mathcal{A}'_n, q) = q(q-2) \prod_{i=2}^{n-1} (q-i). \tag{2}$$

However, Theorem 2.1 is not general enough to prove this innocent looking formula. Thus it is conceivable that a generalization of the result of Blass and Sagan exists in which the assumption that \mathcal{A} is embedded in \mathcal{B}_n is dropped. This is achieved by replacing the cube $[-s, s]^n$ with \mathbb{F}_q^n , where \mathbb{F}_q stands for the finite field with q elements.

Note that a subspace arrangement \mathcal{A} in \mathbb{R}^n , defined over the integers, gives rise to an arrangement over the finite field \mathbb{F}_q . Indeed, any subspace K in \mathcal{A} is the intersection of hyperplanes of the form (1). The corresponding subspace in \mathbb{F}_q^n consists of all n -tuples (x_1, x_2, \dots, x_n) which satisfy the defining equations of K in \mathbb{F}_q . We point out here that, for our purposes, it will suffice to consider q to be a prime number, as opposed to any power of a prime. Therefore, for reasons of simplicity and to avoid any ambiguity with the notation “mod q ,” we restrict our attention to finite fields \mathbb{F}_q for which q is a prime number.

We will denote the arrangement in \mathbb{F}_q^n corresponding to \mathcal{A} by the same symbol, hoping that it will be clear from the context which of the two we are actually considering. Thus, the set $\mathbb{F}_q^n - \bigcup \mathcal{A}$ in the next theorem is the set of all $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ that do not satisfy in \mathbb{F}_q the defining equations of any of the subspaces in \mathcal{A} . The Möbius inversion argument in the proof below is very similar to the one used originally by Crapo and Rota [7, Sect. 16] and later by Blass and Sagan [6]. We include it for the sake of completeness.

THEOREM 2.2. *Let \mathcal{A} be any subspace arrangement in \mathbb{R}^n defined over the integers and q be a large enough prime number. Then*

$$\chi(\mathcal{A}, q) = \# \left(\mathbb{F}_q^n - \bigcup \mathcal{A} \right). \tag{3}$$

Equivalently, identifying \mathbb{F}_q^n with $\{0, 1, \dots, q-1\}^n$, $\chi(\mathcal{A}, q)$ is the number of points with integer coordinates in the cube $[0, q-1]^n$ which do not satisfy mod q the defining equations of any of the subspaces in \mathcal{A} .

Proof. Let $x \in L_{\mathcal{A}}$, the intersection semilattice of the real arrangement and let $\dim x$ be the dimension of x as an affine subspace of \mathbb{R}^n . The proof is based on the fact that $\#x = q^{\dim x}$, viewing x as a subspace of \mathbb{F}_q^n , provided that we choose the prime q to be sufficiently large. This is because

the usual Gaussian elimination algorithm to solve a given linear system with integer coefficients works also over \mathbb{F}_q , if q is large. If not, x might reduce, for example, to the empty set or the whole space \mathbb{F}_q^n . We construct two functions $f, g: L_{\mathcal{A}} \rightarrow \mathbb{Z}$ by

$$f(x) = \#x$$

$$g(x) = \# \left(x - \bigcup_{y>x} y \right),$$

where all cardinalities are taken in \mathbb{F}_q^n . Thus $g(x)$ is the number of elements of x , not in any further intersection strictly contained in x . In particular, $g(\mathbb{R}^n) = \#(\mathbb{F}_q^n - \bigcup \mathcal{A})$. By our first remark above we have $f(x) = q^{\dim x}$. It is clear that $f(x) = \sum_{y \geq x} g(y)$, so by the Möbius inversion theorem [19; 26, Thm. 3.7.1]

$$\begin{aligned} \# \left(\mathbb{F}_q^n - \bigcup \mathcal{A} \right) &= g(\hat{0}) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) f(x) \\ &= \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{\dim x} = \chi(\mathcal{A}, q), \end{aligned}$$

as desired. ■

Zaslavsky notes in [35] the interpretability of his more general chromatic polynomial only for odd arguments. This corresponds to our assumption that q is a large enough prime. Of course, it suffices for Theorem 2.2 that q is a positive integer relatively prime to an integer depending only on the arrangement, once the field \mathbb{F}_q is replaced by the abelian group of integers mod q .

Theoretically, Theorem 2.2 computes the characteristic polynomial only for large prime values of q . For specific arrangements though, when computed for such q , the right hand side of (3) will be a polynomial in q . Since $\chi(\mathcal{A}, q)$ is also a polynomial, the two polynomials will have to agree for all q . It is clear that Theorem 2.2 is equivalent to the result of Blass and Sagan if \mathcal{A} is embedded in \mathcal{B}_n and hence it implies all the specializations of Theorem 2.1 mentioned earlier.

It was pointed out by Richard Stanley that in the special case of hyperplane arrangements, Theorem 2.2 also appeared as Theorem 2.69 in [16], stated again for hyperplane arrangements over finite fields. No consequences of the theorem for real arrangements seem to have been derived in [16] either. The generalization to subspace arrangements was obtained independently by Björner and Ekedahl in their recent work [3].

We also mention that a summation formula for the characteristic polynomial of an arbitrary hyperplane arrangement was recently found by Postnikov [18] (see also the comments in [28, Sect. 1]). This formula generalizes that of Whitney [32] which concerns $\chi(\mathcal{G}, q)$, where \mathcal{G} is a graphical arrangement with associated graph G . Whitney's theorem states that

$$\chi(\mathcal{G}, q) = \sum_{S \subseteq E(G)} (-1)^{\#S} q^{c(S)},$$

where $E(G)$ denotes the set of edges of G and $c(S)$ is the number of connected components of the spanning subgraph G_S of G with edge set S . Postnikov and Stanley use this generalization to study classes of hyperplane arrangements, called *deformations* of \mathcal{A}_n . A deformation of \mathcal{A}_n is an arrangement of the form

$$x_i - x_j = \alpha_{ij}^{(m)}, \quad 1 \leq m \leq m_{ij}, \tag{4}$$

where $\alpha_{ij}^{(m)}$ are arbitrary real numbers. As it turns out, Postnikov's generalization of the Whitney formula (at least for rational hyperplane arrangements) is related to Theorem 2.2 by the famous principle of inclusion-exclusion.

EXAMPLE 2.3. To get a first feeling of the applicability of Theorem 2.2, consider the hyperplane arrangement \mathcal{A}'_n introduced after Theorem 2.1. Theorem 2.2 is saying that for large prime numbers q , $\chi(\mathcal{A}'_n, q)$ counts the number of n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ for which

$$\begin{aligned} x_i &\neq x_j, & 1 \leq i < j \leq n, \\ x_1 - x_n &\neq 1. \end{aligned}$$

There are q ways to choose x_1 , then $q - 2$ ways to choose x_n so that $x_n \neq x_1, x_1 - 1$, $q - 2$ ways to choose x_2 so that $x_2 \neq x_1, x_n$ etc, and finally $q - n + 1$ ways to choose x_{n-1} . This simple count proves (2). It follows from Theorem 1.1 that $r(\mathcal{A}'_n) = (3/2)n!$ and $b(\mathcal{A}'_n) = (n - 2)!$, which can be easily seen otherwise. Note that the normal vectors to the hyperplanes of \mathcal{A}'_n span an $(n - 1)$ -dimensional linear subspace of \mathbb{R}^n and hence the term *bounded* regions has to be interpreted in an $(n - 1)$ -dimensional sense.

In the following sections we will give a large number of examples of more complicated arrangements for which the finite field method is particularly elegant. We reserve for Section 6 a generalization of Theorem 2.2 which gives an interpretation to the Whitney polynomial of a rational subspace arrangement.

We also note that in view of the remark at the end of the introduction, Theorem 2.2 gives a combinatorial method to compute the Euler characteristic of the space $M_{\mathcal{A}} = \mathbb{R}^n - \bigcup \mathcal{A}$ for any rational subspace arrangement \mathcal{A} .

Coxeter hyperplane arrangements. We finally describe a less straightforward example which seems to deserve some mention. We observe that a simple and universal proof of a theorem about the characteristic polynomial of a Coxeter hyperplane arrangement can be derived from Theorem 2.2. The theorem we are referring to is due to Blass and Sagan [6, 20] and independently, in an equivalent form, due to Haiman [10].

We take this opportunity to introduce briefly terminology and notation related to Coxeter arrangements, to be kept throughout the whole paper. We follow Humphreys' exposition [13] and rely on it for background on Coxeter groups.

Let W be a finite Coxeter group, determined by an irreducible crystallographic root system Φ spanning \mathbb{R}^n . The hyperplanes which pass through the origin and are orthogonal to the roots define the *Coxeter arrangement* \mathcal{W} , associated to W . The reflections in these hyperplanes generate the group W . Let $Z(\Phi)$ be the *coweight lattice* associated to Φ , i.e. the set of vectors $x \in \mathbb{R}^n$ satisfying $(\alpha, x) \in \mathbb{Z}$ for all roots $\alpha \in \Phi$. By the term "lattice" here we mean a discrete subgroup of \mathbb{R}^n , not to be confused with a poset whose finite subsets have joins and meets.

For any positive real t we define

$$P_t(\Phi) = \{x \in Z(\Phi) \mid (\alpha, x) < t \text{ for all } \alpha \in \Phi\}.$$

We now fix a simple system

$$\Delta = \{\sigma_1, \dots, \sigma_n\}$$

of Φ . This means that Δ is a linear basis of \mathbb{R}^n and that any root $\alpha \in \Phi$ can be expressed as an integer linear combination

$$\alpha = \sum_{i=1}^n c_i(\alpha) \sigma_i,$$

where the coefficients $c_i(\alpha)$ are either all ≥ 0 or all ≤ 0 . The highest root $\tilde{\alpha}$ is characterized by the conditions $c_i(\tilde{\alpha}) \geq c_i(\alpha)$ for all $\alpha \in \Phi$ and $1 \leq i \leq n$.

Finally, let $L(\Phi^\vee)$ denote the coroot lattice of Φ , i.e. the \mathbb{Z} -span of

$$\Phi^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi \right\}$$

in \mathbb{R}^n . Then the index of $L(\Phi^\vee)$ as a subgroup of $Z(\Phi)$ is called the *index of connection* of Φ and is denoted by f . We are now able to state the result in the language used by Blass and Sagan.

THEOREM 2.4 [6, Thm. 4.1; 10, Thm. 7.4.2; 20, Thm. 2.3]. *Let Φ be an irreducible crystallographic root system for a Weyl group W with associated Coxeter arrangement \mathcal{W} . Let t be a positive integer relatively prime to all the coefficients $c_i = c_i(\tilde{\alpha})$. Then*

$$\chi(\mathcal{W}, t) = \frac{1}{f} \# \left(P_t(\Phi) - \bigcup \mathcal{W} \right).$$

The outlined proof which appears in [6] is a case by case verification, which uses the classification theorem of finite Coxeter groups and some computer calculations for the case of the root systems E_6 , E_7 and E_8 . In both papers [6] and [20], the authors raise the question of finding a simpler, more conceptual proof of the theorem which works simultaneously for all crystallographic root systems. We will show below how Theorem 2.2 can be used to obtain such a proof, after we introduce some more useful notation. This proof turns out to be closely related to an argument already outlined by Haiman in [10] (see the remark following Theorem 7.4.2).

Let W_a be the affine Weyl group associated to Φ and let \mathcal{W}_a be the associated infinite hyperplane arrangement. Thus \mathcal{W}_a is the set of hyperplanes of the form

$$(\alpha, x) = k,$$

where $\alpha \in \Phi$, $k \in \mathbb{Z}$, and W_a is the group generated by the reflections in these hyperplanes.

For $x \in \mathbb{R}^n$, let $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ be defined by

$$x_i^* = (x, \sigma_i).$$

In other words, x^* is the n -tuple of coordinates of x in the dual basis

$$\{\varpi_1^\vee, \varpi_2^\vee, \dots, \varpi_n^\vee\}$$

to Δ , with respect to the standard inner product (\cdot, \cdot) . Note that $x \in Z(\Phi)$ if and only if $x^* \in \mathbb{Z}^n$. Hence the map $x \rightarrow x^*$ defines a vector space isomorphism of \mathbb{R}^n under which the lattice $Z(\Phi)$ corresponds to \mathbb{Z}^n .

Proof of Theorem 2.4. Since

$$(\alpha, x) = \left(\sum_{i=1}^n c_i(\alpha) \sigma_i, x \right) = \sum_{i=1}^n c_i(\alpha) x_i^*,$$

the arrangement \mathcal{W} corresponds, under the isomorphism, to an arrangement \mathcal{W}' defined over the integers. Let t be a large prime. Theorem 2.2 implies that

$$\begin{aligned} \chi(\mathcal{W}, t) &= \# \{x^* \in \{0, 1, \dots, t-1\}^n \mid (\alpha, x) \neq 0, \pm t, \pm 2t, \dots \text{ for all } \alpha \in \Phi\} \\ &= \# \left\{ x^* \in \{0, 1, \dots, t-1\}^n \mid \frac{x}{t} \text{ is not in } \bigcup \mathcal{W}_a \right\} \\ &= \# \left(R_t - \bigcup \mathcal{W}_a \right), \end{aligned}$$

where

$$R_t = R \cap \frac{1}{t} Z(\Phi)$$

and R is the parallelepiped

$$\left\{ \sum_{i=1}^n y_i \varpi_i^\vee \mid 0 \leq y_i \leq 1 \right\}.$$

It is known that $R - \bigcup \mathcal{W}_a$ has $(\#W)/f$ connected components [13, p. 99] and that W_a acts transitively on them. It is also clear that W_a preserves the points of R_t . It follows that each connected component of $R - \bigcup \mathcal{W}_a$ has the same number of points belonging to R_t . Hence, if

$$A_0 = \{x \in \mathbb{R}^n \mid 0 < (\alpha, x) < 1 \text{ for all } \alpha \in \Phi\}$$

is the fundamental alcove of the affine Coxeter arrangement W_a , then

$$\begin{aligned} \chi(\mathcal{W}, t) &= \frac{\#W}{f} \# \left(A_0 \cap \frac{1}{t} Z(\Phi) \right) \\ &= \frac{\#W}{f} \# \{x \in Z(\Phi) \mid 0 < (\alpha, x) < t \text{ for all } \alpha \in \Phi\} \\ &= \frac{1}{f} \# \left(P_t(\Phi) - \bigcup \mathcal{W} \right). \end{aligned}$$

The last equality follows from the fact that

$$\{x \in \mathbb{R}^n \mid (\alpha, x) < t \text{ for all } \alpha \in \Phi\} - \bigcup \mathcal{W}$$

has $\#W$ connected components and W acts simply transitively on them and preserves $Z(\Phi)$. This proves the result, at least for large primes t .

Let \mathbb{P} denote the set of positive integers. Note that, by the defining property of the highest root we have

$$\begin{aligned} & \# \{x \in Z(\Phi) \mid 0 < (\alpha, x) < t \text{ for all } \alpha \in \Phi\} \\ &= \# \{x \in Z(\Phi) \mid 0 < (\alpha, x) \text{ for all } \alpha \in \Phi \text{ and } (\tilde{\alpha}, x) < t\} \\ &= \# \left\{ x^* \in \mathbb{P}^n \mid \sum_{i=1}^n c_i(\alpha) x_i^* < t \right\}, \end{aligned}$$

which is the *Ehrhart quasi-polynomial* of the open simplex bounded by the coordinate hyperplanes and the hyperplane $\sum c_i x_i^* = t$. This quasi-polynomial is a polynomial in t for t relatively prime to the c_i . Hence the expressions $\chi(\mathcal{W}, t)$ and $(1/f) \#(P_t(\Phi) - \cup \mathcal{W})$ agree for all such t . ■

3. THE SHI ARRANGEMENTS

In this section we focus on some hyperplane arrangements, first introduced by Shi [21, 22], related to the affine Weyl groups [13, Ch. 4] and several variations. Theorem 2.2 applied to these arrangements leads to some interesting elementary counting problems. The solutions to these problems give easy proofs of results obtained in the past by much more complicated methods and suggest some generalizations.

Let Φ be an irreducible crystallographic root system Φ spanning \mathbb{R}^l with associated Weyl group W . We use the letter l instead of n for the rank of Φ , i.e. the dimension of the linear span of Φ , for reasons which will be apparent below. For any $\alpha \in \mathbb{R}^l$ and $k \in \mathbb{R}$, we denote by $H_{\alpha, k}$ the hyperplane defined by the linear equation $(\alpha, x) = k$. Fix a set of positive roots $\Phi^+ \subset \Phi$ once and for all, as in [13, Sect. 2.10]. We define the *Shi* arrangement corresponding to Φ as the collection of hyperplanes

$$\{H_{\alpha, k} \mid \alpha \in \Phi^+ \text{ and } k = 0, 1\}.$$

We denote it by $\hat{\mathcal{W}}$ except that, in order to be consistent with our earlier notation, we denote the Shi arrangement

$$x_i - x_j = 0, 1 \quad \text{for } 1 \leq i < j \leq n,$$

corresponding to the root system A_{n-1} , by $\hat{\mathcal{A}}_n$ rather than $\hat{\mathcal{A}}_{n-1}$. Thus, for the arrangement $\hat{\mathcal{A}}_n$ we should keep in mind that $l = n - 1$, although we consider it to be an arrangement in \mathbb{R}^n . This arrangement was denoted by \mathcal{S}_n in [28]. We choose the notation $\hat{\mathcal{A}}_n$ here to be consistent with other root systems. Shi [21] proved that the number of regions of $\hat{\mathcal{A}}_n$ is

$$r(\hat{\mathcal{A}}_n) = (n + 1)^{n-1} \tag{5}$$

using group-theoretic techniques and later [22] generalized his result to show that

$$r(\hat{\mathcal{W}}) = (h + 1)^l,$$

where h is the *Coxeter number* of W [13, p. 75]. Shi's proof is universal for all root systems, but still quite complicated. For an outline of a bijective proof of (5) and a refinement see the discussion in [28, Sect. 5]. Assuming Shi's result, Headley computed the characteristic polynomial of $\hat{\mathcal{W}}$ as follows.

THEOREM 3.1 [11; 12, Ch. VI]. *Let Φ , W , $\hat{\mathcal{W}}$ and l be as above. Let h be the Coxeter number of W . Then*

$$\chi(\hat{\mathcal{W}}, q) = (q - h)^l.$$

Since $h = n$ for the Coxeter group A_{n-1} , it follows that the characteristic polynomial of $\hat{\mathcal{A}}_n$ is $q(q - n)^{n-1}$. The extra factor of q corresponds to the fact that the dimension of the ambient Euclidean space of $\hat{\mathcal{A}}_n$ exceeds by one the dimension of the corresponding space in Theorem 3.1.

Headley's proof was done case by case. Stanley [28, Sect. 5] noted for the arrangement $\hat{\mathcal{A}}_n$ that Headley's proof can be simplified using an exponential generating function argument. This approach still has the disadvantage that it relies on Shi's result, whose proof is quite involved. Note that Theorem 2.2 implies the following for a general root system. The reasoning is the same with that in the proof of Theorem 2.4.

COROLLARY 3.2. *Let Φ and $\hat{\mathcal{W}}$ be given, as before. For large primes q we have*

$$\chi(\hat{\mathcal{W}}, q) = \#\{x^* \in \{0, 1, \dots, q-1\}^n \mid (\alpha, x) \not\equiv 0, 1 \pmod{q} \text{ for all } \alpha \in \Phi^+\},$$

where the notation is from Section 2.

It would be highly desirable to obtain Theorem 3.1 directly from Corollary 3.2 (compare also with the remarks before Theorem 5.5).

We will give a simple proof of Theorem 3.1, as well as several generalizations, for the case of the four infinite families of root systems. This enables us to obtain Shi's result as a corollary, via Theorem 1.1.

The Shi arrangement of type A. We first discuss the interpretation for $\chi(\mathcal{A}, q)$ given by Theorem 2.2 when \mathcal{A} is a deformation of \mathcal{A}_n . Suppose that \mathcal{A} is as in (4) and that the $\alpha_{ij}^{(m)}$ are integers. For large primes q ,

$\chi(\mathcal{A}_n, q)$ counts the number of n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ which satisfy conditions of the form

$$x_i - x_j \neq \alpha \tag{6}$$

in \mathbb{F}_q , where $\alpha = \alpha_{ij}^{(m)}$ for some m . We think of such an n -tuple as a map from $[n] = \{1, 2, \dots, n\}$ to \mathbb{F}_q , sending i to the class $x_i \in \mathbb{F}_q$. We think of the elements of \mathbb{F}_q as boxes arranged and labeled cyclically with the classes mod q . The top box is labeled with the zero class, the clockwise next box is labeled with the class $1 \pmod q$ etc. The n -tuples in \mathbb{F}_q^n become placements of the integers from 1 to n in the q boxes and $\chi(\mathcal{A}, q)$ counts the number of placements which satisfy certain restrictions prescribed by conditions (6).

The restriction prescribed by $x_i - x_j \neq 0$ is that i and j are not allowed to be placed in the same box. In general, the restriction prescribed by (6) is that i cannot be placed in the box labeled with $x_j + \alpha$, where x_j is the label of the box that j occupies. In other words, i cannot follow j clockwise “by α boxes,” if $\alpha > 0$ and cannot precede j clockwise by $-\alpha$ boxes, if $\alpha < 0$. We call the pair (i, j) a *descent* of the placement if $i < j$ and $x_i - x_j = 1$. This means that i immediately follows j clockwise in the placement.

Since it is only the relative positions of the integers from 1 to n that matters, we can remove the labels from the boxes. We refer to the boxes in this case as “unlabeled,” implying that they are indistinguishable. The placements that satisfy the restrictions prescribed by (6) are counted by $\tilde{\chi}(\mathcal{A}, q)$, where we have used the $\tilde{\chi}$ notation of the introduction. For each such placement we have q choices to decide where the zero class mod q will be, so that $\chi(\mathcal{A}, q) = q\tilde{\chi}(\mathcal{A}, q)$.

If \mathcal{A} contains \mathcal{A}_n , then no two distinct integers are allowed to be placed in the same box, so we are counting placements without repetitions. When dealing with unlabeled boxes, we can disregard the occupied boxes in the placement. Thus $\tilde{\chi}(\mathcal{A}, q)$ simply counts the number of appropriate circular placements of the integers from 1 to n and $q - n$ unlabeled boxes. The restriction imposed by the condition $x_i - x_j \neq \alpha$ is that i cannot follow j clockwise by α objects, where an object is either an integer or an unlabeled box.

As a first application of these ideas we consider the Shi arrangement $\hat{\mathcal{A}}_n$.

THEOREM 3.3. *The characteristic polynomial of $\hat{\mathcal{A}}_n$ is*

$$\chi(\hat{\mathcal{A}}_n, q) = q(q - n)^{n-1}.$$

In particular, $r(\hat{\mathcal{A}}_n) = (n + 1)^{n-1}$ and $b(\hat{\mathcal{A}}_n) = (n - 1)^{n-1}$.

Proof. By the previous discussion, for large primes q , $\tilde{\chi}(\hat{\mathcal{A}}_n, q)$ counts the number of circular placements of the integers from 1 to n and $q - n$

unlabeled boxes, such that no descent occurs. Equivalently, any string of consecutive integers, with no boxes in between, must be clockwise increasing.

To count these placements, let's consider first the $q - n$ unlabeled boxes, placed around a circle. There are $(q - n)^{n - 1}$ ways to place the elements of $[n]$ in the $q - n$ spaces between the boxes. Here we consider that there is only one way to place the first element of $[n]$ because of the cyclic symmetry of the arrangement of the $q - n$ unlabeled boxes. There is one way to order the elements placed in each space in clockwise increasing order. This gives the desired value for $\tilde{\chi}(\mathcal{A}_n, q)$. ■

To generalize our previous result, let S be any subset of the edge set

$$E_n = \{ij \mid 1 \leq i < j \leq n\}$$

of the complete graph on n vertices. Here and in what follows, we denote the two element set $\{i, j\}$, where $i < j$ by ij for simplicity. Thus such an S defines a simple graph on the vertex set $[n]$. To every $S \subset E_n$ we assign the hyperplane arrangement

$$\begin{aligned} x_i - x_j &= 0 && \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 && \text{for } 1 \leq i < j \leq n, \quad ij \in S \end{aligned}$$

and denote it by $\mathcal{A}_{n,S}$. The arrangement \mathcal{A}_n corresponds to the complete graph and the arrangement \mathcal{A}_n to the empty graph. In general, $\mathcal{A}_{n,S}$ interpolates between the two arrangements. The following generalization of Theorem 3.3 produces a new large class of hyperplane arrangements whose characteristic polynomials factor completely over the nonnegative integers. Compare also with the definition and corresponding property of chordal graphs [24, Example 4.6]. Recall that the notation $ij \in S$ implies that $i < j$.

THEOREM 3.4. *Suppose that the set $S \subseteq E_n$ has the following property: if $ij \in S$, then $ik \in S$ for all $j < k \leq n$. Then*

$$\chi(\mathcal{A}_{n,S}, q) = q \prod_{1 < j \leq n} (q - n + j - a_j - 1),$$

where $a_j = \#\{i < j \mid ij \in S\}$. In particular,

$$r(\mathcal{A}_{n,S}) = \prod_{1 < j \leq n} (n - j + a_j + 2)$$

and

$$b(\mathcal{A}_{n,S}) = \prod_{1 < j \leq n} (n - j + a_j).$$

Proof. The idea is similar to that in the proof of Theorem 3.3. By Theorem 2.2, $\tilde{\chi}(\mathcal{A}_{n,S}, q)$ counts the number of circular placements of the integers from 1 to n and $q-n$ unlabeled boxes, such that no “ S -descent” occurs. An S -descent is a descent (i, j) of the placement with $ij \in S$ (hence $1 \leq i < j \leq n$).

We consider again the $q-n$ unlabeled boxes, placed around a circle. We will now enter the integers $1, 2, \dots, n$ into the $q-n$ spaces between the boxes one by one, in the order indicated. We claim that, for each $j \geq 2$, after having inserted $1, \dots, j-1$ to obtain a circular placement of the $q-n$ boxes and the first $j-1$ positive integers, there are $q-n+j-a_j-1$ ways to insert j . This is because there are $q-n+j-1$ spaces in all and the a_j spaces immediately before the a_j positive integers $i < j$ for which $ij \in S$ are forbidden. Indeed, if j is placed immediately before i , where $ij \in S$, then by construction, the element immediately preceding i in the final placement will be some $k > i$. This will produce an S -descent, since by the assumption on S , $ik \in S$. ■

We now mention some corollaries of Theorem 3.4 which demonstrate its wide applicability.

COROLLARY 3.5. *Let $1 \leq k \leq n$ be an integer. The arrangement $\hat{\mathcal{A}}_{n,S}$*

$$x_i - x_j = 0 \quad \text{for } 1 \leq i < j \leq n,$$

$$x_i - x_j = 1 \quad \text{for } 1 \leq i < j \leq k,$$

corresponding to $S = \{ij \mid 1 \leq i < j \leq k\}$, has characteristic polynomial

$$\chi(\hat{\mathcal{A}}_{n,S}, q) = q(q-k)^{k-1} \prod_{k \leq j \leq n-1} (q-j).$$

In particular,

$$r(\hat{\mathcal{A}}_{n,S}) = \frac{n!}{k!} (k+1)^{k-1} \quad \text{and} \quad b(\hat{\mathcal{A}}_{n,S}) = \frac{(n-2)!}{(k-1)!} (k-1)^k.$$

Proof. Clearly, the characteristic polynomial is the same with that of $\hat{\mathcal{A}}_{n,T}$, where $T = \{ij \mid n-k+1 \leq i < j \leq n\}$. This choice of T satisfies the hypothesis of Theorem 3.4. We have $a_j = 0$ for $2 \leq j \leq n-k$ and $a_{n-k+j} = j-1$ for $1 \leq j \leq k$. The result follows from Theorem 3.4. ■

For $k=1$ and $k=n$ we obtain again the characteristic polynomials of \mathcal{A}_n and $\hat{\mathcal{A}}_n$ respectively.

COROLLARY 3.6. Let $0 \leq k \leq n-1$ and $0 \leq l \leq n-k-1$ be integers. Let $S \subseteq E_n$ be

$$S = \{ij \mid i < j, 1 \leq i \leq k\} \cup \{k+1j \mid n-l+1 \leq j \leq n\}.$$

Then

$$\chi(\hat{\mathcal{A}}_{n,S}, q) = q(q-n)^{k-1} (q-k-l-1) \prod_{k+1 < j \leq n} (q-j).$$

In particular,

$$r(\hat{\mathcal{A}}_{n,S}) = \frac{(n+1)!}{(k+2)!} (k+l+2)(n+1)^{k-1}$$

and

$$b(\hat{\mathcal{A}}_{n,S}) = \frac{(n-1)!}{k!} (k+l)(n-1)^{k-1}.$$

Proof. It follows directly from Theorem 3.4 since, for the given S , $a_j = j-1$ for $2 \leq j \leq k$, $a_{k+1} = \dots = a_{n-l} = k$ and $a_{n-l+1} = \dots = a_n = k+1$. ■

For $k=n-1, l=0$ the formulas check with Theorem 3.3 once more. For $k=0, l=1$ we get the result for the arrangement \mathcal{A}'_n of Section 2. More generally, for $k=0$ and any $0 \leq l \leq n-1$ we get the following specialization of Corollary 3.6.

COROLLARY 3.7. The arrangement

$$\begin{aligned} x_i - x_j &= 0 & \text{for } 1 \leq i < j \leq n, \\ x_1 - x_j &= 1 & \text{for } n-l+1 \leq j \leq n \end{aligned}$$

has characteristic polynomial

$$q(q-l-1) \prod_{1 < j \leq n-1} (q-j).$$

In particular, for this arrangement $r = (1/2)(l+2)n!$ and $b = l(n-2)!$.

Finally, we mention separately the special case $l=0$ of Corollary 3.6.

COROLLARY 3.8. The arrangement

$$\begin{aligned} x_i - x_j &= 0 & \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 & \text{for } i < j, \quad 1 \leq i \leq k \end{aligned}$$

has characteristic polynomial

$$q(q - n)^{k-1} \prod_{k < j \leq n} (q - j).$$

In particular, for this arrangement

$$r = \frac{(n + 1)!}{(k + 1)!} (n + 1)^{k-1}$$

and

$$b = \frac{(n - 1)!}{(k - 1)!} (n - 1)^{k-1}.$$

Considering also affine hyperplanes which correspond to negative roots, we obtain the following straightforward generalization of Theorem 3.4. For further generalizations, see [1, Ch. 6]. Here, S is a subset of

$$\mathcal{E}_n = \{(i, j) \in [n] \times [n] \mid i \neq j\},$$

the edge set of the complete directed graph on n vertices, having no loops.

THEOREM 3.9. *Suppose that the set $S \subseteq \mathcal{E}_n$ has the following properties:*

- (i) *If $i, j < k, i \neq j$ and $(i, j) \in S$, then $(i, k) \in S$ or $(k, j) \in S$.*
- (ii) *If $i, j < k, i \neq j$ and $(i, k) \in S, (k, j) \in S$, then $(i, j) \in S$.*

Then the characteristic polynomial of the arrangement

$$\begin{aligned} x_i - x_j &= 0 && \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 && \text{for } (j, i) \in S \end{aligned}$$

factors as in Theorem 3.4, where

$$a_j = \#\{i < j \mid (j, i) \in S\} + \#\{i < j \mid (i, j) \in S\}.$$

Shi Arrangements for other root systems. We now consider $\hat{\mathcal{B}}_n, \hat{\mathcal{C}}_n, \hat{\mathcal{D}}_n$ and related arrangements. $\hat{\mathcal{D}}_n$ is the arrangement

$$\begin{aligned} x_i - x_j &= 0, 1 && \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &= 0, 1 && \text{for } 1 \leq i < j \leq n. \end{aligned}$$

For simplicity, we first consider the arrangement obtained from $\hat{\mathcal{D}}_n$ by adding the hyperplanes $x_i = 0$. We denote this arrangement by $\hat{\mathcal{D}}_n^0$. The argument in the proof of the next theorem is a variation of the one used for deformations of \mathcal{A}_n .

THEOREM 3.10. *The characteristic polynomial of $\hat{\mathcal{D}}_n^0$ is*

$$\chi(\hat{\mathcal{D}}_n^0, q) = (q - 2n + 1)^n.$$

In particular, $r(\hat{\mathcal{D}}_n^0) = (2n)^n$ and $b(\hat{\mathcal{D}}_n^0) = (2n - 2)^n$.

Proof. By Theorem 2.2, $\chi(\hat{\mathcal{D}}_n^0, q)$ counts the number of n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ which satisfy the conditions

$$\begin{aligned} x_i &\neq 0 && \text{for } 1 \leq i \leq n, \\ x_i - x_j &\neq 0, 1 && \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &\neq 0, 1 && \text{for } 1 \leq i < j \leq n. \end{aligned}$$

As with the deformations of \mathcal{A}_n , we can think of such an n -tuple as a map from $[n]$ to \mathbb{F}_q , sending i to the class $x_i \in \mathbb{F}_q$. It is more convenient now to think of this n -tuple as a map from $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$ to \mathbb{F}_q , sending i to the class $x_i \in \mathbb{F}_q$ and $-i$ to the class $-x_i$. We call the elements of $\pm[n]$ the signed integers from 1 to n .

The conditions $x_i \neq 0$, $x_i \pm x_j \neq 0$ impose the restriction that no signed integer is sent to the zero class and no two distinct signed integers are sent to the same class. The conditions $x_i - x_j \neq 1$ for $1 \leq i < j \leq n$ imply that no $i < j$ can be placed to the class $x_j + 1$, i.e. immediately after j and that no $-i$, where $i < j$, can be placed to the class $-x_j - 1$, i.e. immediately before $-j$. The condition $x_i + x_j \neq 1$ for $i \neq j$ implies that no $-i$, with $i \neq j$, can be placed to the class $x_j - 1$, i.e. immediately before j . In other words, no negative integer $-i$ can immediately precede a positive one j with $i \neq j$.

Overall, we want to place the signed integers in the nonzero classes mod q without repetitions and under the symmetry condition explained above, so that the following is true: Any two integers i, j with $i \neq \pm j$ that appear consecutively in the placement, with no other objects in between, should be clockwise ordered according to the linear order $1 \triangleleft 2 \triangleleft \dots \triangleleft n \triangleleft -n \triangleleft \dots \triangleleft -1$. Again, we think of the classes mod q as boxes arranged along a circle, with the top box labeled with the zero class, the clockwise next box labeled with the class $1 \bmod q$ etc.

We can now concentrate on what happens only on the right half of the circle, i.e. the classes from 0 to $(1/2)(q-1)$, included. Indeed, if a signed integer is placed in one of these classes, say a , then its negative is placed in the class $-a$ and an allowable placement on the right half gives an allowable placement on the left half. For each pair $(i, -i)$, where $i \in [n]$, exactly one of $i, -i$ should appear in the right semicircle. So we are looking for the number of placements of the elements of $[n]$ in the $(1/2)(q-1)$ boxes on this semicircle, each element with a \pm sign, subject to the restrictions in the previous paragraph. Now there are $(1/2)(q+1) - n$ boxes

which will be unoccupied in the end, starting with the top box labeled with the zero class, and $(1/2)(q+1) - n$ spaces, starting with the space to the right of the top box. There are $q - 2n + 1$ choices to place each element of $[n]$, which is twice the number of possible spaces, accounting for the freedom to choose one of two possible signs. There is one way to order the integers in each space, prescribed by \triangleleft . Hence, there are $(q - 2n + 1)^n$ placements in all. ■

As in the case with \mathcal{A}_n , we can extend the previous argument to get a generalization of Theorem 3.10. We denote by $\hat{\mathcal{G}}_{n,S,T}^0$ the arrangement

$$\begin{aligned} x_i &= 0 && \text{for } 1 \leq i \leq n, \\ x_i \pm x_j &= 0 && \text{for } 1 \leq i < j \leq n, \\ x_i - x_j &= 1 && \text{for } 1 \leq i < j \leq n, \quad ij \in S, \\ x_i + x_j &= 1 && \text{for } 1 \leq i < j \leq n, \quad ij \in T, \end{aligned}$$

where $S, T \subseteq E_n$, the edge set of the complete graph on the vertex set $[n]$. This arrangement interpolates between \mathcal{B}_n and $\hat{\mathcal{G}}_n^0$.

THEOREM 3.11. *Suppose that the sets $S, T \subseteq E_n$ have the following properties:*

- (i) *If $ij \in S$ and $i < j < k$, then $ik \in S \cap T$.*
- (ii) *If $ij \in T$ and $i < j < k$, then $ik \in S$ or $jk \in T$ and also, $ik \in T$ or $jk \in S$.*
- (iii) *If $ik \in S, jk \in T$ and $i < j < k$, then $ij \in T$ and similarly, if $jk \in S, ik \in T$ and $i < j < k$, then $ij \in T$.*

Then

$$\chi(\hat{\mathcal{G}}_{n,S,T}^0, q) = \prod_{j=1}^n (q - 2n + 2j - 1 - a_j - b_j),$$

where $a_j = \#\{i < j \mid ij \in S\}$ and $b_j = \#\{i < j \mid ij \in T\}$. In particular,

$$r(\hat{\mathcal{G}}_{n,S,T}^0) = \prod_{j=1}^n (2n - 2j + a_j + b_j + 2)$$

and

$$b(\hat{\mathcal{G}}_{n,S,T}^0) = \prod_{j=1}^n (2n - 2j + a_j + b_j).$$

Proof. We follow the argument in the proof of Theorem 3.10. The conditions $x_i \pm x_j \neq 1$ are now replaced by $x_i - x_j \neq 1$ if $ij \in S$ and $x_i + x_j \neq 1$ if $ij \in T$. To count the corresponding number of placements of signed integers and unlabeled boxes on a semicircle, we insert the integers $1, 2, \dots, n$ in this order, each with a sign, in the $(1/2)(q+1) - n$ spaces between the boxes. These spaces include the one to the right of the last box. We claim that we have $q - 2n + 2j - 1 - a_j - b_j$ choices to insert j . This is because there are $(1/2)(q+1) - n + j - 1$ spaces between the empty boxes and the first $j - 1$ integers already inserted, and we have two choices for the sign of j , giving a total of $q - 2n + 2j - 1$ choices. Some of these choices though are forbidden. For each $i < j$ with $ij \in S$, we should avoid the patterns ji and $-i - j$, due to the restriction imposed by $x_i - x_j \neq 1$. This accounts for a_j choices. The condition $x_i + x_j \neq 1$ implies that we should avoid the patterns $-ji$ and $-ij$ and excludes b_j more possibilities. Assumption (iii) guarantees that the forbidden choices to insert j are distinct. Assumptions (i) and (ii) ensure that, once we create a forbidden pattern when inserting j , a (possibly different) forbidden pattern will still exist after we have inserted the rest of the integers. ■

For $S = T = E_n$, the previous theorem reduces to Theorem 3.10. Note that for $S = \emptyset$, $T = \{n - 1n\}$, $\mathcal{D}_{n,S,T}^0$ has the same intersection lattice with the arrangement obtained from \mathcal{B}_n by adding the hyperplane $x_1 + x_2 = 1$. This is the B_n analogue of the arrangement \mathcal{A}'_n , considered early in Section 2, since $e_1 + e_2$ is the highest root in B_n . Theorem 3.11 implies that the characteristic polynomial of this arrangement is $(q - 2)(q - 3)(q - 5) \cdots (q - 2n + 1)$. To give a more general example, we mention the following specialization, obtained in the same way as Corollary 3.5 was obtained from Theorem 3.4.

COROLLARY 3.12. *Let $1 \leq k \leq n$ be an integer. The arrangement*

$$\begin{aligned} x_i &= 0 & \text{for } 1 \leq i \leq n, \\ x_i \pm x_j &= 0 & \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &= 1 & \text{for } 1 \leq i < j \leq k \end{aligned}$$

has characteristic polynomial

$$\prod_{j=k}^{n-1} (q - 2j - 1) \prod_{j=k}^{2k-1} (q - j).$$

In particular, for this arrangement

$$r = 2^{n-k} n! \binom{2k}{k}$$

and

$$b = 2^{n-k}(n-1)! (k-1) \binom{2k-2}{k-1}.$$

Note that \mathcal{B}_n and \mathcal{C}_n coincide over a field of characteristic different from 2. This does not happen for $\hat{\mathcal{B}}_n$ and $\hat{\mathcal{C}}_n$, so we consider them separately. The arrangements $\hat{\mathcal{B}}_n, \hat{\mathcal{C}}_n$ are obtained from $\hat{\mathcal{D}}_n$, mentioned before Theorem 3.10, by adding the hyperplanes $x_i = 0, 1$ for $1 \leq i \leq n$ and $2x_i = 0, 1$ for $1 \leq i \leq n$, respectively. The following theorem computes the characteristic polynomials of $\hat{\mathcal{B}}_n, \hat{\mathcal{C}}_n$ and $\hat{\mathcal{D}}_n$. It verifies Theorem 3.1 since the Weyl groups B_n, C_n and D_n have Coxeter numbers $2n, 2n$ and $2n - 2$ respectively.

THEOREM 3.13. *We have*

$$\chi(\mathcal{W}, q) = \begin{cases} (q - 2n)^n, & \text{if } \Phi = B_n \text{ or } C_n; \\ (q - 2n + 2)^n, & \text{if } \Phi = D_n. \end{cases}$$

In particular, $r(\hat{\mathcal{B}}_n) = r(\hat{\mathcal{C}}_n) = (2n + 1)^n$, $b(\hat{\mathcal{B}}_n) = b(\hat{\mathcal{C}}_n) = r(\hat{\mathcal{D}}_n) = (2n - 1)^n$ and $b(\hat{\mathcal{D}}_n) = (2n - 3)^n$.

Proof. We modify the argument in the proof of Theorem 3.10. If $\Phi = B_n$, we have the extra conditions $x_i \neq 1$. This means that the class 1 mod q should either be empty or occupied by a negative integer $-i$, in case some $x_i = -1$. Hence, by the restrictions on the order of consecutive integers, the space immediately following the zero class should contain only negative integers and their order is prescribed. The choice of sign is arbitrary for the remaining $(1/2)(q - 1) - n$ spaces. Thus, for each $i \in [n]$, we have $1 + 2((q - 1)/2 - n) = q - 2n$ choices to place i and hence $(q - 2n)^n$ placements in all.

If $\Phi = C_n$, we have the extra conditions $2x_i \neq 1$, i.e. $x_i \neq (1/2)(q + 1)$, or $-x_i \neq (1/2)(q - 1)$. So now the last class $(1/2)(q - 1)$ in the right semicircle should either be empty or occupied by a positive integer, which implies that we are forced to choose the positive sign when inserting integers in the last space. The rest of the reasoning is as before.

If $\Phi = D_n$, the conditions $x_i \neq 0$ are missing. We have now one more allowable space between the $(1/2)(q + 1) - n$ unlabeled boxes, namely the one immediately to the left of the first box. This space will be nonempty if $x_i = 0$ for some i . In this case $-x_i = 0$, so i and $-i$ are both placed in the zero class and the rest of the integers (if any) in the first space should be negative. Thus, sign and order are prescribed for placing integers in the first space. Hence, for each $i \in [n]$ we have $1 + 2((q + 1)/2 - n) = q - 2n + 2$ placement choices, giving again the desired result. ■

There are some further variations of the results in this section which can be obtained using the same reasoning. We give one such next and refer the interested reader to [1, Ch. 6] for more details.

THEOREM 3.14. *The arrangement*

$$\begin{aligned} x_i &= 0, 1 && \text{for } 1 \leq i \leq n, \\ x_i \pm x_j &= 0 && \text{for } 1 \leq i < j \leq n, \\ x_i + x_j &= 1 && \text{for } 1 \leq i < j \leq n \end{aligned}$$

has characteristic polynomial

$$\prod_{j=n+1}^{2n} (q-j).$$

In particular, for this arrangement $r = (2n+1)/(n+1)!$ and $b = (2n-1)/(n-1)!$.

Proof. We have $q-2n$ choices to insert 1, as for the \mathcal{B}_n arrangement. Now we can only choose the negative sign in the space to the right of the zero class and should avoid the patterns $-ji$ and $-ij$ when $i < j$, in all other spaces. The number of choices increases by one after each insertion, regardless of the sign and space we choose at each stage. ■

4. THE LINIAL ARRANGEMENT

In this section we will be primarily concerned with the *Linial* arrangement

$$x_i - x_j = 1 \quad \text{for } 1 \leq i < j \leq n.$$

Following [28], we denote this arrangement by \mathcal{L}_n and let $r(\mathcal{L}_n) = g_n$. The number g_n has a surprising combinatorial interpretation, initially conjectured by Stanley on the basis of data provided by Linial and Ravid, and recently proved by Postnikov [28, Sect. 4]. We will give another proof of Stanley's conjecture based on Theorem 2.2. We start with the necessary definitions and refer the reader to [28, Sect. 4] for more information and other combinatorial interpretations of g_n .

An *alternating tree* or *intransitive tree* on $n+1$ vertices is a labeled tree with vertices $0, 1, \dots, n$, such that no i, j, k with $i < j < k$ are consecutive vertices of a path in the tree. In other words, for any path in the tree with

consecutive vertices a_0, a_1, \dots, a_l we have $a_0 < a_1 > a_2 < a_3 \cdots a_l$ or $a_0 > a_1 < a_2 > a_3 \cdots a_l$. Alternating trees first arose in the context of hypergeometric functions [9]. In the paper [17], Postnikov proved that if f_n denotes the number of alternating trees on $n + 1$ vertices and if

$$y = \sum_{n \geq 0} f_n \frac{x^n}{n!},$$

then

$$y = e^{(x/2)(y+1)}$$

and

$$f_{n-1} = \frac{1}{n2^{n-1}} \sum_{k=1}^n \binom{n}{k} k^{n-1}.$$

Postnikov’s theorem (initially conjectured by Stanley) can be stated as follows.

THEOREM 4.1 [18; 28, Theorem 4.1]. *For all $n \geq 0$ we have $f_n = g_n$.*

In the following theorem we give a new, explicit formula for the characteristic polynomial of \mathcal{L}_n which implies Theorem 4.1, via Theorem 1.1.

THEOREM 4.2. *For all $n \geq 1$ we have*

$$\chi(\mathcal{L}_n, q) = \frac{q}{2^n} \sum_{j=0}^n \binom{n}{j} (q-j)^{n-1}. \tag{7}$$

In particular,

$$g_n = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (j+1)^{n-1} = f_n$$

and

$$b(\mathcal{L}_n) = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (j-1)^{n-1}.$$

We postpone the proof of Theorem 4.2 until the end of this section. We remark here that it would be interesting to find a combinatorial interpretation for the numbers $b(\mathcal{L}_n)$, similar to the one that Theorem 4.1 gives for g_n .

More generally, for any nonnegative integer k we consider the arrangement with hyperplanes

$$x_i - x_j = 1, 2, \dots, k \quad \text{for } 1 \leq i < j \leq n.$$

We denote this arrangement by $\mathcal{A}_n^{[k]}$. Note that for $k=0$ it reduces to the empty arrangement in \mathbb{R}^n , with characteristic polynomial q^n , and for $k=1$ to the Linial arrangement \mathcal{L}_n . We also denote by $\mathcal{A}_n^{[0, k]}$ the arrangement

$$x_i - x_j = 0, 1, \dots, k \quad \text{for } 1 \leq i < j \leq n,$$

obtained from $\mathcal{A}_n^{[k]}$ by adding the hyperplanes $x_i - x_j = 0$ for $1 \leq i < j \leq n$. This arrangement provides another generalization of the Shi arrangement \mathcal{S}_n .

We now show that the characteristic polynomials of $\mathcal{A}_n^{[k]}$ and $\mathcal{A}_n^{[0, k]}$ are closely related.

THEOREM 4.3. *For all $n, k \geq 1$ we have*

$$\tilde{\chi}(\mathcal{A}_n^{[0, k]}, q) = \tilde{\chi}(\mathcal{A}_n^{[k-1]}, q-n).$$

Proof. Let q be a large prime. Using Theorem 2.2 as in Section 3, $\tilde{\chi}(\mathcal{A}_n^{[0, k]}, q)$ counts the number of circular placements of the integers from 1 to n and $q-n$ unlabeled boxes, such that at least k boxes separate an integer j from an integer $i < j$ which follows j in the clockwise direction, except for the boxes in between. Such a pair (i, j) is said to form a *weak descent* of the placement. We call these placements of type α . It follows easily (see (8) in the proof of the next theorem) that the number of placements of type α is a polynomial in q , so the previous interpretation for $\tilde{\chi}(\mathcal{A}_n^{[0, k]}, q)$ is true for all $q > n$.

Using Theorem 2.2 again, for large primes q , $\tilde{\chi}(\mathcal{A}_n^{[k-1]}, q-n)$ counts the number of placements of the integers from 1 to n into $q-n$ cyclically arranged unlabeled boxes, such that at least $k-1$ empty boxes separate the box of an integer j from that of an integer $i < j$ which follows j in the clockwise direction, except for the empty boxes in between. Note that we are now allowed to place many integers in the same box. We linearly order the integers in any occupied box to be increasing in the clockwise direction. We call these placements of type β .

To prove the result, it suffices to establish a bijection between the placements of type α and those of type β . Starting with a placement of type α , remove a box from each maximal string of consecutive unlabeled boxes. If a single box forms such a string by itself, i.e. is preceded and followed by integers (not defining a weak descent), then place a bar between these integers after removing the box. The maximal clockwise increasing strings of consecutive integers, with no boxes or bars in between define the

occupied boxes of the placement of type β thus produced. This placement has now $q - n$ boxes, since the correspondence described reduces the number of objects by one between each of the n pairs of weakly consecutive integers (with possibly boxes in between) of the placement of type α , and we had q objects to start with. For example, if i was followed clockwise by j with no boxes in between in the placement of type α (hence $i < j$), then i and j will be placed in the same box in the placement of type β , decreasing the total number of objects by one. It is easy to see that this correspondence is indeed a bijection. ■

Note that for $k = 1$, Theorem 4.3 reduces to Theorem 3.3. We now give some formulas for the characteristic polynomial and number of regions of $\mathcal{A}_n^{[0, k]}$. Here and in what follows, we denote by $[x^n] f(x)$ the coefficient of x^n in a formal series $f(x)$ of the form $f(x) = \sum_{n \geq n_0} c_n x^n$, where $n_0 \in \mathbb{Z}$.

THEOREM 4.4. *For all $n, k \geq 1, q > n$ we have*

$$\chi(\mathcal{A}_n^{[0, k]}, q) = q[y^{q-n}](1 + y + y^2 + \dots + y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}$$

and

$$r(\mathcal{A}_n^{[0, k]}) = [y^{kn+1}](1 + y + y^2 + \dots + y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}.$$

In particular,

$$r(\mathcal{A}_n^{[0, 2]}) = \sum_{j=0}^{\lfloor n-1/2 \rfloor} \binom{n}{2j+1} (n-j)^{n-1}.$$

Proof. To count the placements of type α described in the proof of Theorem 4.3, we first choose a cyclic placement w of the integers from 1 to n . Then we distribute the $q - n$ unlabeled boxes in the n spaces between the integers, placing at least k boxes in each space with a descent. A descent of w is a pattern ji with $i < j$, i.e. a pair (i, j) with $i < j$, such that i immediately follows j clockwise in w . Let $d(w)$ be the number of descents of w . Given w , there are $\binom{q - kd(w) - 1}{n - 1}$ ways to distribute the $q - n$ boxes according to the above restriction. Hence

$$\tilde{\chi}(\mathcal{A}_n^{[0, k]}, q) = \sum_{w \in P_n} \binom{q - kd(w) - 1}{n - 1}, \tag{8}$$

where P_n stands for the set of $(n - 1)!$ cyclic placements of the elements of $[n]$. Note that the cyclic placements of $[n]$ with j descents correspond to

permutations of $[n-1]$ with $j-1$ descents. Indeed, we can remove the largest entry n of the placement and unfold to get a linear permutation with one less descent than before. Thus,

$$\begin{aligned}\tilde{\chi}(\mathcal{A}_n^{[0,k]}, q) &= \sum_{w \in S_{n-1}} \binom{q - kd(w) - k - 1}{n-1} = [y^{q-n}] \frac{\sum_{w \in S_{n-1}} y^{k+kd(w)}}{(1-y)^n} \\ &= [y^{q-n}] (1+y+y^2+\dots+y^{k-1})^n \frac{\sum_{w \in S_{n-1}} y^{k(1+d(w))}}{(1-y^k)^n}.\end{aligned}$$

The proposed formula for $\chi(\mathcal{A}_n^{[0,k]}, q)$ follows from the well known identity

$$\frac{\sum_{w \in S_{n-1}} \lambda^{1+d(w)}}{(1-\lambda)^n} = \sum_{j=0}^{\infty} j^{n-1} \lambda^j. \quad (9)$$

A proof and generalization of this identity is provided by Stanley's theory of P -partitions [23] (see also [26, Thm. 4.5.14]).

To obtain the formula for $r(\mathcal{A}_n^{[0,k]})$ we use Theorem 1.1 in the first summation formula for $\tilde{\chi}(\mathcal{A}_n^{[0,k]}, q)$ after (8). Here $a(w) = n-2-d(w)$ stands for the number of ascents of w .

$$\begin{aligned}r(\mathcal{A}_n^{[0,k]}, q) &= (-1)^{n-1} \sum_{w \in S_{n-1}} \binom{-kd(w) - k - 2}{n-1} = \sum_{w \in S_{n-1}} \binom{n+kd(w)+k}{n-1} \\ &= [y^{kn+1}] \frac{\sum_{w \in S_{n-1}} y^{k+ka(w)}}{(1-y)^n} \\ &= [y^{kn+1}] (1+y+y^2+\dots+y^{k-1})^n \sum_{j=0}^{\infty} j^{n-1} y^{jk}.\end{aligned}$$

The specialization for $k=2$ mentioned at the end of the theorem is an immediate consequence of the result above. ■

We are now able to prove Theorem 4.2.

Proof of Theorem 4.2. Theorems 4.3 and 4.4 for $k=2$ yield

$$\begin{aligned}\tilde{\chi}(\mathcal{L}_n, q) &= [y^q] (1+y)^n \sum_{j=0}^{\infty} j^{n-1} y^{2j} = \frac{1}{2^{n-1}} [y^q] (1+y)^n \sum_{j=0}^{\infty} (2j)^{n-1} y^{2j} \\ &= \frac{1}{2^{n-1}} \sum_{\substack{j=0 \\ j \equiv q \pmod{2}}}^n \binom{n}{j} (q-j)^{n-1} \\ &= \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (q-j)^{n-1},\end{aligned}$$

as desired. The last equality follows from the fact that the quantity

$$\sum_{j=0}^n (-1)^j \binom{n}{j} (q-j)^{n-1}$$

is identically zero, as an n th finite difference of a polynomial of degree $n-1$. ■

5. OTHER INTERESTING HYPERPLANE ARRANGEMENTS

In this section we give a few other examples of hyperplane arrangements, related to the ones we have discussed so far, on which Theorem 2.2 sheds light. We will focus mainly on deformations of \mathcal{A}_n , as defined in (4). For a more systematic approach, we refer the interested reader to [1, Ch. 7].

Fix a nonnegative integer k . We first consider the arrangement in \mathbb{R}^n with hyperplanes

$$x_i - x_j = 0, 1, \dots, k \quad \text{for all } i \neq j. \tag{10}$$

It was noted by Stanley [28, Sect. 2] that the number of regions of this arrangement can be shown to equal

$$\frac{n!}{kn+1} \binom{(k+1)n}{n}.$$

Stanley states this result in the context of generalized interval orders. See [28] for special cases that have appeared earlier in the literature. We remark below that a simple explicit product formula for the characteristic polynomial, which implies the above formula via Theorem 1.1, can be derived from Theorem 2.2. A stronger version of the next theorem was obtained by Edelman and Reiner (see Theorem 3.2 in [8]).

THEOREM 5.1. *The arrangement (10) has characteristic polynomial*

$$q \prod_{j=1}^{n-1} (q - kn - j).$$

In particular, the number of regions is $(kn+n)!/(kn+1)!$ and the number of bounded regions $(kn+n-2)!/(kn-1)!$.

Proof. Recall the discussion before Theorem 3.3. Except for the factor of q , the characteristic polynomial of (10), evaluated at a large prime q ,

counts the number of circular placements of the elements of $[n]$ and $q-n$ unlabeled boxes such that no two integers are separated by less than k objects, in either direction. To construct these placements, we first cyclically permute the integers in $[n]$ in $(n-1)!$ ways. Then we distribute the $q-n$ unlabeled boxes in the n spaces between the integers, placing at least k boxes in each space. This can be done in $\binom{q-kn-1}{n-1}$ ways. ■

More generally, let $l = \{l_1, l_2, \dots, l_m\}$ be a set of m positive integers satisfying $l_1 < l_2 < \dots < l_m$. We denote by $\mathcal{A}_n^{l \cup 0}$ the following deformation of \mathcal{A}_n :

$$x_i - x_j = 0, l_1, l_2, \dots, l_m \quad \text{for all } i \neq j. \quad (11)$$

If l is empty ($m=0$), this arrangement reduces to \mathcal{A}_n and if $l = \{1, 2, \dots, k\}$, we get the arrangement (10). The following theorem gives an expression for the characteristic polynomial and number of regions of $\mathcal{A}_n^{l \cup 0}$, under a certain assumption on l .

THEOREM 5.2. *Suppose that the set of positive integers not in l is closed under addition. Let $p_l(x) = \sum_{j=1}^m x^{l_j-1}$. Then, for all integers $q > nl_m$,*

$$\chi(\mathcal{A}_n^{l \cup 0}, q) = q(n-1)! [x^{q-n}] \left(\frac{1 + (x-1) p_l(x)}{1-x} \right)^n.$$

Moreover, let

$$(1 + (x-1) p_l(x))^n = (1-x)^n Q_n(x) + R_n(x),$$

where Q_n and R_n are polynomials with $\deg R_n < n$. Then

$$r(\mathcal{A}_n^{l \cup 0}) = (n-1)! [x^n] \frac{x^{n-1} R_n(1/x)}{(1-x)^n}$$

and

$$b(\mathcal{A}_n^{l \cup 0}) = (n-1)! [x^{n-2}] \frac{x^{n-1} R_n(1/x)}{(1-x)^n}.$$

Proof. We follow the reasoning in the proof of Theorem 5.1. We first cyclically permute the integers from 1 to n in $(n-1)!$ ways. We want to insert $q-n$ unlabeled boxes between them, so that no two integers are separated by l_j-1 objects, where $1 \leq j \leq m$. Because of the assumption

on l , it suffices not to insert $l_j - 1$ boxes between any two consecutive integers. Hence by Theorem 2.2, for large primes q we have

$$\tilde{\chi}(\mathcal{A}_n^{l \cup 0}, q) = (n - 1)! [x^{q-n}] \left(\sum_k x^k \right)^n,$$

where, in the sum, k ranges over all nonnegative integers different from $l_j - 1$ for $1 \leq j \leq m$. This is equivalent to the proposed formula for the characteristic polynomial. The result holds for all $q > nl_m$ since the coefficient of x^k in the rational function on the right is a polynomial in k , say $P(k)$, for $k > nl_m - n$ (see [26, Prop. 4.2.2, Cor. 4.3.1]).

To obtain the value of $\chi(\mathcal{A}_n^{l \cup 0}, q)$ at -1 , we need to evaluate $P(k)$ at $k = -n - 1$. By construction,

$$\sum_{k=0}^{\infty} P(k) x^k = \frac{R_n(x)}{(1-x)^n}.$$

Proposition 4.2.3 in [26] implies that

$$\sum_{k=0}^{\infty} P(-k) x^k = -\frac{R_n(1/x)}{(1-1/x)^n}.$$

Since

$$r(\mathcal{A}_n^{l \cup 0}) = (-1)^n \chi(\mathcal{A}_n^{l \cup 0}, -1) = (-1)^{n+1} (n - 1)! P(-n - 1),$$

the result for the number of regions follows. Similarly we get the formula for the number of bounded regions by evaluating $P(k)$ at $k = -n + 1$. ■

If $l = (1, 2, \dots, k)$ then the condition in Theorem 5.2 is trivially satisfied, $1 + (x - 1) p_l(x) = x^k$ and one can easily deduce the formula for the number of regions mentioned earlier directly from Theorem 5.2.

For general l , let \mathcal{A}_n^l be the arrangement obtained from $\mathcal{A}_n^{l \cup 0}$ by dropping the hyperplanes $x_i - x_j = 0$. The characteristic polynomials of \mathcal{A}_n^l were related to those of $\mathcal{A}_n^{l \cup 0}$ by Postnikov and Stanley via an exponential generating function identity. The following theorem is equivalent (at least for integer l_j 's) to Theorem 2.3 in [28] (see also [28, Thm. 1.2]), which is proved in [18]. We remark here that this theorem is quite easy to derive, once the characteristic polynomials are interpreted combinatorially as in the proof of Theorems 5.1 and 5.2.

THEOREM 5.3. *Let*

$$F_l(q, t) = \sum_{n=0}^{\infty} \chi(\mathcal{A}_n^l, q) \frac{t^n}{n!}$$

and

$$F_l^0(q, t) = \sum_{n=0}^{\infty} \chi(\mathcal{A}_n^{l \cup 0}, q) \frac{t^n}{n!}.$$

Then,

$$F_l(q, t) = F_l^0(q, e^t - 1). \quad (12)$$

Proof. Using the combinatorial interpretation of the characteristic polynomials, $\chi(\mathcal{A}_n^l, q)$ counts the number of ways to partition $[n]$ into blocks and place a structure on the set of blocks. If the number of blocks of the partition is k , then the number of possible structures is counted by $\chi(\mathcal{A}_k^{l \cup 0}, q)$. Thus, the result follows from standard properties of exponential generating functions [29, Ch. 5, Sect. 1]. ■

The reasoning in the previous proof can be applied to more general situations. We give an example below. For a more detailed discussion see [1, Ch. 7].

THEOREM 5.4. *Consider the following arrangement in \mathbb{R}^n , which we denote by \mathcal{P}_n :*

$$x_i + x_j = 0, 1 \quad \text{for all } 1 \leq i \leq j \leq n.$$

It has characteristic polynomial

$$\chi(\mathcal{P}_n, q) = \sum_{k=1}^n S(n, k)(q - 2k)(q - 2k + 1) \cdots (q - k - 1),$$

where $S(n, k)$ denotes a Stirling number of the second kind. In particular,

$$\sum_{n=0}^{\infty} r(\mathcal{P}_n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{2n+1}{n} (1 - e^{-t})^n.$$

Proof. The argument in the proof of Theorem 5.3 remains valid if \mathcal{A}_n^l is replaced by \mathcal{P}_n and $\mathcal{A}_n^{l \cup 0}$ is replaced by

$$\begin{aligned} x_i - x_j &= 0 && \text{for all } 1 \leq i < j \leq n, \\ x_i + x_j &= 0, 1 && \text{for all } 1 \leq i \leq j \leq n. \end{aligned}$$

An argument similar to the one given in the proof of Theorem 3.14 shows that this arrangement also has characteristic polynomial

$$\prod_{j=n+1}^{2n} (q - j).$$

The result follows easily from the new version of (12). ■

We now generalize the result of Theorem 5.1 for the arrangement (10) to other root systems. Let Φ be an irreducible crystallographic root system spanning \mathbb{R}^l with corresponding Weyl group W . Let $l = \{l_1, l_2, \dots, l_m\}$ be a set of m nonnegative integers. We denote by \mathcal{W}^l the collection of hyperplanes

$$\{H_{\alpha, k} \mid \alpha \in \Phi \text{ and } k = l_1, l_2, \dots, l_m\},$$

where $H_{\alpha, k}$ has the same meaning as in Section 3. Thus, if $\Phi = A_{n-1}$ and $l = [0, k]$, \mathcal{W}^l is the arrangement (10) restricted in an $(n-1)$ -dimensional Euclidean space. This creates some slight ambiguity with the notation \mathcal{A}_n^l , as happens with the Shi arrangement \mathcal{A}_n .

If $\Phi = B_n, C_n$ or D_n , we denote \mathcal{W}^l by $\mathcal{B}_n^l, \mathcal{C}_n^l$ and \mathcal{D}_n^l respectively. Thus, $\mathcal{D}_n^{[0, k]}$ has hyperplanes

$$x_i - x_j = 0, \pm 1, \dots, \pm k \quad \text{for } 1 \leq i < j \leq n,$$

$$x_i + x_j = 0, \pm 1, \dots, \pm k \quad \text{for } 1 \leq i < j \leq n.$$

The arrangements $\mathcal{B}_n^{[0, k]}, \mathcal{C}_n^{[0, k]}$ can be obtained from $\mathcal{D}_n^{[0, k]}$ by adding the hyperplanes $x_i = 0, \pm 1, \dots, \pm k$ and $2x_i = 0, \pm 1, \dots, \pm k$ for $1 \leq i \leq n$, respectively.

Theorem 5.1 gives the characteristic polynomial of \mathcal{W}^l for $\Phi = A_{n-1}, l = [0, k]$ as

$$(q - kn - 1)(q - kn - 2) \cdots (q - (k + 1)n + 1).$$

This product is equal to $\tilde{\chi}(\mathcal{A}_n, q - kn)$. The following theorem is quite easy to derive case by case, using arguments analogous to the one in the proof of Theorem 5.1 and Theorem 3.13. A detailed proof and a generalization appears in [1, Ch. 7]. A stronger statement appears as Conjecture 3.3 in [8]. As with Headley's result (Theorem 3.1), it would be interesting to find a case-free, simple proof, based on Theorem 2.2.

THEOREM 5.5. *Suppose Φ is a root system of type A, B, C or D and that h is the Coxeter number of the associated Weyl group W . Let $\chi(\mathcal{W}, q)$ be the characteristic polynomial of the corresponding Coxeter arrangement \mathcal{W} . Then*

$$\chi(\mathcal{W}^{[0, k]}, q) = \chi(\mathcal{W}, q - kh).$$

Lastly, we consider the arrangement $\mathcal{A}_{n, S}$, where S is the path $\{12, 23, \dots, n-1n\}$. The hyperplanes of $\mathcal{A}_{n, S}$ are

$$x_i - x_j = 0 \quad \text{for } 1 \leq i < j \leq n,$$

$$x_i - x_{i+1} = 1 \quad \text{for } 1 \leq i \leq n - 1.$$

THEOREM 5.6. For S as above we have

$$\chi(\hat{\mathcal{A}}_{n,S}, q) = q \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} (q-k)(q-k-1) \cdots (q-n+1). \quad (13)$$

In particular,

$$r(\hat{\mathcal{A}}_{n,S}) = \sum_{k=1}^n \frac{n!}{k!} \binom{n-1}{k-1}$$

is the number of ways to partition a set with n elements and linearly order each block.

Proof. We use Theorem 2.2 and the inclusion–exclusion principle. We will first count the number of n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ satisfying $x_i - x_j \neq 0$ for $1 \leq i < j \leq n$ and a given set of k of the conditions $x_i - x_{i+1} = 1$ for $1 \leq i \leq n-1$, where $0 \leq k \leq n-1$.

In terms of circular placements, imposing the condition $x_i - x_{i+1} = 1$ means that i has to be preceded by $i+1$. Imposing k of these conditions splits $[n]$ into blocks of the form $j, j-1, \dots, i$, whose entries have to appear in order, with no boxes in between. Clearly, the number of these blocks is $n-k$. There are $(n-k-1)!$ ways to cyclically permute these blocks, $\binom{q-k-1}{n-k-1}$ ways to place $q-n$ unlabeled boxes in the $n-k$ spaces between the blocks and q ways to choose the box with label the zero class of \mathbb{F}_q . This gives a total of

$$q(q-k-1)(q-k-2) \cdots (q-n+1)$$

ways and (13) follows by inclusion–exclusion, once k is replaced by $k-1$.

Finally, it is well known and easy to see that the expression for the number of regions, resulting from (13) and Zaslavsky's Theorem, has the suggested combinatorial interpretation. ■

Some variations of Theorem 5.6 appear in [1, Ch. 7].

6. THE WHITNEY POLYNOMIAL

In this section we discuss a two variable generalization of the characteristic polynomial of a subspace arrangement (cf. Zaslavsky's Möbius polynomial). We give an interpretation of this polynomial for rational subspace arrangements, thus generalizing Theorem 2.2. For a hyperplane arrangement \mathcal{A} , a specialization of this polynomial gives the face numbers of \mathcal{A} , or equivalently the f -polynomial of a certain polyhedral complex associated to the arrangement.

Let \mathcal{A} be any subspace arrangement in \mathbb{R}^n . For $x \in L_{\mathcal{A}}$, we denote by \mathcal{A}^x the arrangement whose elements are the proper subspaces of x obtained by intersecting the subspaces of \mathcal{A} with x . Thus, the ambient Euclidean space for \mathcal{A}^x is the space x . The intersection semilattice of \mathcal{A}^x is the dual order ideal of $L_{\mathcal{A}}$ corresponding to x . Theorem 2.2 can be stated more generally for \mathcal{A}^x as follows.

COROLLARY 6.1. *If \mathcal{A} is defined over the integers, $x \in L_{\mathcal{A}}$ and q is a large enough prime, then*

$$\chi(\mathcal{A}^x, q) = \# \left(\mathbb{F}_q^n \cap x - \bigcup \mathcal{A}^x \right).$$

Proof. With the notation in the proof of Theorem 2.2,

$$\begin{aligned} \# \left(\mathbb{F}_q^n \cap x - \bigcup \mathcal{A}^x \right) &= g(x) = \sum_{z \geq x} \mu(x, z) f(z) \\ &= \sum_{z \in L_{\mathcal{A}^x}} \mu(x, z) q^{\dim z} = \chi(\mathcal{A}^x, q). \quad \blacksquare \end{aligned}$$

The generalization of the characteristic polynomial we are concerned with is the *Whitney polynomial*, denoted by $w(\mathcal{A}, t, q)$. This polynomial was defined by Zaslavsky in [33, Sect. 1] for hyperplane arrangements and called the Möbius polynomial. It was further investigated by the same author in [35, Sect. 2] for hyperplane arrangements defined by signed graphs, i.e. arrangements contained in \mathcal{B}_n , and named the Whitney polynomial of the signed graph. We give the definition for an arbitrary subspace arrangement \mathcal{A} .

DEFINITION 6.2. The Whitney polynomial of \mathcal{A} is the two variable polynomial

$$\begin{aligned} w(\mathcal{A}, t, q) &= \sum_{x \leq L_{\mathcal{A}}} \mu(x, z) t^{n - \dim x} q^{\dim z} \\ &= \sum_{x \in L_{\mathcal{A}}} t^{n - \dim x} \chi(\mathcal{A}^x, q). \end{aligned}$$

Since $\dim x = n$ if and only if $x = \hat{0} = \mathbb{R}^n$, the Whitney polynomial $w(\mathcal{A}, t, q)$ specializes to the characteristic polynomial $\chi(\mathcal{A}, q)$ for $t = 0$. In the case that \mathcal{A} is the central hyperplane arrangement corresponding to a signed graph Σ on n vertices, Zaslavsky [35, Sect. 2] interpreted the Whitney polynomial as the generating function of all colorings of Σ ,

classified by the rank of the set of “impropriety.” The following theorem extends Zaslavsky’s observation to rational subspace arrangements. For $t=0$ it reduces to Theorem 2.2.

THEOREM 6.3. *Suppose that \mathcal{A} is a subspace arrangement defined over the integers. For any point p , we denote by x_p the intersection of all elements of \mathcal{A} which contain p . If q is a large enough prime, then*

$$w(\mathcal{A}, t, q) = \sum_{p \in \mathbb{F}_q^n} t^{n - \dim x_p}.$$

Proof. Given any $x \in L_{\mathcal{A}}$, we have $x_p = x$ if and only if p lies in x but in no further intersection $y > x$. Thus by Corollary 6.1, the number of $p \in \mathbb{F}_q^n$ which satisfy $x_p = x$ is $\chi(\mathcal{A}^x, q)$. Hence,

$$\begin{aligned} \sum_{p \in \mathbb{F}_q^n} t^{n - \dim x_p} &= \sum_{x \in L_{\mathcal{A}}} \sum_{\substack{p \in \mathbb{F}_q^n \\ x_p = x}} t^{n - \dim x} \\ &= \sum_{x \in L_{\mathcal{A}}} t^{n - \dim x} \chi(\mathcal{A}^x, q) = w(\mathcal{A}, t, q). \quad \blacksquare \end{aligned}$$

We now describe an interesting application of the previous theorem to the computation of the face numbers of a rational hyperplane arrangement. We begin with a few definitions.

Suppose \mathcal{A} is a hyperplane arrangement in \mathbb{R}^n . The arrangement \mathcal{A} defines a cellular decomposition of \mathbb{R}^n . The cells are the regions of the arrangements \mathcal{A}^x , where x ranges over the elements of $L_{\mathcal{A}}$. Let $0 \leq k \leq n$ be an integer. The cells of dimension k , which are the regions of \mathcal{A}^x corresponding to all $x \in L_{\mathcal{A}}$ of dimension k , were called by Zaslavsky the *faces of dimension k* of the arrangement \mathcal{A} . Following [33, Sect. 2], we denote by $f_k(\mathcal{A})$ the number of k -dimensional faces of \mathcal{A} . Thus, $f_n(\mathcal{A}) = r(\mathcal{A})$. By Theorem 1.1, the number of k -dimensional faces of \mathcal{A} is given by

$$f_k(\mathcal{A}) = \sum_{\dim x = k} (-1)^k \chi(\mathcal{A}^x, -1). \quad (14)$$

It is also common in the literature to consider the f -vector of the dual complex of \mathcal{A} , instead of its face numbers. Suppose first that \mathcal{A} is central, with hyperplanes $a_i \cdot x = 0$ for $1 \leq i \leq N$. The *dual complex* of \mathcal{A} is the zonotope

$$Z[\mathcal{A}] = \sum_{i=1}^N S_i,$$

where $S_i = \text{conv}\{\pm a_i\}$ is the segment joining a_i and $-a_i$. For an exposition of the theory of zonotopes see [14]. The dual complex was considered by Zaslavsky in [33, Sect. 6] and also in [37] for the arrangements that correspond to a signed graph Σ . The vertices of the zonotope $Z[\mathcal{A}]$ correspond to the components of the complement $\mathbb{R}^n - \bigcup \mathcal{A}$. In general, the $(n - k)$ -dimensional faces of $Z[\mathcal{A}]$ correspond to the k -dimensional faces of \mathcal{A} .

If \mathcal{A} is any hyperplane arrangement in \mathbb{R}^n , the dual complex $Z[\mathcal{A}]$ of \mathcal{A} is a zonotopal complex, i.e. a polyhedral complex all of whose facets are zonotopes. Each facet of $Z[\mathcal{A}]$ is the zonotope corresponding to a maximal centered arrangement contained in \mathcal{A} . The correspondence of the faces of $Z[\mathcal{A}]$ and \mathcal{A} described above carries through. In fact it is inclusion reversing, so that the face poset of the $Z[\mathcal{A}]$ is the dual of the poset of closures of the faces of \mathcal{A} , ordered by inclusion. The number $f_{n-k}(Z[\mathcal{A}])$ of $(n - k)$ -dimensional faces of $Z[\mathcal{A}]$ satisfies

$$f_{n-k}(Z[\mathcal{A}]) = f_k(\mathcal{A}) \tag{15}$$

and hence is given by (14).

The following theorem is due to Zaslavsky [33, Sect. 2, Corollary 6.3]. It is an immediate consequence of (14), (15) and Definition 6.2.

THEOREM 6.4. *Let \mathcal{A} be any hyperplane arrangement in \mathbb{R}^n and let $f_i(Z[\mathcal{A}])$ denote the number of i -dimensional faces of $Z[\mathcal{A}]$. Then the f -polynomial of $Z[\mathcal{A}]$ satisfies*

$$\sum_{i=0}^n f_i(Z[\mathcal{A}]) t^i = (-1)^n w(\mathcal{A}, -t, -1).$$

Thus, Theorem 6.3 gives a way to compute the f -polynomial of $Z[\mathcal{A}]$ when \mathcal{A} is rational, extending Zaslavsky’s method [35, Sect. 2, Cor. 4.1’] which applies to hyperplane arrangements defined by signed graphs.

As an application, we compute the f -vector of $Z[\hat{\mathcal{A}}_n]$, the dual complex of the Shi arrangement of type A_{n-1} . For a generalization and more computations see [1, Ch. 8]. Stanley used Zaslavsky’s interpretation [35, Sect. 2] to compute the f -vector of a zonotope related to graphical degree sequences [25, Thm. 4.2].

THEOREM 6.5. *The coefficients in t of the Whitney polynomial of $\hat{\mathcal{A}}_n$ are given by*

$$[t^k] w(\hat{\mathcal{A}}_n, t, q) = \binom{n}{k} q \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (q - n + i)^{n-1}$$

for $0 \leq k \leq n$. In particular, the f -vector (f_0, f_1, \dots, f_n) of $Z[\hat{\mathcal{A}}_n]$ is given by

$$f_k = \binom{n}{k} \sum_{i=0}^k (-1)^i \binom{k}{i} (n-i+1)^{n-1}.$$

In other words,

$$f_k = \binom{n}{k} \# \{f: [n-1] \rightarrow [n+1] \mid [k] \subseteq \text{Im } f\}$$

for $0 \leq k \leq n-1$ and $f_n = 0$.

Proof. We compute

$$[t^k] w(\hat{\mathcal{A}}_n, t, q)$$

for a large prime q , using the interpretation of Theorem 6.3. As in Section 3, we think of a point $p = (x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ as a placement of the elements of $[n]$ into q boxes arranged and labeled cyclically with the classes mod q . The integer i is placed in the box labeled with the class x_i . Consider the following relation on $[n]$: i and j are related if, say, $i < j$ and x_i, x_j satisfy one of the defining equations

$$x_i - x_j = 0, 1$$

in \mathbb{F}_q . This means that either i and j are placed in the same box or that they form a descent, i.e. i is placed in the box labeled with $x_j + 1$, where $i < j$ and x_j is the label of the box that j occupies. Call an equivalence class of the transitive closure of this relation a *block* of the placement. Then $\dim x_p$ is the number of blocks and we want to count all such placements with $n-k$ blocks.

We linearly order the elements in each occupied box to make them strictly increasing, clockwise. This defines a cyclic placement of the integers from 1 to n . To count the placements with $n-k$ blocks, we start with any cyclic placement w of the integers from 1 to n , insert $n-k$ bars in the spaces between them without repetitions to form the $n-k$ blocks and then insert empty boxes in the places where the bars were inserted to construct the placement. In the end, any string of consecutive integers in increasing order with no bars (or boxes) in between forms an occupied box.

Let $d(w)$ be the number of descents of w , as defined in the proof of Theorem 4.4. There are $\binom{n}{k}$ ways to insert the bars. Suppose that r of the bars are inserted in places where w has a descent, called descent cuts and the rest $s = n-k-r$ in places with ascents, called ascent cuts. The boxes that we have defined so far, that is strings of consecutive integers in

increasing order with no bars in between, are $d(w) + s$. Thus we need to insert $q - d(w) - s$ more empty boxes. We can insert any number of boxes in an ascent cut, but at least one in every descent cut. Hence, the number of ways to distribute the $q - d(w) - s$ boxes is $\binom{q-d(w)-1}{n-k-1}$ and

$$\begin{aligned} [t^k] w(\mathcal{A}_n, t, q) &= \binom{n}{k} q \sum_{w \in P_n} \binom{q-d(w)-1}{n-k-1} \\ &= \binom{n}{k} q \sum_{w \in P_n} [y^{q-n+k}] y^{d(w)} (1-y)^{-(n-k)}, \end{aligned}$$

where, as in the proof of Theorem 4.4, P_n stands for the set of $(n-1)!$ cyclic placements of the elements of $[n]$ and q accounts for the number of ways to decide where the zero class mod q will be. We switch cyclic placements of $[n]$ to permutations of $[n-1]$, as in the proof of Theorem 4.4. Now $d(w)$ stands for the number of descents of the permutation w . Using the identity (9) once more, we get

$$\begin{aligned} [t^k] w(\mathcal{A}_n, t, q) &= \binom{n}{k} q [y^{q-n+k}] \frac{\sum_{w \in S_{n-1}} y^{1+d(w)} (1-y)^k}{(1-y)^n} \\ &= \binom{n}{k} q [y^{q-n+k}] (1-y)^k \sum_{j=0}^{\infty} j^{n-1} y^j \end{aligned}$$

and the proposed formula follows.

The result about the f -vector follows from Theorem 6.4 and the formula obtained for the coefficients of the Whitney polynomial by setting $q = -1$. The combinatorial interpretation of f_k given in the end follows by inclusion-exclusion. ■

The formula for f_0 obtained agrees, once more, with the result of Theorem 3.3.

7. FURTHER DIRECTIONS

Free hyperplane arrangements were introduced by Terao in [30]. The basic result of Terao [31] (also [16, Thm. 4.137]) about free arrangements implies that their characteristic polynomials factor completely over the nonnegative integers. The roots are the generalized “exponents” of the arrangement. We have already noted that most of the hyperplane arrangements considered in Sections 3 and 5 have characteristic polynomials which factor completely over the nonnegative integers. One of the natural questions that the present work raises is the question of freeness for the centralizations of these arrangements. This question seems to be interesting in view

of the algebraic structure associated to a free hyperplane arrangement [16, Ch. 4; 20, Sect. 3; 31, Sect. 2]. The centralization of \mathcal{A} is obtained by homogenizing each hyperplane (1) of \mathcal{A} to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = dx_{n+1}$$

and adding the hyperplane $x_{n+1} = 0$. This operation makes the arrangement central and multiplies the characteristic polynomial by $q - 1$. Some results and conjectures in this direction have already appeared in [8].

The question of direct combinatorial proofs of our results for the number of regions and the number of bounded regions of the arrangements we have considered also arises naturally. For the number of regions of the Shi arrangement $\hat{\mathcal{A}}_n$, such a proof can be obtained by combining a bijection due to Kreweras and one due to Pak and Stanley (see the discussion in [28, Sect. 5]). Combinatorial proofs of Theorem 6.5 and related results would also be desirable.

We would also like to use Theorem 2.2 to study specific examples or classes of subspace (as opposed to hyperplane) arrangements.

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