Random points in isotropic unconditional convex bodies

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Abstract

We study three questions about independent random points uniformly distributed in isotropic symmetric convex bodies $K, T_1, \dots, T_s$: (a) Let $\varepsilon \in (0,1)$ and let $x_1, \dots, x_N$ be chosen from K. Is it true that if $N \geq C(\varepsilon) n \log n$ then

$$\Big\| I - \frac{1}{NL_K^2} \sum_{i=1}^N x_i \otimes x_i \Big\| < \varepsilon$$

with probability greater than $1-\varepsilon$? (b) Let x_i be chosen from T_i. Is it true that the unconditional norm

$$\|\mathbf{t}\| = \int_{T_1} \cdots \int_{T_s} \Big\| \sum_{i=1}^s t_i x_i \Big\|_K \, dx_s \cdots dx_1$$

is well-comparable to the Euclidean norm in $\mathbb{R}^s$? (c) Let $x_1, \dots, x_N$ be chosen from K. Let $\mathbb{E}\,(K,N) := \mathbb{E}\,|\mathrm{conv}\{x_1, \dots, x_N\}|^{1/n}$ be the expected volume radius of their convex hull. Is it true that $\mathbb{E}\,(K,N) \simeq \mathbb{E}\,(B(n),N)$ for all N, where $B(n)$ is the Euclidean ball of volume 1?

We prove that the answer to these questions is affirmative if we restrict ourselves to the class of unconditional convex bodies. Our main tools come from recent work of Bobkov and Nazarov. Some observations about the general case are also included.

1 Introduction

In this article we study three problems about random points in isotropic convex bodies. Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and there is a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \tag{1.1}$$

for every θ in the unit sphere S^{n-1}. It is not hard to see that for every convex body K in $\mathbb{R}^n$ with center of mass at the origin, there exists $S \in GL(n)$ such that $S(K)$ is isotropic. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, one may define the isotropic constant L_K as an invariant of the linear class of K.

We consider the following questions about independent random points which are uniformly distributed in convex bodies.

I Approximation of the identity operator. The isotropic condition (1.1) may be equivalently written in the form

$$I = \frac{1}{L_K^2}\int_K x\otimes x dx, \tag{1.2}$$

where I is the identity operator. Let $\varepsilon \in (0,1)$ and consider N independent random points $x_1,\dots,x_N$ uniformly distributed in K. The question is to find N_0, as small as possible, for which the following holds true: if $N\geq N_0$ then with probability greater than $1-\varepsilon$ one has

$$(1-\varepsilon)L_K^2 \leq \frac{1}{N}\sum_{i=1}^N \langle x_i,\theta\rangle^2 \leq (1+\varepsilon)L_K^2 \tag{1.3}$$

for every $\theta\in S^{n-1}$. This question has its origin in the problem of finding a fast algorithm for the computation of the volume of a given convex body. Kannan, Lovász and Simonovits (see [20]) proved that one can take $N_0 = C(\varepsilon)n^2$ for some constant $C(\varepsilon)>0$ depending only on ε. This was later improved to $N_0\simeq C(\varepsilon)n(\log n)^3$ by Bourgain [10] and to $N_0\simeq C(\varepsilon)n(\log n)^2$ by Rudelson [26]. One can actually check (see [15]) that this last estimate can be recovered if we incorporate a result of Alesker [1] into Bourgain's argument. It is quite probable that the best estimate for N_0 is $C(\varepsilon)n\log n$. We prove this in the unconditional case, and we show the connection of the general problem to some recent conjectures about the central limit properties of isotropic convex bodies.

II A multi-integral norm. Let K and T_i $(i=1,\dots,s)$ be symmetric convex bodies in $\mathbb{R}^n$ with $|K|=|T_1|=\cdots=|T_s|=1$. The unconditional norm

$$\|\mathbf{t}\| = \int_{T_1}\cdots\int_{T_s}\left\|\sum_{i=1}^s t_ix_i\right\|_K dx_s\cdots dx_1 \tag{1.4}$$

on $\mathbb{R}^s$ was studied in [11], where it was proved that, in the case $s=n$,

$$\|\mathbf{t}\| \geq c\sqrt{n}\left(\prod_{i=1}^n |t_i|\right)^{1/n} \tag{1.5}$$

for every $\mathbf{t}=(t_1,\dots,t_n)\in\mathbb{R}^n$. It was conjectured that if $K=T_1=\cdots=T_s$ and if $\|\cdot\|_K$ satisfies some cotype condition, then $\|\cdot\|$ is equivalent to the ℓ_2^s norm. Recently, Gluskin and Milman (see [18] and [19]) showed that the lower bound holds true in full generality (the proof uses the rearrangement inequality of Brascamp, Lieb and Luttinger [12]): There exists a sequence $c(n)$ of positive reals with $c(n) \to 1/\sqrt{2}$ such that: if K is a star body with $0 \in \mathrm{int}(K)$ and if T_i $(i = 1, \ldots, s)$ are measurable sets in $\mathbb{R}^n$ with $|K| = |T_1| = \cdots = |T_s| = 1$, then

$$\|\mathbf{t}\| := \int_{T_1} \cdots \int_{T_s} p_K\left(\sum_{i=1}^{s} t_i x_i\right)\, dx_s \cdots dx_1 \geq c(n)\left(\sum_{i=1}^{s} t_i^2\right)^{1/2} \tag{1.6}$$

for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where p_K is the Minkowski functional of K.

It should be noted that, in the case $K = T_1 = \cdots = T_s$, Theorem 1.4 from [11] establishes a second lower bound for the norm (1.4), which involves L_K. Therefore, upper bounds for this norm may give upper bounds for the isotropic constant.

III Volume radius of a random polytope. Let K be a convex body in $\mathbb{R}^n$ with volume 1. We fix $N \geq n+1$ and consider N independent random points $x_1, \ldots, x_N$ uniformly distributed in K. Let $\mathrm{conv}(x_1, \ldots, x_N)$ be their convex hull. The question is to estimate the expected volume radius

$$\mathbb{E}\,(K, N) = \int_K \cdots \int_K |\mathrm{conv}(x_1, \ldots, x_N)|^{1/n} dx_N \cdots dx_1 \tag{1.7}$$

of this random polytope. Observe that $\mathbb{E}\,(K, N)$ is invariant under volume preserving affine transformations, so we may also assume that K has its center of mass at the origin. When $N = n+1$, this quantity is an exact function of the isotropic constant of K. To see this, one can use the identity (see [21])

$$L_K^{2n} = n! S_2^2(K), \tag{1.8}$$

where

$$S_2^2(K) := \int_K \ldots \int_K |\mathrm{conv}(0, x_1, \ldots, x_n)|^2 dx_n \ldots dx_1. \tag{1.9}$$

Combining this fact with Khintchine type inequalities for linear functionals on convex bodies (see [21]) one can show that

$$\mathbb{E}\,(K, n+1) \simeq \frac{L_K}{\sqrt{n}}. \tag{1.10}$$

In [16] it was proved that for every isotropic convex body K in $\mathbb{R}^n$ and every $N \geq n+1$,

$$\mathbb{E}\,(B(n), N) \leq \mathbb{E}\,(K, N) \leq cL_K \frac{\log(2N/n)}{\sqrt{n}}, \tag{1.11}$$

where $B(n)$ is a ball of volume 1. Moreover, it was shown that if $N \geq cn(\log n)^2$ then $\mathbb{E}\,(B(n), N) \geq c(\log(N/n)/n)^{1/2}$. A strong conjecture is that

$$\mathbb{E}\,(K, N) \simeq \min\left\{\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, 1\right\} \tag{1.12}$$

for every convex body K of volume 1 in $\mathbb{R}^n$ and every $N \geq n+1$. We prove this in the unconditional case, and we show the connection of the general problem to the "ψ_2-behaviour" of linear functionals on isotropic convex bodies.

IV Results. Consider the class of symmetric convex bodies which generate a norm with unconditional basis. After a linear transformation, we may assume that the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ is an unconditional basis for $\|\cdot\|_K$. That is, for every choice of real numbers $t_1, \ldots, t_n$ and every choice of signs $\varepsilon_i = \pm 1$,

$$\left\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\right\|_K = \left\|t_1 e_1 + \cdots + t_n e_n\right\|_K. \tag{1.13}$$

We will prove the following three facts:

Theorem A *Let $\varepsilon \in (0,1)$ and let $\rho > 2$. Assume that $n \geq n_0(\rho)$ and let K be an isotropic unconditional convex body in $\mathbb{R}^n$. If $N \geq c\varepsilon^{-\rho} n \log n$, where $c > 0$ is an absolute constant, and if $x_1, \ldots, x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have*

$$(1-\varepsilon)L_K^2 \leq \frac{1}{N}\sum_{i=1}^{N}\langle x_i, \theta\rangle^2 \leq (1+\varepsilon)L_K^2, \tag{1.14}$$

for every $\theta \in S^{n-1}$.

The proof of Theorem A is based on the following observation: if K is an isotropic unconditional convex body in $\mathbb{R}^n$ and if N is polynomial in n, then $\mathbb{E}\max_{i\leq N}\|x_i\|_2^{2q} \leq (cn)^q$ for large values of q. This follows from a strong dimension-dependent concentration estimate of Bobkov and Nazarov (Theorem 2.2) and suggests that the general problem is related to some recent conjectures about the central limit properties of isotropic convex bodies. In Section 3 we provide some evidence for a general affirmative answer. Let the parameter σ_K be defined by $\sigma_K^2 = \mathrm{Var}(\|x\|_2^2)/(nL_K^4)$. Then, the following statement holds true: Let $\rho > 2$ and $\varepsilon \in (0,1)$, and assume that $n \geq n_0(\rho)$. For every isotropic convex body K in $\mathbb{R}^n$ and every $N \geq c\varepsilon^{-\rho}(\sigma_K+1)^2 n\log n$, where $c > 0$ is an absolute constant, if $x_1, \ldots, x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have (1.14) for every $\theta \in S^{n-1}$. It is conjectured (see [7]) that there exists an absolute constant $C > 0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K. If this is true then, for every K, we have ε-approximation of the identity operator with $N \simeq C(\varepsilon) n \log n$.

Theorem B *There exists an absolute constant $C > 0$ with the following property: if K and T_i $(i = 1, \ldots, s)$ are isotropic convex bodies in $\mathbb{R}^n$ which satisfy* (1.13)*, then*

$$\|\mathbf{t}\| := \int_{T_1}\cdots\int_{T_s}\Big\|\sum_{i=1}^{s} t_i x_i\Big\|_K \, dx_s \cdots dx_1 \leq C\sqrt{\log n}\max\left\{\|\mathbf{t}\|_2, \sqrt{\log n}\|\mathbf{t}\|_\infty\right\} \tag{1.15}$$

for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$.

The proof of Theorem B is based on a comparison theorem of Bobkov and Nazarov (Theorem 2.4) which asserts that the integral of a symmetric and coordinatewise increasing absolutely continuous function over an isotropic unconditional convex body K in $\mathbb{R}^n$ is (roughly speaking) maximal when K is the normalized ℓ_1^n–ball. The estimate on the right hand side of (1.15) is sharp: we give examples showing that the terms $\sqrt{\log n}\|\cdot\|_2$ and $\log n\|\cdot\|_\infty$ are both needed. The situation is less clear in the very interesting special case $K = T_1 = \cdots = T_s$.

Theorem C *Let K be an unconditional convex body of volume 1 in $\mathbb{R}^n$. If $n+1 \leq N \leq \exp(cn)$, then*

$$c_1 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \leq \mathbb{E}\,(K, N) \leq c_2 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \tag{1.16}$$

where $c, c_1, c_2 > 0$ are absolute constants.

For the proof of the upper bound in Theorem C we follow the general argument from [16]. However, in order to obtain the optimal upper bound in (1.16) we need two properties of isotropic unconditional convex bodies: dimension-dependent volume concentration (Theorem 2.2) and the good "ψ_2-behaviour" of linear functionals (Theorem 2.5). Also, the lower bound in (1.16) was proved in [16] under the restriction $N \geq n(\log n)^2$. Here, we provide a different proof of the lower bound for $\mathbb{E}\,(B(n), N)$, which is based on an idea of Dyer, Füredi and McDiarmid from [14]. Because of this, we are able to remove the restriction on N.

V Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. The unit ball of ℓ_p^n is denoted by B_p^n. Volume is denoted by $|\cdot|$. We write σ for the rotationally invariant probability measure on S^{n-1} and ω_n for the volume of B_2^n.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters c, c', C, c_1, c_2 etc., denote absolute positive constants which may change from line to line. We refer to the book of Milman and Schechtman [23] for basic facts from the asymptotic theory of finite dimensional normed spaces and to the paper [21] of Milman and Pajor for background information about isotropic convex bodies.

2 Isotropic unconditional convex bodies

Let K be an unconditional convex body in $\mathbb{R}^n$. Without loss of generality we may assume that the norm $\|\cdot\|_K$ satisfies (1.13), where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Then, it is easily checked that one can bring K to the isotropic position by a diagonal operator. Therefore, an isotropic unconditional body K in $\mathbb{R}^n$ is characterized by the following two properties:

(a) For every $x = (x_1, \ldots, x_n) \in K$ the parallelepiped $\prod_{i=1}^n [-|x_i|, |x_i|]$ is contained in K.

(b) For every $j = 1, \ldots, n$,

$$\int_K x_j^2 dx = L_K^2. \tag{2.1}$$

It will be convenient to consider the normalized part $K^+ = 2K \cap \mathbb{R}_+^n$ of K in $\mathbb{R}_+^n = [0, +\infty)^n$. It is easy to check that K^+ has volume 1 and satisfies the following:

(a+) If $x = (x_1, \ldots, x_n) \in K^+$, then $\prod_{i=1}^n [0, x_i] \subseteq K^+$.

(b+) For every $j = 1, \ldots, n$,

$$\int_{K^+} x_j^2 dx = 4L_K^2. \tag{2.2}$$

We write μ_K for the uniform distribution on K and μ_K^+ for the uniform distribution on K^+. Notice that if $x = (x_1, \ldots, x_n)$ is uniformly distributed in K then $(2|x_1|, \ldots, 2|x_n|)$ is uniformly distributed in K^+.

It is not hard to prove that the isotropic constant of any unconditional convex body satisfies $L_K \simeq 1$. The upper bound follows from the Loomis-Whitney inequality; see also [8] where the inequality $2L_K^2 \leq 1$ is proved. On the other hand, for every convex body K in $\mathbb{R}^n$ one has $L_K \geq L_{B_2^n} \geq c$, where $c > 0$ is an absolute constant (see [21]).

Bobkov and Nazarov have recently given a complete picture of the volume distribution on isotropic unconditional convex bodies. In the case of the ℓ_p^n-balls, very precise estimates on volume concentration were previously given in [28], [27], [30] and [29]. All the results which are stated in this section come from [8] and [9]. The starting point is the next inequality (see [8], Proposition 2.3).

Theorem 2.1 *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mu_K^+\big(x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\big) \leq \left(1 - \frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{6}n}\right)^n, \tag{2.3}$$

for all $(\alpha_1, \ldots, \alpha_n) \in K^+$. □

As a direct consequence we see that

$$\mu_K^+\big(x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\big) \leq \exp(-c(\alpha_1 + \cdots + \alpha_n)), \tag{2.4}$$

for all $\alpha_1, \ldots, \alpha_n \geq 0$, where $c = 1/\sqrt{6}$. Using this fact, Bobkov and Nazarov established a striking dimension-dependent concentration estimate for the Euclidean norm.

Theorem 2.2 *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then,*

$$\mu_K\big(\|x\|_2 \geq \sqrt{6}t\sqrt{n}\big) \leq \exp\left(-t\sqrt{n}/2\right), \tag{2.5}$$

for every $t \geq 4$. □

Analogous concentration results hold true if we replace the Euclidean norm by any ℓ_p-norm. For example,

$$\mu_K\big(\|x\|_1 \geq 2tn\big) \leq \exp\left(-\frac{c_3 tn}{\log n + 1}\right), \tag{2.6}$$

for all $t \geq 1$, where $c_3 > 0$ is an absolute constant.

Another consequence of Theorem 2.1 is that an interior point $(\alpha_1, \ldots, \alpha_n)$ of K^+ necessarily satisfies

$$\alpha_1 + \cdots + \alpha_n < \sqrt{6}n. \tag{2.7}$$

This observation proves one part of the next Proposition.

Proposition 2.3 *Let K be an isotropic unconditional convex body K in $\mathbb{R}^n$. Then,*

$$cB_\infty^n \subseteq K \subseteq V_n \tag{2.8}$$

where $V_n = \sqrt{3/2}nB_1^n$ and $c > 0$ is an absolute constant. □

Theorem 2.1 and Proposition 2.3 lead to a very useful comparison theorem. Let $F : \mathbb{R}^n \to \mathbb{R}$. We say that F is *symmetric* if $F(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = F(x_1, \ldots, x_n)$ for all choices of signs. We also say that F is *coordinatewise increasing* if $F(y) \leq F(x)$ for all x and y which satisfy $0 \leq y_i \leq x_i$ for all $i \leq n$. On observing that

$$\mu_{V_n}^+\big(x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\big) = \left(1 - \frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{6}n}\right)^n, \tag{2.9}$$

for all $(\alpha_1, \ldots, \alpha_n) \in V_n^+$, one has:

Theorem 2.4 *For every isotropic unconditional convex body K in $\mathbb{R}^n$ and every $\alpha_1, \ldots, \alpha_n \geq 0$,*

$$\mu_K\big(x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\big) \leq \mu_{V_n}\big(x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n\big). \tag{2.10}$$

Consequently,

$$\int F(x) d\mu_K(x) \leq \int F(x) d\mu_{V_n}(x) \tag{2.11}$$

for every function $F : \mathbb{R}^n \to \mathbb{R}$ which is symmetric, coordinatewise increasing and absolutely continuous. □

The comparison theorem can be used for the study of linear functionals on isotropic unconditional convex bodies. Define

$$C_n(\theta) = \|\theta\|_\infty \sqrt{n/\log n} \tag{2.12}$$

for $\theta \in \mathbb{R}^n$ and $n \geq 2$. Since the expectation of $\|\theta\|_\infty$ on S^{n-1} is of the order of $\sqrt{\log n/n}$, for a random $\theta \in S^{n-1}$ we have $C_n(\theta) \simeq 1$. Computing on B_1^n and using Theorem 2.4 one gets the following (see [9]):

Theorem 2.5 *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Let $\theta \in S^n$ and set $f_\theta(x) = \langle x, \theta\rangle$. For every $p \geq 2$,*

$$\|f_\theta\|_{L^p(\mu_K)} \leq c\sqrt{p}\max\{1, C_n(\theta)\sqrt{\log p}\}, \tag{2.13}$$

where $c > 0$ is an absolute constant. □

Define the Orlicz norm $\|\cdot\|_{L^{\psi_2}(\mu_K)}$ of a measurable function $f : K \to \mathbb{R}$ by

$$\|f\|_{L^{\psi_2}(\mu_K)} = \inf\left\{t > 0 : \int e^{(f/t)^2} d\mu_K \leq 2\right\}. \tag{2.14}$$

By Theorem 2.5 and (2.8) we have

$$\|f_\theta\|_{L^p(\mu_K)} \leq c\sqrt{p}\sqrt{n}\|\theta\|_\infty \tag{2.15}$$

for all $p \geq 2$. Since

$$\|f\|_{L^{\psi_2}(\mu_K)} \leq c\sup_{p\geq 2}\|f\|_{L^p(\mu_K)},$$

this shows that

$$\|f_\theta\|_{L^{\psi_2}(\mu_K)} \leq c\sqrt{n}\|\theta\|_\infty \tag{2.16}$$

for every $\theta \in \mathbb{R}^n$.

3 Random isotropic and unconditional vectors

The proof of Theorem A will be based on Rudelson's approach to the general case. The main lemma in [26] is the following.

Theorem 3.1 (Rudelson) *Let $x_1, \ldots, x_N$ be vectors in $\mathbb{R}^n$ and let $\varepsilon_1, \ldots, \varepsilon_N$ be independent Bernoulli random variables which take the values ± 1 with probability $1/2$. Then, for all $p \geq 1$,*

$$\left(\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i \otimes x_i\right\|^p\right)^{1/p} \leq C\sqrt{p + \log n}\cdot\max_{i\leq N}\|x_i\|_2 \cdot \left\|\sum_{i=1}^N x_i \otimes x_i\right\|^{1/2}, \tag{3.1}$$

where $C > 0$ is an absolute constant. □

Proof of Theorem A. Let $\varepsilon \in (0,1)$ and let $p \geq 1$. We first estimate the expectation of $\max_{i\leq N}\|x_i\|_2^{2p}$, where $x_1, \ldots, x_N$ are independent random points uniformly distributed in K. Fix $\alpha \geq 4$ which will be suitably chosen. Theorem 2.2 shows that

$$\begin{aligned}
\mathbb{E}\max_{i\leq N}\|x_i\|_2^{2p} &= (6n)^p \int_0^\infty 2pt^{2p-1}\mu_K\left(\max_{i\leq N}\|x_i\|_2 \geq \sqrt{6}t\sqrt{n}\right)dt \\
&\leq (6n)^p\left(\alpha^{2p} + (2p)N\int_\alpha^\infty t^{2p-1}\exp(-t\sqrt{n}/2)dt\right) \\
&= (6n)^p\left(\alpha^{2p} + (2p)Ne^{-\alpha\sqrt{n}/2}\int_0^\infty (s+\alpha)^{2p-1}\exp(-s\sqrt{n}/2)ds\right).
\end{aligned}$$

Since $(s+\alpha)^{2p-1} \leq 2^{2p-2}(s^{2p-1}+\alpha^{2p-1})$, we have

$$\begin{aligned}
\mathbb{E}\max_{i\leq N}\|x_i\|_2^{2p} &\leq (6n)^p(2p)2^{2p-2}Ne^{-\alpha\sqrt{n}/2}\left(\int_0^\infty (s^{2p-1}+\alpha^{2p-1})e^{-s\sqrt{n}/2}ds\right) \\
&\quad +(6n)^p\alpha^{2p} \\
&= (6n)^p\left(\alpha^{2p} + (2p)2^{2p-2}Ne^{-\alpha\sqrt{n}/2}\left(\frac{4^p\Gamma(2p)}{n^p} + \frac{2\alpha^{2p-1}}{\sqrt{n}}\right)\right).
\end{aligned}$$

It follows that, if $N \leq \exp(\alpha\sqrt{n}/2)$ and $p \leq \sqrt{n}$ then

$$\mathbb{E}\max_{i\leq N}\|x_i\|_2^{2p} \leq (C_1\alpha^2 n)^p \tag{3.2}$$

where $C_1 > 0$ is an absolute constant.

If $N > \exp(\alpha\sqrt{n}/2)$ we use Proposition 2.3: Since $K \subseteq V_n$, we have $\|x\|_2 \leq \sqrt{3/2n}$ for every $x \in K$. Therefore, in this case we have the trivial estimate

$$\mathbb{E}\max_{i\leq N}\|x_i\|_2^{2p} \leq (C_2 n)^{2p}. \tag{3.3}$$

We now follow Rudelson's argument: if $x_1', \ldots, x_N'$ are independent random points from K which are chosen independently from the x_i's, then

$$\begin{aligned}
A^p &:= \mathbb{E}_{\mathbf{x}}\left\|I - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i\right\|^p \\
&\leq \mathbb{E}_{\mathbf{x}}\mathbb{E}_{\mathbf{x}'}\left\|\frac{1}{NL_K^2}\sum_{i=1}^N x_i\otimes x_i - \frac{1}{NL_K^2}\sum_{i=1}^N x_i'\otimes x_i'\right\|^p \\
&= \mathbb{E}_\varepsilon\mathbb{E}_{\mathbf{x}}\mathbb{E}_{\mathbf{x}'}\left\|\frac{1}{NL_K^2}\sum_{i=1}^N \varepsilon_i(x_i\otimes x_i - x_i'\otimes x_i')\right\|^p \\
&\leq 2^{p-1}\mathbb{E}_{\mathbf{x}}\mathbb{E}_\varepsilon\left\|\frac{1}{NL_K^2}\sum_{i=1}^N \varepsilon_i x_i\otimes x_i\right\|^p,
\end{aligned}$$

and using Theorem 3.1 we get

$$\begin{aligned}
A^p &\le (2C)^p \frac{(p+\log n)^{p/2}}{N^p L_K^{2p}} \, \mathbb{E}_{\mathbf{x}} \left(\max_{i\le N} \|x_i\|_2^p \cdot \left\| \sum_{i=1}^N x_i \otimes x_i \right\|^{p/2} \right) \\
&\le (2C)^p \frac{(p+\log n)^{p/2}}{N^{p/2} L_K^{p}} \left(\mathbb{E}_{\mathbf{x}} \max_{i\le N} \|x_i\|_2^{2p} \right)^{1/2} \left(\mathbb{E}_{\mathbf{x}} \left\| \frac{1}{NL_K^2} \sum_{i=1}^N x_i \otimes x_i \right\|^{p} \right)^{1/2} \\
&\le (4C)^p \frac{(p+\log n)^{p/2}}{N^{p/2} L_K^{p}} \left(\mathbb{E}_{\mathbf{x}} \max_{i\le N} \|x_i\|_2^{2p} \right)^{1/2} \sqrt{A^p+1}.
\end{aligned}$$

We choose $p = \log n$ and $\alpha^2 = 16\varepsilon^{-1/\log n}$, and we distinguish two cases:

Case 1: If $N \le \exp(\alpha\sqrt{n}/2)$, then using (3.2) and the fact that $L_K \simeq 1$, we get

$$A^p \le \left(C_3 \frac{\alpha^2 n \log n}{N} \right)^{p/2} \sqrt{A^p+1}. \tag{3.4}$$

If

$$\left(C_3 \frac{\alpha^2 n \log n}{N} \right)^{p/2} < \frac{\varepsilon^{p+1}}{2}, \tag{3.5}$$

then (3.4) implies that

$$\mathbb{E}_{\mathbf{x}} \left\| I - \frac{1}{NL_K^2} \sum_{i=1}^N x_i \otimes x_i \right\|^p = A^p < \varepsilon^{p+1}. \tag{3.6}$$

If $2 < \rho < 3$ then (3.5) is satisfied provided that $n \ge n_0(\rho)$ and $N \ge N_0 = C_4\varepsilon^{-\rho} n \log n$. Observe that $\exp(\alpha\sqrt{n}/2) > C_4\varepsilon^{-\rho} n\log n$ if $n \ge n_0(\rho)$.

Then, Markov's inequality shows that, if $N_0 \le N \le \exp(\alpha\sqrt{n}/2)$,

$$\mathrm{Prob}\left(\left\| I - \frac{1}{NL_K^2} \sum_{i=1}^N x_i \otimes x_i \right\| > \varepsilon \right) < \varepsilon. \tag{3.7}$$

Case 2: Assume that $N > \exp(\alpha\sqrt{n}/2)$. Then, using (3.3) and the fact that $L_K \simeq 1$, we get

$$A^p \le \left(C_5 \frac{n^2 \log n}{N} \right)^{p/2} \sqrt{A^p+1} \le \left(C_5 \frac{n^2 \log n}{e^{\alpha\sqrt{n}/2}} \right)^{p/2} \sqrt{A^p+1}. \tag{3.8}$$

We need to check that

$$\left(C_5 \frac{n^2 \log n}{e^{\alpha\sqrt{n}/2}} \right)^{p/2} < \frac{\varepsilon^{p+1}}{2}, \tag{3.9}$$

and then we get (3.7) exactly as in Case 1. Now, (3.9) is equivalent to

$$\varepsilon^{\frac{1}{2p}} C_6 \log n < \sqrt{n} - \frac{4(p+1)}{p} \varepsilon^{\frac{1}{2p}} \log \frac{1}{\varepsilon} \tag{3.10}$$

and the maximum of $\varepsilon^{\frac{1}{2p}} \log \frac{1}{\varepsilon}$ on $(0,1)$ is attained at $\varepsilon = e^{-2p}$ and equals $2p/e \simeq \log n$. Therefore, (3.10) is clearly satisfied if $n \geq n_0$. This completes the proof. □

The general problem is connected to some recent conjectures about the central limit properties of convex bodies: Let K be an isotropic convex body in $\mathbb{R}^n$. In [7] the parameter σ_K of K is defined by

$$\sigma_K^2 = \frac{\mathrm{Var}(\|x\|_2^2)}{nL_K^4} = \frac{n\mathrm{Var}(\|x\|_2^2)}{\left(\mathbb{E}\,\|x\|_2^2\right)^2}. \tag{3.11}$$

The second expression has the advantage of being invariant under homotheties, and hence, easier to compute. It is easily checked that $\sigma^2_{B(n)} = \frac{4}{n+4}$. Actually, in [7] Bobkov and Koldobsky show that σ_K is minimal when K is a Euclidean ball.

A question which has attracted much attention is whether there exists an absolute constant $C > 0$ such that $\sigma_K^2 \leq C$ for every isotropic convex body K. From (3.11) one can check that $\sigma_K^2 \leq cn$ for every isotropic convex body K in $\mathbb{R}^n$. The subindependence of coordinate slabs theorem of Anttila, Ball and Perissinaki (see [4] and [2]) shows that $\sigma_{B_p^n}$ remains bounded by a constant independent of n and $p \in [1, \infty]$.

Upper bounds for σ_K are related to Theorem A because of the following proposition (see [25] for a different proof).

Proposition 3.2 *If K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\mu_K\big(\|x\|_2 \geq (1+\sigma_K)\sqrt{n}L_K t\big) \leq n^{-t/2} \tag{3.12}$$

for every $t \geq 1$.

Proof. A simple application of Chebyshev's inequality shows that, for every $\varepsilon > 0$,

$$\begin{aligned} \mu_K\big(\big|\,\|x\|_2 - \sqrt{n}L_K\,\big| \geq \varepsilon\sqrt{n}L_K\big) &\leq \mu_K\left(\big|\,\|x\|_2^2 - nL_K^2\,\big| \geq \varepsilon nL_K^2\right) \\ &\leq \frac{\mathrm{Var}(\|x\|_2^2)}{\varepsilon^2 n^2 L_K^4} = \frac{\sigma_K^2}{\varepsilon^2 n}. \end{aligned}$$

Therefore,

$$\mu_K\big(\|x\|_2 \geq (1+\sigma_K)\sqrt{n}L_K\big) \leq \frac{1}{n}. \tag{3.13}$$

Applying Borell's lemma (see [23, Appendix III]) we see that

$$\mu_K\big(\|x\|_2 \geq (1+\sigma_K)\sqrt{n}L_K t\big) \leq \frac{n-1}{n}\left(\frac{1}{n-1}\right)^{\frac{t+1}{2}} \leq n^{-t/2} \tag{3.14}$$

for all $t \geq 1$ (we only need to check (3.14) for $t \ll \sqrt{n}$; it is well-known that $K \subseteq (n+1)L_K B_2^n$ for every isotropic convex body K in $\mathbb{R}^n$, and hence, the left hand side of (3.14) is equal to zero if $t(1+\sigma_K)\sqrt{n} > n+1$). This concludes the proof. $\square$

We can now repeat the argument of the proof of Theorem A. Let $x_1, \ldots, x_N$ be independent random points uniformly distributed in K. Let $\varepsilon \in (0,1)$ and set $\sigma = \sigma_K + 1$. We fix $p \geq 1$ and $\alpha > 1$ which will be suitably chosen. Using Proposition 3.2 we write

$$\begin{aligned} \mathbb{E} \max_{i \leq N} \|x_i\|_2^{2p} &\leq (\sigma^2 n L_K^2)^p \left(\alpha^{2p} + N \int_\alpha^\infty 2pt^{2p-1} n^{-t/2} dt \right) \\ &\leq (4\sigma^2 n L_K^2)^p \left(\alpha^{2p} + (2p) N e^{-\alpha \log n/2} \left(\frac{4^p \Gamma(2p)}{(\log n)^{2p}} + \frac{2\alpha^{2p-1}}{\log n} \right) \right). \end{aligned}$$

It follows that if $N \leq n^{\alpha/2}$ and $p \leq \log n$ then

$$\mathbb{E} \max_{i \leq N} \|x_i\|_2^{2p} \leq (C_1 \sigma^2 \alpha^2 L_K^2 n)^p. \tag{3.15}$$

If $N > n^{\alpha/2}$ we use the bound

$$\mathbb{E} \max_{i \leq N} \|x_i\|_2^{2p} \leq (C_2 L_K^2 n^2)^p, \tag{3.16}$$

which follows from the inclusion $K \subseteq (n+1)L_K B_2^n$. Set

$$A^p := \mathbb{E}_{\mathbf{x}} \left\| I - \frac{1}{N L_K^2} \sum_{i=1}^N x_i \otimes x_i \right\|^p. \tag{3.17}$$

Following the proof of Theorem A, we see that

$$A^p \leq (4C)^p \frac{(p + \log n)^{p/2}}{N^{p/2} L_K^p} \left(\mathbb{E}_{\mathbf{x}} \max_{i \leq N} \|x_i\|_2^{2p} \right)^{1/2} \sqrt{A^p + 1}. \tag{3.18}$$

Let $\rho > 2$. We choose $p = \log n$ and $\alpha^2 = D^2 \varepsilon^{-1/\log n}$ (where $D > 1$ is an absolute constant) and we distinguish two cases:

Case 1: If $N \leq n^{\alpha/2}$, then using (3.15) we get

$$A^p \leq \left(C_3 \frac{\sigma^2 \alpha^2 n \log n}{N} \right)^{p/2} \sqrt{A^p + 1}. \tag{3.19}$$

Therefore, if $n \geq n_0(\rho)$ and $N \geq c\sigma^2 \varepsilon^{-\rho} n \log n$ we see that

$$\left(C_3 \frac{\sigma^2 \alpha^2 n \log n}{N} \right)^{p/2} < \frac{\varepsilon^{p+1}}{2}, \tag{3.20}$$

which implies that

$$\mathbb{E}_{\mathbf{x}}\Big\| I - \frac{1}{NL_K^2}\sum_{i=1}^{N} x_i\otimes x_i\Big\|^p < \varepsilon^{p+1}. \tag{3.21}$$

Case 2: If $N > n^{\alpha/2}$, then using (3.16) we get

$$A^p \leq \left(C_4\frac{n^2\log n}{n^{\alpha/2}}\right)^{p/2}\sqrt{A^p+1} < \frac{\varepsilon^{p+1}}{2}\sqrt{A^p+1} \tag{3.22}$$

provided that $n \geq n_0$ and

$$3\log n < \frac{\alpha}{2}\log n - \frac{2(p+1)}{p}\log\frac{1}{\varepsilon}. \tag{3.23}$$

This is satisfied if

$$\frac{2(p+1)}{p}\sup_{\varepsilon\in(0,1)}\varepsilon^{\frac{1}{2p}}\log\frac{1}{\varepsilon} = 4e^{-1}(\log n+1) < \left(\frac{D}{2}-3\right)\log n, \tag{3.24}$$

which is true if $D > 1$ is large enough. Therefore, (3.21) is verified in this case as well. An application of Markov's inequality shows the following.

Theorem 3.3 *Let $\varepsilon \in (0,1)$ and let $\rho > 2$. Assume that $n \geq n_0(\rho)$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $N \geq c\varepsilon^{-\rho}(\sigma_K+1)^2 n\log n$, where $c > 0$ is an absolute constant, and if $x_1,\dots,x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have*

$$(1-\varepsilon)L_K^2 \leq \frac{1}{N}\sum_{i=1}^{N}\langle x_i,\theta\rangle^2 \leq (1+\varepsilon)L_K^2 \tag{3.25}$$

for every $\theta \in S^{n-1}$. □

4 A geometric inequality

In this Section we study the multi-integral norm (1.4) in the case where K and $T_1,\dots,T_s$ are isotropic and unconditional with respect to the standard orthonormal basis of $\mathbb{R}^n$. Our estimate is stated in the next theorem.

Theorem 4.1 *There exists an absolute constant $C > 0$ with the following property: if K and T_i $(i=1,\dots,s)$ are isotropic convex bodies in $\mathbb{R}^n$ which satisfy (1.13), then*

(4.1)

$$\|\mathbf{t}\| := \int_{T_1}\cdots\int_{T_s}\Big\|\sum_{i=1}^{s} t_i x_i\Big\|_K dx_s\cdots dx_1 \leq C\sqrt{\log n}\max\left\{\|\mathbf{t}\|_2, \sqrt{\log n}\|\mathbf{t}\|_\infty\right\}$$

for every $\mathbf{t} = (t_1,\dots,t_s) \in \mathbb{R}^s$.

The proof will be based on the comparison Theorem 2.4. We write μ_n for the uniform distribution on B_1^n. The density of μ_n is given by

$$\frac{d\mu_n(x)}{dx} = \frac{n!}{2^n}\chi_{B_1^n}(x). \tag{4.2}$$

We also define $\Delta_n = \{x \in \mathbb{R}_+^n : x_1 + \cdots + x_n \leq 1\}$. A simple computation shows that for every n-tuple of non-negative integers $p_1, \ldots, p_n$,

$$\int_{\Delta_n} x_1^{p_1} \ldots x_n^{p_n} dx = \frac{p_1! \cdots p_n!}{(n + p_1 + \cdots + p_n)!}. \tag{4.3}$$

Proof of Theorem 4.1: From Proposition 2.3 we have $\| \cdot \|_K \leq c_1 \| \cdot \|_\infty$ where $c_1 > 0$ is an absolute constant, so it is enough to consider the case $K = B_\infty^n$, the unit cube in $\mathbb{R}^n$. Since $\| \cdot \|_\infty \leq \| \cdot \|_{2q}$ for every integer $q \geq 1$, our problem is to give upper bounds for the norm

$$\|\mathbf{t}\| := \int_{T_1} \cdots \int_{T_s} \Big\| \sum_{i=1}^{s} t_i x_i \Big\|_{2q} \, dx_s \cdots dx_1 \tag{4.4}$$

where $q \geq 1$ is an integer. We write $x_i = (x_{i1}, \ldots, x_{in})$ and define $y_j = (x_{1j}, \ldots, x_{sj})$ for all $j = 1, \ldots, n$. Then, Hölder's inequality shows that

$$\|\mathbf{t}\|^{2q} \leq \int_{T_1} \cdots \int_{T_s} \sum_{j=1}^{n} \langle \mathbf{t}, y_j \rangle^{2q} \, dx_s \cdots dx_1. \tag{4.5}$$

Observe that

$$\int_{T_1} \cdots \int_{T_s} \langle \mathbf{t}, y_j \rangle^{2q} \, dx_s \cdots dx_1 = \sum_{q_1 + \cdots + q_s = q} \frac{(2q)!}{(2q_1)! \cdots (2q_s)!} \prod_{i=1}^{s} t_i^{2q_i} \int_{T_i} x_{ij}^{2q_i} \, dx_i. \tag{4.6}$$

Applying Theorem 2.4 we get

$$\int_{T_i} x_{ij}^{2q_i} \, dx_i \leq \int_{V_n} x_1^{2q_i} d\mu_{V_n}(x) \leq (c_1 n)^{2q_i} n! \int_{\Delta_n} x_1^{2q_i} dx = (c_1 n)^{2q_i} \frac{n!(2q_i)!}{(n + 2q_i)!}, \tag{4.7}$$

where $c_1 = \sqrt{3/2}$. From (4.5), (4.6) and (4.7) it follows that

$$\|\mathbf{t}\|^{2q} \leq n(n!)^s (c_1 n)^{2q} (2q)! \sum_{q_1 + \cdots + q_s = q} \frac{t_1^{2q_1} \cdots t_s^{2q_s}}{(n + 2q_1)! \cdots (n + 2q_s)!}. \tag{4.8}$$

Since $(n + 2r)! \geq n! n^{2r}$ for every $r \geq 0$, we can write

$$\|\mathbf{t}\|^{2q} \leq n c_1^{2q} (2q)! \sum_{q_1 + \cdots + q_s = q} t_1^{2q_1} \cdots t_s^{2q_s}. \tag{4.9}$$

We now use the following lemma from [9]:

Lemma 4.2 *Let $q \geq 1$ be an integer and define*

$$P_q(y) = \sum_{q_1+\cdots+q_s=q} y_1^{q_1} \cdots y_s^{q_s} \tag{4.10}$$

on $\mathbb{R}_+^s$. If $y \in \mathbb{R}_+^s$ and $y_1 + \cdots + y_s = 1$, then

$$P_q(y) \leq (2e \max\{1/q, \|y\|_\infty\})^q . \tag{4.11}$$

□

Applying Lemma 4.2 to the s-tuple $y = \frac{1}{\|\mathbf{t}\|_2^2}\left(t_1^2, \ldots, t_s^2\right)$ we get

$$\begin{aligned} \|\mathbf{t}\| &\leq c_1 n^{\frac{1}{2q}} \sqrt[2q]{(2q)!} \left(2e \max\{\|\mathbf{t}\|_2^2/q, \|\mathbf{t}\|_\infty^2\}\right)^{1/2} \\ &\leq C n^{\frac{1}{2q}} \sqrt{q} \max\{\|\mathbf{t}\|_2, \sqrt{q}\|\mathbf{t}\|_\infty\}. \end{aligned}$$

Choosing $q \simeq \log n$ we conclude the proof. □

Remark 4.3 The ℓ_∞–term in the estimate provided by Theorem 4.1 is necessary. This can be seen for the case in which $T_1 = \cdots = T_s = W_n = \delta_n n B_1^n$ and $K = \frac{1}{2} B_\infty^n$, where $\delta_n \to \frac{1}{2e}$ is chosen so that $|W_n| = 1$. For these bodies consider the vector $\mathbf{t}_0 = (1, 0, 0, \ldots, 0)$. We then have,

$$\|\mathbf{t}_0\| = \int_{W_n} 2\|x\|_\infty \, dx = 2(\delta_n n)^{n+1} \int_{B_{\ell_1^n}} \|x\|_\infty \, dx. \tag{4.12}$$

It is enough to show that $\|\mathbf{t}_0\| \geq c \log n$ for some absolute constant $c > 0$. Let $F_n(t)$ be the proportion of the volume of the ℓ_1^n ball inside the cube $[-t, t]^n$. This quantity was studied in [4] where it was shown that it is dominated by the function $f_n(t) = \left(1 - (1-t)^n\right)^n$. Using this, and writing λ for the Lebesgue measure in $\mathbb{R}^n$, we get

$$\begin{aligned} \|\mathbf{t}_0\| &\geq 2(\delta_n n)^{n+1} \int_0^{\log n/n} \lambda(x \in B_1^n \, : \, \|x\|_\infty > t) \, dt \\ &\geq 2(\delta_n n)^{n+1} \int_0^{\log n/n} \left(1 - F_n(t)\right) |B_1^n| \, dt \\ &= 2\delta_n n \int_0^{\log n/n} \left(1 - F_n(t)\right) dt \\ &\geq 2\delta_n \log n \left(1 - F_n\left(\frac{\log n}{n}\right)\right). \end{aligned}$$

Since $n\left(1 - \frac{\log n}{n}\right)^n \to 1$ as $n \to \infty$, it is now easy to check that

$$F_n\left(\frac{\log n}{n}\right) \leq f_n\left(\frac{\log n}{n}\right) \leq c_1 \tag{4.13}$$

for some universal constant $0 < c_1 < 1$, from which it follows that $\|\mathbf{t}_0\| \geq c \log n$.

It should be also noticed that if s is large enough then the directions $\mathbf{t} \in S^{s-1}$ for which $\sqrt{\log n}\|\mathbf{t}\|_\infty > 1$ form a set of small measure. Since $\|\cdot\|_\infty$ is a 1-Lipschitz function on S^{n-1}, we have

$$\sigma\Big(\mathbf{t} \in S^{s-1} : \big|\, \|\mathbf{t}\|_\infty - \mathbb{E}\,\|\cdot\|_\infty \big| \geq r\Big) \leq \exp(-c_1 r^2 s) \tag{4.14}$$

for all $r > 0$. A simple computation shows that $\mathbb{E}\,\|\cdot\|_\infty \simeq (\log s/s)^{1/2}$. Assume that $s \gg \log^2 n$. From (4.14) we see that

$$\sigma\big(\mathbf{t} \in S^{s-1} : \sqrt{\log n}\|\mathbf{t}\|_\infty > 1\big) \leq \exp(-c_2 s/\log n). \tag{4.15}$$

Remark 4.4 A modification of the example in Remark 4.3 shows that Theorem 4.1 is optimal even for the case in which all T_i and K are equal up to a permutation of coordinates. Consider the orthogonal operator $U = \sum_{j=0}^{n-1} e_j \otimes e_{n-j}$. Assume that n is even and consider the bodies $T_i = \alpha W_{n/2} \times \beta B_\infty^{n/2}$ $(i = 1, \ldots, s)$ where $\alpha, \beta \simeq 1$ are chosen so that T_i is isotropic. Define $K = U(T_i)$ and let $\mathbf{t}_0 = (1, 0, 0, \ldots, 0)$. We have

$$\|\mathbf{t}_0\| = \int\limits_{\alpha W_{n/2}\times\beta B_\infty^{n/2}} \|x\|_{\beta B_\infty^{n/2}\times\alpha W_{n/2}}\, dx \geq \int\limits_{\alpha W_{n/2}\times\beta B_\infty^{n/2}} \|Px\|_{\beta B_\infty^{n/2}}\, dx \tag{4.16}$$

where P is the orthogonal projection onto the first $n/2$ coordinates. Now, Fubini's theorem implies that the last quantity equals

$$\int_{\alpha W_{n/2}} \|x\|_{\beta B_\infty^{n/2}}\, dx \tag{4.17}$$

and by Remark 4.3 this is greater than $c \log n$.

Remark 4.5 Consider the case $K = T_1 = \cdots = T_s$. Then, we do not know if the ℓ_∞–term is really needed in Theorem 4.1. However, the example of the cube shows that the term $\sqrt{\log n}\|\mathbf{t}\|_2$ is necessary: In [19] it is proved that if $K = T_1 = \cdots = T_n = \frac{1}{2}B_\infty^n$ then

$$\|\mathbf{t}\| \simeq q_n(\mathbf{t}) = \sum_{i=1}^{u} t_i^* + \sqrt{u}\left(\sum_{i=u+1}^{n} (t_i^*)^2\right)^{1/2} \tag{4.18}$$

where $u \simeq \log n$ and $(t_i^*)_{i\leq n}$ is the decreasing rearrangement of $(|t_i|)_{i\leq n}$. From (4.18) one can check that

$$\int_{S^{n-1}} \|\mathbf{t}\|\, \sigma(d\mathbf{t}) \geq c\sqrt{\log n}. \tag{4.19}$$

Indeed, using [24, Lemma 2] we may write

$$q_n(\mathbf{t}) \geq c_1 \|\mathbf{t}\|_{P(u)} \tag{4.20}$$

where

$$\|\mathbf{t}\|_{P(u)} = \sup \left\{ \sum_{m=1}^{u} \left(\sum_{i \in B_m} t_i^2 \right)^{1/2} \right\} \tag{4.21}$$

and the supremum is taken over all disjoint subsets $B_1, \ldots, B_u$ of $\{1, \ldots, n\}$. Choose a partition of $\{1, \ldots, n\}$ into successive intervals B_m so that $k := \min |B_m|$ is greater than $\frac{n}{2u}$. Then,

$$\begin{aligned} \int_{S^{n-1}} \|\mathbf{t}\| \, d\sigma &\geq c_1 \int_{S^{n-1}} \sum_{m=1}^{u} \left(\sum_{i \in B_m} t_i^2 \right)^{1/2} d\sigma \geq c_1 u \int_{S^{n-1}} \left(\sum_{i=1}^{k} t_i^2 \right)^{1/2} d\sigma \\ &\geq c_2 u \left(\int_{S^{n-1}} \sum_{i=1}^{k} t_i^2 d\sigma \right)^{1/2} = c_2 u (k/n)^{1/2} \geq c_3 \sqrt{u}. \end{aligned}$$

Since $u \simeq \log n$ we get (4.19).

5 Volume radius of random polytopes

Recall the definition of $\mathbb{E}\,(K, N)$: if $x_1, \ldots, x_N$ are independent random points uniformly distributed in a convex body K of volume 1 in $\mathbb{R}^n$, we define

$$\mathbb{E}(K, N) = \mathbb{E}\,|\mathrm{conv}(x_1, \ldots, x_N)|^{1/n}. \tag{5.1}$$

Assume that K is isotropic and unconditional. We will prove an optimal upper bound for $\mathbb{E}\,(K, N)$.

Theorem 5.1 *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then, for every $N \geq n + 1$,*

$$\mathbb{E}\,(K, N) \leq C \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, \tag{5.2}$$

where $C > 0$ is an absolute constant.

For the proof, we will use two deterministic results on the volume of convex hulls of N points and on the volume of the intersection of N symmetric strips in $\mathbb{R}^n$. The first one was proved independently by Bárány and Füredi [5] or [6], Carl and Pajor [13], and Gluskin [17]:

Lemma 5.2 *There exists an absolute constant $c_1 > 0$ such that: if $N \geq n + 1$ and $x_1, \ldots, x_N \in \mathbb{R}^n$, then*

$$|\mathrm{conv}(x_1, \ldots, x_N)|^{1/n} \leq c_1 \max_{i \leq N} \|x_i\|_2 \, \frac{\sqrt{\log(2N/n)}}{n}. \tag{5.3}$$

The second one is a result of Ball and Pajor [3]:

Lemma 5.3 *Let $x_1, \ldots, x_N \in \mathbb{R}^n \setminus \{0\}$ and let $1 \le q < \infty$. If*

$$W = \{z \in \mathbb{R}^n : |\langle z, x_j\rangle| \le 1,\ j = 1, \ldots, N\}, \tag{5.4}$$

then

$$|W|^{1/n} \ge 2 \left(\frac{n+q}{n} \sum_{j=1}^{N} \frac{1}{|B_q^n|} \int_{B_q^n} |\langle z, x_j\rangle|^q dz \right)^{-1/q}. \tag{5.5}$$

Proof of Theorem 5.1: We distinguish two cases (small and large N):
Case 1: $\mathbf{N \le n^2}$: Fix $t \ge 4$ which will be suitably chosen. We define

$$A := \left\{(x_1, \ldots, x_N) \in K^N :\ \exists i \le N \text{ such that } \|x_i\|_2 \ge \sqrt{6}t\sqrt{n}\right\}. \tag{5.6}$$

From Theorem 2.2 we have

$$\mathrm{Prob}\,(A) \le N \exp\left(-t\sqrt{n}/2\right). \tag{5.7}$$

Using Lemma 5.2 we write

$$\begin{aligned} \mathbb{E}\,(K, N) &= \int_A |\mathrm{conv}(x_1, \ldots, x_N)|^{1/n} + \int_{A^c} |\mathrm{conv}(x_1, \ldots, x_N)|^{1/n} \\ &\le \mathrm{Prob}(A) + \mathrm{Prob}(A^c)\, \sqrt{6}c_1 t\sqrt{n}\, \frac{\sqrt{\log(2N/n)}}{n} \\ &\le N \exp\left(-\frac{t\sqrt{n}}{2}\right) + c_2 t\, \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \end{aligned}$$

We have assumed that $N \le n^2$, which implies

$$\exp\left(\frac{t\sqrt{n}}{2}\right) \ge \frac{t^6 n^3}{6!2^6} \ge \frac{t^6 nN}{6!2^6} \ge \frac{nN}{c_2 t\sqrt{\log 2}} \tag{5.8}$$

if $t \ge 4$ is chosen large enough (and independent of n and N). Then,

$$\mathbb{E}\,(K, N) \le (2c_2 t)\, \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{5.9}$$

Thus, Case 1 is settled.
Case 2: $\mathbf{N \ge n^2}$: We write K_N for the absolute convex hull $\mathrm{conv}(\pm x_1, \ldots, \pm x_N)$ of N independent random points uniformly distributed in K. By the Blaschke-Santaló inequality,

$$\mathbb{E}\,(K, N) \le \mathbb{E}\,|K_N|^{1/n} \le \omega_n^{2/n} \cdot \mathbb{E}\,|K_N^\circ|^{-1/n} \tag{5.10}$$

where K_N° is the polar body of K_N. Lemma 5.3 shows that

$$|K_N^\circ|^{-1/n} \leq \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^{N}\frac{1}{|B_q^n|}\int_{B_q^n}|\langle z, x_j\rangle|^q dz\right)^{1/q} \tag{5.11}$$

for every $q \geq 1$. Consider the convex body $T = K \times \cdots \times K$ (N times) in $\mathbb{R}^{Nn}$. We apply Hölder's inequality and change the order of integration:

$$\begin{aligned}\mathbb{E}|K_N^\circ|^{-1/n} &\leq \int_T \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^{N}\frac{1}{|B_q^n|}\int_{B_q^n}|\langle z, x_j\rangle|^q\, dz\right)^{1/q} dx_N \dots dx_1 \\ &\leq \frac{1}{2}\left(\frac{n+q}{n}\sum_{j=1}^{N}\frac{1}{|B_q^n|}\int_{B_q^n}\int_T|\langle z, x_j\rangle|^q\, dx_N \dots dx_1\, dz\right)^{1/q}.\end{aligned}$$

It follows that

$$\mathbb{E}\,(K,N) \leq \frac{\omega_n^{2/n}}{2}\left(\frac{N(n+q)}{n}\frac{1}{|B_q^n|}\int_{B_q^n}\int_K|\langle z, x\rangle|^q dx\, dz\right)^{1/q}. \tag{5.12}$$

Now, we use (2.15). For every $z \in B_q^n$ we have

$$\int_K |\langle z, x\rangle|^q dx \leq \left(C_1\sqrt{q}\sqrt{n}\|z\|_\infty\right)^q. \tag{5.13}$$

Therefore,

$$\mathbb{E}\,(K,N) \leq \frac{\omega_n^{2/n}}{2}C_1\sqrt{q}\sqrt{n}\left(\frac{N(n+q)}{n}\frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_\infty^q dz\right)^{1/q}. \tag{5.14}$$

Observe that $\|z\|_\infty \leq \|z\|_q$ and

$$\int_{B_q^n}\|z\|_q^q dz = \frac{n}{n+q}|B_q^n|. \tag{5.15}$$

It follows that

$$\frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_\infty^q dz \leq \frac{1}{|B_q^n|}\int_{B_q^n}\|z\|_q^q dz = \frac{n}{n+q}. \tag{5.16}$$

Combining the above we get

$$\mathbb{E}\,(K,N) \leq C_2\frac{\sqrt{q}}{\sqrt{n}}\,N^{1/q} \tag{5.17}$$

for every $q \geq 1$. Choose $q = \log(2N/n)$. Since $N \geq n^2$, we have

$$N^{1/q} = \exp\left(\frac{\log N}{\log(2N/n)}\right) \leq \exp\left(\frac{\log N}{\log(2\sqrt{N})}\right) \leq e^2. \tag{5.18}$$

Therefore,

$$\mathbb{E}\,(K,N) \leq C_3 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{5.19}$$

This completes the proof. □

A lower bound for $\mathbb{E}\,(K,N)$ was given in [16] as a consequence of the following facts. If K is a convex body in $\mathbb{R}^n$ with volume 1 and $B(n)$ is a ball in $\mathbb{R}^n$ with volume 1, then it is proved in [16] that

$$\mathbb{E}\,(K,N) \geq \mathbb{E}\,(B(n),N). \tag{5.20}$$

On the other hand, there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that: if $n \geq n_0$ and $n(\log n)^2 \leq N \leq \exp(cn)$ then, for N independent random points $x_1, \dots, x_N$ uniformly distributed in $B(n)$ we have

$$\mathrm{conv}(x_1, \dots, x_N) \supseteq \frac{\sqrt{\log(N/n)}}{6\sqrt{n}} B(n) \tag{5.21}$$

with probability greater than $1 - \exp(-n)$. We will give a different argument which proves (5.21) for all N satisfying $c_1 n \leq N \leq \exp(c_2 n)$. The idea comes from the work of Dyer, Füredi and McDiarmid in [14].

Proposition 5.4 *There exist $c_1, c_2 > 0$ which satisfy the following: Let $B(n)$ be the centered ball of volume 1 in $\mathbb{R}^n$. If $N \geq c_1 n$ and $x_1, \dots, x_N$ are independent random points uniformly distributed in $B(n)$, then*

$$\mathrm{conv}(x_1, \dots, x_N) \supseteq c_2 \min\left\{\frac{\sqrt{\log(N/n)}}{\sqrt{n}}, 1\right\} B(n) \tag{5.22}$$

with probability greater than $1 - \exp(-n)$.

Proof. Let r_n be the radius of $B(n)$ and let $\alpha \in (0,1)$ be a constant which will be suitably chosen. Consider the random polytope $K_N := \mathrm{conv}(x_1, \dots, x_N)$. With probability equal to one, K_N has non-empty interior and, for every $J = \{j_1, \dots, j_n\} \subset \{1, \dots, N\}$, the points $x_{j_1}, \dots, x_{j_n}$ are affinely independent. Write H_J for the affine subspace determined by $x_{j_1}, \dots, x_{j_n}$ and H_J^+, H_J^- for the two closed halfspaces whose bounding hyperplane is H_J.

If $\alpha B(n) \not\subseteq K_N$, then there exists $x \in \alpha B(n) \setminus K_N$, and hence, there is a facet of K_N which is contained in some H_J and satisfies the following: either $x \in H_J^-$ and $K_N \subset H_J^+$, or, $x \in H_J^+$ and $K_N \subset H_J^-$. Observe that, for every J,

$$\max\bigl\{\mathrm{Prob}(K_N \subseteq H_J^+), \mathrm{Prob}(K_N \subseteq H_J^-)\bigr\} \leq \bigl(\mu_{B(n)}(\{x_1 \leq \alpha r_n\})\bigr)^{N-n}. \tag{5.23}$$

It follows that

$$\mathrm{Prob}(\alpha B(n) \not\subseteq K_N) \leq 2\binom{N}{n}\big(\mu_{B(n)}(\{x_1 \leq \alpha r_n\})\big)^{N-n}. \tag{5.24}$$

A simple calculation shows that if $\frac{c}{\sqrt{n}} \leq \alpha \leq \frac{1}{4}$ then

$$\begin{aligned}\mu_{B(n)}\big(\{x_1 \geq \alpha r_n\}\big) &= \omega_{n-1} r_n^n \int_\alpha^1 (1-t^2)^{(n-1)/2} dt \geq \frac{\omega_{n-1}}{\omega_n}\alpha(1-4\alpha^2)^{(n-1)/2} \\ &\geq \exp\big(-4(n-1)\alpha^2\big) \geq \exp(-4\alpha^2 n)\end{aligned}$$

since $\sqrt{n}\omega_n \leq c\omega_{n-1}$ for some absolute constant $c > 0$. Going back to (5.24) we get

$$\begin{aligned}\mathrm{Prob}(\alpha B(n) \not\subseteq K_N) &\leq 2\binom{N}{n}\big(1-\exp(-4\alpha^2 n)\big)^{N-n} \\ &\leq \left(\frac{2eN}{n}\right)^n \exp\left(-(N-n)e^{-4\alpha^2 n}\right).\end{aligned}$$

With an easy computation we get that this probability is smaller than $\exp(-n)$ if $\alpha \simeq \min\left\{\sqrt{\log\left(N/n\right)}\Big/\sqrt{n},\ 1\right\}$, for all $N \geq c_1 n$, where $c_1 > 0$ is a (large enough) absolute constant (for this choice of α the restriction $\frac{c}{\sqrt{n}} \leq \alpha \leq \frac{1}{4}$ is also satisfied). This completes the proof. □

Observe that $\mathbb{E}\left(B(n), N\right)$ is increasing in N. It follows from (1.10) that if $n+1 \leq N \leq c_1 n$ then

$$\mathbb{E}\left(B(n), N\right) \geq \mathbb{E}(B(n), n+1) \simeq \frac{1}{\sqrt{n}} \geq c\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{5.28}$$

In view of Proposition 5.4 this shows that, for every $N > n$,

$$\mathbb{E}\left(B(n), N\right) \geq c\min\left\{\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}, 1\right\} \tag{5.29}$$

where $c > 0$ is an absolute constant. Combining with Theorem 5.1 we get Theorem C.

Remark 5.5 The referee informed us that Proposition 5.4 follows from an analogous result of Gluskin for gaussian random vectors. In [17] it is proved that the probability

$$\mathrm{Prob}\left(\mathrm{conv}\left(g_1, g_2, \ldots, g_N\right) \supseteq c\frac{\sqrt{\log(1+N/n)}}{\sqrt{n}}B(n)\right)$$

is close to 1. See also [22].

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