COMBINATORIAL THEORY

Fall Semester 2022
Problem Set #1

1. For which formal power series $G(x) \in \mathbb{C}[[x]]$ does there exist a formal power series $F(x) \in \mathbb{C}[[x]]$ such that $(F(x))^3 = G(x)$? You may use the fact that the equation $z^3 = a$ has a solution in $\mathbb{C}$ for every $a \in \mathbb{C}$.

2. Consider the formal power series $F(x) = \sum_{k \geq 0} (x + x^2 - x^3)^k \in \mathbb{C}[[x]]$.

(a) Compute $F(x)$ as a rational function of x.

(b) Show that the coefficients of $F(x)$ are positive integers.

(c) Compute the coefficient of x^n in $(F(x))^2$ for every $n \in \mathbb{N}$.

3. Let m be a positive integer and let $c(n, m)$ denote the number of compositions of n whose parts are odd integers $\leq 2m - 1$. Prove that

$$\sum_{n \geq 0} c(n, m) x^n \; = \; \frac{1 - x^2}{1 - x - x^2 + x^{2m+1}},$$

where $c(0, m) = 1$ by convention.

4. Prove that for every positive integer n, the number of partitions λ of n no part of which appears with multiplicity one in λ is equal to the number of partitions of n with parts not congruent to $\pm 1 \pmod 6$.

5. Consider the product

$$\mathcal{A}_n \; = \; \{1\} \times \{1, 2\} \times \cdots \times \{1, 2, \ldots, n\}$$

and for $\sigma = (a_1, a_2, \ldots, a_n) \in \mathcal{A}_n$ define the set of descents $\mathrm{Des}(\sigma) := \{i \in [n-1] : a_i \geq a_{i+1}\}$ of σ.

(a) Prove that the number of sequences $\sigma \in \mathcal{A}_n$ having descent set $\mathrm{Des}(\sigma) = S$ is equal to the number of permutations $w \in \mathfrak{S}_n$ having descent set $\mathrm{Des}(w) = S$ for every $S \subseteq [n-1]$.

(b) Deduce that

$$A_n(x) \; = \; \sum_{\sigma \in \mathcal{A}_n} x^{\mathrm{des}(\sigma)}$$

for every $n \in \mathbb{Z}_{>0}$, where $\mathrm{des}(\sigma)$ stands for the number of descents of $\sigma \in \mathcal{A}_n$ and $A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$.

Deadline: Sunday 23 October