Comparing the M-position with some classical positions of convex bodies

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Abstract

The purpose of this article is to compare some classical positions of convex bodies. We provide exact quantitative results which show that the minimal surface area position and the minimal mean width position are not necessarily M-positions. We also construct examples of unconditional convex bodies of minimal surface area that exhibit the worst possible behavior with respect to their mean width or their minimal hyperplane projection.

1 Introduction

The following theorem of Milman ([14], see also [15]) establishes the existence of "M-ellipsoids" associated to any convex body: There exists an absolute constant $C > 0$ such that for every convex body K in $\mathbb{R}^n$ with center of mass at the origin, there exists an origin symmetric ellipsoid E_K such that $|K| = |E_K|$ and for every convex body T in $\mathbb{R}^n$ one has $\frac{1}{C}\big|E_K + T\big|^{1/n} \leqslant \big|K + T\big|^{1/n} \leqslant C\big|E_K + T\big|^{1/n}$ and $\frac{1}{C}|E_K^\circ + T|^{1/n} \leqslant |K^\circ + T|^{1/n} \leqslant C|E_K^\circ + T|^{1/n}$, where A° is the polar body of A.

Interchanging the roles of K and E_K, let us assume that $|K| = 1$ and the previous statement is satisfied by $E_K = D_n$, the Euclidean ball of volume 1 in $\mathbb{R}^n$. This is always possible if we apply a linear transformation to K. Then, setting $T = D_n$ we get $|K+D_n|^{1/n} \leqslant 2C$. In other words, there exists an absolute constant $\beta > 0$ such that every convex body K in $\mathbb{R}^n$ with center of mass at the origin has a linear image $\tilde{K}$ with $|\tilde{K}| = 1$ which satisfies

$$|K + D_n|^{1/n} \leqslant \beta. \tag{1.1}$$

We say that a convex body K in $\mathbb{R}^n$ which has volume 1, center of mass at the origin and satisfies (1.1) is in *M-position with constant β*. If K_1 and K_2 are two such convex bodies, then it is easily checked that $|K_1 + K_2|^{1/n} \leqslant C(\beta)\left(|K_1|^{1/n} + |K_2|^{1/n}\right)$ and $|K_1^\circ + K_2^\circ|^{1/n} \leqslant C(\beta)\left(|K_1^\circ|^{1/n} + |K_2^\circ|^{1/n}\right)$, where $C(\beta)$ is a constant depending only on β. This statement is the reverse Brunn-Minkowski inequality.

Recall the definition of the covering number $N(A,B)$ of two convex bodies A and B: this is the least integer N for which there exist N translates of B whose union covers A. It is quite easy to check that $|A + B| \leqslant N(A,B)|2B|$ and if B

is symmetric, $|A + B/2| \geqslant N(A, B)|B/2|$. Pisier (see [19, Chapter 7]) has given a different approach to Milman's theorem, which provides a special M-position of any convex body K with regularity estimates on the covering numbers $N(K, tB_2^n)$. The precise statement is as follows: For every $\alpha \in (0, 2)$ and every convex body K in $\mathbb{R}^n$, there exists an affine image $\tilde{K}$ of K which satisfies $|\tilde{K}| = |B_2^n|$ and

$$\max\{N(\tilde{K}, tB_2^n), N(B_2^n, t\tilde{K}), N(\tilde{K}^\circ, tB_2^n), N(B_2^n, t\tilde{K}^\circ)\} \leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right) \tag{1.2}$$

for every $t \geqslant 1$, where $c(\alpha)$ is a constant depending only on α, with $c(\alpha) = O\big(\frac{1}{2-\alpha}\big)$ as $\alpha \to 2$. We then say that $\tilde{K}$ is in M-position of order α (or α-regular M-position).

The purpose of this article is to compare some classical positions of a convex body with the M-position. A first example is the minimal surface area position. We say that K has minimal surface area if the surface area $\partial(K)$ of K is minimal among those of its affine images of the same volume. Petty [18] (see also [10]) gave a characterization of the minimal surface area position: K has minimal surface area if and only if the surface area measure σ_K of K is isotropic. In [8] similar characterizations were investigated for the extremal positions which correspond to other quermassintegrals. For example, it was proved that K has minimal mean width if and only if the measure ν_K with density h_K (where h_K is the support function of K) with respect to the rotationally invariant probability measure σ on S^{n-1} is isotropic. See Section 2 for notation and background information.

The question whether minimal surface area position is an M-position was posed in [8] and was recently answered in the negative by the third named author in [20]. We provide an alternative proof of this result in Section 4.

Theorem 1.1. *There exists an absolute constant $c > 0$ such that, for every $n \in \mathbb{N}$ there exists an unconditional convex body K of volume 1 in $\mathbb{R}^n$ which is in minimal surface area position and*

$$|K + D_n|^{1/n} \geqslant c\sqrt[8]{n}. \tag{1.3}$$

We also show that, up to the value of the isotropic constant L_K of K (see §3 for background information) the exponent $1/8$ in Theorem 1.1 is optimal in the symmetric case.

Theorem 1.2. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$|K + D_n|^{1/n} \leqslant C\sqrt[8]{n}L_K, \tag{1.4}$$

where $C > 0$ is an absolute constant.

We believe that the method which is presented in this article gives a natural explanation for this situation. Our example is the minimal surface position of the product of two convex bodies of volume 1; both of them have minimal surface area but of different order with respect to the dimension. In Section 3 we explain the main idea which, in the setting of the standard isotropic position, has its origin in a work of Bourgain, Klartag and Milman [4]. Using the same method, in Section 5 we show that the minimal mean width position is not an M-position.

Theorem 1.3. *There exists an absolute constant $c > 0$ such that, for every $n \in \mathbb{N}$ there exists an unconditional convex body K of volume 1 in $\mathbb{R}^n$ which is in minimal mean width position and*

$$|K + D_n|^{1/n} \geqslant c \sqrt[8]{\log n}. \tag{1.5}$$

We discuss two more questions about the geometry of convex bodies with minimal surface area. The first one concerns their hyperplane projections. K. Ball [2] has proved that every convex body K has an affine image $\tilde{K}$ of volume 1 such that for every unit vector θ,

$$|P_{\theta^\perp}(\tilde{K})| \geqslant 1. \tag{1.6}$$

In this result, $\tilde{K}$ is chosen so that the ellipsoid of minimal volume containing the polar projection body $\Pi^*(\tilde{K})$ is a Euclidean ball. It was proved in [10] that if K has minimal surface area and volume 1 then, with probability greater than $1-2^{-n}$ a hyperplane projection of K has volume greater than c, where $c > 0$ is an absolute constant. Actually, the statement is stronger as it depends on the value of the surface area of K; see Section 6 for details. In [10] it was asked if $|P_{\theta^\perp}(K)| \geqslant c$ holds true for *every* $\theta \in S^{n-1}$ in the minimal surface area position. We provide an optimal negative answer to this question.

Theorem 1.4. *There exists an unconditional convex body K of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies*

$$\min_{\theta \in S^{n-1}} |P_{\theta^\perp}(K)| \leqslant \frac{C}{\sqrt{n}}, \tag{1.7}$$

where $C > 0$ is an absolute constant.

In Section 7 we give an upper bound for the mean width of a symmetric convex body which has minimal surface area.

Theorem 1.5. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$w(K) \leqslant C \frac{n^{3/2}}{\partial_K}, \tag{1.8}$$

where ∂_K is the (minimal) surface area of K and $C > 0$ is an absolute constant.

Since $\partial_K \geqslant \partial_{D_n} \geqslant c\sqrt{n}$, an immediate consequence of Theorem 1.5 is the general upper bound $w(K) \leqslant Cn$. We don't know what is the optimal upper bound for $w(K)$ in the minimal surface area position. Nevertheless, in Section 7 we provide an example of an unconditional convex body K of volume 1 in $\mathbb{R}^n$ which has minimal surface area and mean width $w(K) \geqslant cn/\log n$, where $c > 0$ is an absolute constant. In other words, Theorem 1.5 is almost optimal.

In Section 7, we also give the upper bound $|K + D_n|^{1/n} \leqslant c\sqrt[4]{n}$ in the case of a symmetric convex body of volume 1 with minimal surface area: the exponent is worse than the one in Theorem 1.2, but we avoid the appearance of the isotropic constant L_K.

Finally, in Section 8 we show that John's position is also not an M-position. More precisely, we prove that there exists an unconditional convex body K in $\mathbb{R}^n$ which is in the "normalized John's position", such that $|K + D_n|^{1/n} \geqslant c\sqrt[8]{n}$, where $c > 0$ is an absolute constant.

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2 Classical positions and isotropic measures on the sphere

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, D_n for the Euclidean ball of volume 1 and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. Let $k \leqslant n$ and $F \in G_{n,k}$. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also define $B_F := B_2^n \cap F$ and $S_F := S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the books [6] and [21] for basic facts from the Brunn–Minkowski theory and to the books [17] and [19] for basic facts from the local theory of normed spaces. We also refer to [16] and [7] for more information on isotropic convex bodies.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$. We say that K is centered if it has center of mass at the origin, i.e. $\int_K \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$. In this paper, we will say that a symmetric convex body K in $\mathbb{R}^n$ is unconditional if $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for the norm $\| \cdot \|_K$ induced to $\mathbb{R}^n$ by K: this means that for every choice of real numbers $t_1, \ldots, t_n$ and every choice of signs $\varepsilon_j = \pm 1$ we have $\left\| \varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n \right\|_K = \left\| t_1 e_1 + \cdots + t_n e_n \right\|_K$. We will say that K is 1-symmetric if for every choice of real numbers $t_1, \ldots, t_n$, for every permutation σ of

$\{1,\dots,n\}$ and every choice of signs $\varepsilon_j=\pm 1$ we have $\big\|\varepsilon_1 t_{\sigma(1)}e_1+\cdots+\varepsilon_n t_{\sigma(n)}e_n\big\|_K = \big\|t_1e_1+\cdots+t_ne_n\big\|_K$.

The support function of a convex body K is defined by

$$h_K(y)=\max\{\langle x,y\rangle : x\in K\}, \tag{2.1}$$

and the mean width of K is

$$w(K)=\int_{S^{n-1}} h_K(\theta)d\sigma(\theta). \tag{2.2}$$

The radius of K is the quantity $R(K)=\max\{\|x\|_2 : x\in K\}$ and, if the origin is an interior point of K, we write $r(K)$ for the inradius of K (the largest $r>0$ for which $rB_2^n\subseteq K$) and we define the polar body K° of K by

$$K^\circ := \{y\in\mathbb{R}^n : \langle x,y\rangle\leqslant 1 \text{ for all } x\in K\}. \tag{2.3}$$

A Borel measure μ on S^{n-1} is called isotropic if

$$\int_{S^{n-1}}\langle x,\theta\rangle^2 d\mu(x)=\frac{\mu(S^{n-1})}{n} \tag{2.4}$$

for every $\theta\in S^{n-1}$. We will make frequent use of the next standard Lemma.

Lemma 2.1. *Let μ be a Borel measure on S^{n-1}. The following are equivalent:*

(i) *μ is isotropic.*

(ii) *For every $i,j=1,\dots,n$,*

$$\int_{S^{n-1}}\phi_i\phi_j d\mu(\phi)=\frac{\mu(S^{n-1})}{n}\delta_{i,j}. \tag{2.5}$$

(iii) *For every linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$,*

$$\int_{S^{n-1}}\langle\phi,T\phi\rangle d\mu(\phi)=\frac{\mathrm{tr}(T)}{n}\mu(S^{n-1}). \tag{2.6}$$

Proof. Setting $\theta=e_i$ and $\theta=\frac{e_i+e_j}{\sqrt{2}}$ in (2.4) we get (2.5). On observing that if $T=(t_{ij})_{i,j=1}^n$ then $\langle\phi,T\phi\rangle=\sum_{i,j=1}^n t_{ij}\phi_i\phi_j$, we readily see that (2.5) implies (2.6). Finally, note that applying (2.6) with $T(\phi)=\langle\phi,\theta\rangle\theta$ we get (2.4). $\square$

Next, we introduce the classical positions that we are going to discuss; we set the notation and provide some background information.

§1. Minimal surface area position. The surface measure of a convex body K is the Borel measure σ_K on S^{n-1} which is defined by

$$\sigma_K(A)=\nu(\{x\in \mathrm{bd}(K):\ u_K(x)\in A\}), \tag{2.7}$$

where $u_K(x)$ is the outer unit normal vector to K at x, and ν is the $(n-1)$-dimensional Lebesgue measure on $\mathrm{bd}(K)$. The surface area of K is equal to $\partial(K) = \sigma_K(S^{n-1})$. We say that a convex body K of volume 1 has *minimal surface area* if $\partial(K) \leqslant \partial(TK)$ for every $T \in SL(n)$. Petty ([18], see also [10]) gave the following characterization of the minimal surface area position: a convex body K of volume 1 in $\mathbb{R}^n$ has minimal surface area if and only if σ_K is isotropic. Equivalently, if

$$\partial(K) = n \int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) \tag{2.8}$$

for every $\theta \in S^{n-1}$. Moreover, this position is uniquely determined up to orthogonal transformations. For a convex body K of volume 1, we define the *minimal surface area parameter* ∂_K of K by

$$\partial_K = \min\{\partial(TK) : T \in SL(n)\}. \tag{2.9}$$

We also set

$$\overline{\partial}(n) = \max_{|K|=1} \partial_K \text{ and } \underline{\partial}(n) = \min_{|K|=1} \partial_K. \tag{2.10}$$

The isoperimetric inequality shows that $\underline{\partial}(n) = \partial(D_n) = n\omega_n^{1/n} \geqslant c_1\sqrt{n}$ where $c_1 > 0$ is an absolute constant. A sharp upper bound for $\overline{\partial}(n)$ was given by K. Ball in [1]. The extremal bodies are the cube C_n in the symmetric case and the regular simplex S_n in the general case. Since $\partial_{C_n} = 2n$ and $\partial_{S_n} \simeq n$, we see that $\overline{\partial}(n) \simeq n$.

§2. Minimal mean width position. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. We say that K is in *minimal mean width position* if $w(K) \leqslant w(TK)$ for every $T \in SL(n)$. It was proved in [8] that K has minimal mean width if and only if

$$w(K) = n \int_{S^{n-1}} \langle u, \theta\rangle^2 h_K(u)\sigma(du) \tag{2.11}$$

for every $\theta \in S^{n-1}$. Equivalently, K has minimal mean width if and only if the measure ν_K on S^{n-1} which has density h_K with respect to σ is isotropic. Moreover, this minimal mean width position is uniquely determined up to orthogonal transformations. For every centered convex body K of volume 1 we define the *minimal mean width parameter* w_K of K by

$$w_K = \min\{w(TK) : T \in SL(n)\}. \tag{2.12}$$

We also set

$$\overline{w}(n) = \max_{|K|=1} w_K \text{ and } \underline{w}(n) = \min_{|K|=1} w_K. \tag{2.13}$$

Urysohn's inequality states that among all convex bodies which have volume 1, the Euclidean ball D_n has the smallest mean width: it follows that $\underline{w}(n) = \omega_n^{-1/n} \simeq \sqrt{n}$, where $c_2 > 0$ is an absolute constant. It is also known that every centered convex body K has a linear image $\tilde{K}$ of volume 1 with $w(\tilde{K}) \leqslant c_3\sqrt{n}\log n$, where $c_3 > 0$ is an absolute constant. This follows from work of Lewis, Figiel-Tomczak-Jaegermann and Pisier (see [19] and [9] for references). Since $w_K \simeq \sqrt{n\log n}$ when $K = \overline{B}_1^n$ is the normalized ℓ_1^n-ball, we have

$$c_4\sqrt{n\log n} \leqslant \overline{w}(n) \leqslant c_3\sqrt{n}\log n. \tag{2.14}$$

3 Isotropic position and the main idea

The main idea behind the construction of our examples has its origin in the work [4] of Bourgain, Klartag and Milman about the classical isotropic position. We first recall some basic facts. If K is a centered convex body in $\mathbb{R}^n$ with center of mass at the origin, then there exists an ellipsoid $E_L(K)$ which satisfies

$$\int_{E_L(K)} \langle x, y\rangle^2 dx = \int_K \langle x, y\rangle^2 dx \tag{3.1}$$

for all $y \in \mathbb{R}^n$; in other words, $E_L(K)$ has the same moments of inertia as K ($E_L(K)$ is the Legendre ellipsoid of K). We say that K is in isotropic position if $|K| = 1$ and $E_L(K)$ is a multiple of B_2^n. This means that there exists a constant $L_K > 0$ with the property

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{3.2}$$

for all $\theta \in S^{n-1}$. Every convex body K has an isotropic position, which is uniquely determined up to orthogonal transformations. Therefore, the isotropic constant L_K of K is uniquely determined for the class $\{T(K) : T \in GL(n)\}$. The isotropic position of K is characterized as an extremal position in the following sense: K is isotropic if and only if

$$\int_K \|x\|_2^2 dx \leqslant \int_{T(K)} \|x\|_2^2 dx \tag{3.3}$$

for every $T \in SL(n)$. It is easily checked that $L_K \geqslant L_{D_n} \geqslant c > 0$ for every convex body K in $\mathbb{R}^n$, where $c > 0$ is an absolute constant. The question whether there exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every centered convex body K is open. Bourgain [3] proved that $L_K \leqslant c\sqrt[4]{n}\log n$ for every symmetric convex body K in $\mathbb{R}^n$. The best known general estimate is currently $L_K \leqslant c\sqrt[4]{n}$; this was proved by Klartag [12] – see also [13].

If we define $\overline{L}(n) = \max_K L_K$ and $\underline{L}(n) = \min_K L_K = L_{D_n}$, then the question is whether

$$\overline{L}(n) \leqslant C\underline{L}(n) \tag{3.4}$$

for some absolute constant $C > 0$. An equivalent question is if the quantity

$$I(n) = \max_{|K|=1} \min_{T\in SL(n)} \int_{TK} \|x\|_2^2 dx \tag{3.5}$$

is bounded by Cn, where $C > 0$ is an absolute constant. Our starting point is the following observation about the isotropic position (see [4], also [7]).

Lemma 3.1. *Let K and T be two isotropic convex bodies in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then, $W := (L_T/L_K)^{\frac{m}{n+m}} K \times (L_K/L_T)^{\frac{n}{n+m}} T$ is an isotropic convex body in $\mathbb{R}^{n+m}$, and*

$$L_{K\times T} = L_K^{\frac{n}{n+m}} L_T^{\frac{m}{n+m}}. \tag{3.6}$$

Using this fact one can obtain some information on the question if the isotropic position is an M-position. The next lemma contains a useful observation which will be used several times in the sequel.

Lemma 3.2. *Let W be a convex body of volume 1 in $\mathbb{R}^{2n}$. For any n-dimensional subspace F of $\mathbb{R}^{2n}$ one has*

$$|W + D_{2n}|^{\frac{1}{2n}} \geq c|P_F(W)|^{\frac{1}{2n}}, \tag{3.7}$$

where $c > 0$ is an absolute constant.

Proof. Since $N(W, D_{2n}) \leqslant 2^{2n}|W + D_{2n}|$ and $N(P_F(W), P_F(D_{2n})) \leqslant N(W, D_{2n})$, we may write

$$\begin{aligned} |W + D_{2n}|^{\frac{1}{2n}} &\geqslant \tfrac{1}{2}[N(W, D_{2n}]^{\frac{1}{2n}} \geqslant \tfrac{1}{2}[N(P_F(W), P_F(D_{2n}))]^{\frac{1}{2n}} \\ &\geqslant \tfrac{1}{2}\left(\frac{|P_F(W)|}{|P_F(D_{2n})|}\right)^{\frac{1}{2n}} \geqslant c|P_F(W)|^{\frac{1}{2n}}, \end{aligned}$$

because $|P_F(D_{2n})|^{\frac{1}{2n}} \simeq 1$. □

Proposition 3.3. *Let K and T be isotropic convex bodies in $\mathbb{R}^n$ with $L_K = \overline{L}(n)$ and $L_T = \underline{L}(n)$. Consider the isotropic convex body $W = aK \times bT$ in $\mathbb{R}^{2n}$, where $a = \sqrt{\frac{L_T}{L_K}}$ and $b = \sqrt{\frac{L_K}{L_T}}$. Then,*

$$|W + D_{2n}|^{\frac{1}{2n}} \geqslant c\left(\frac{\overline{L}(n)}{\underline{L}(n)}\right)^{\frac{1}{4}}. \tag{3.8}$$

Proof. Let E be the subspace spanned by the first n standard unit vectors in $\mathbb{R}^{2n}$ and let $F = E^{\perp}$. Then, $P_F(W) = bT$ and Lemma 3.2 shows that

$$|W + D_{2n}|^{\frac{1}{2n}} \geqslant c\sqrt{b} = c\left(\frac{L_K}{L_T}\right)^{\frac{1}{4}}. \tag{3.9}$$

Since $L_K = \overline{L}(n)$ and $L_T = \underline{L}(n)$, the result follows. $\square$

Proposition 3.3 shows that if there exists a constant $\beta_n > 0$ such that every isotropic convex body K in $\mathbb{R}^n$ is in M-position with constant β_n in the sense of (1.1), then

$$\overline{L}(n)/\underline{L}(n) \leqslant C\beta_n^4. \tag{3.10}$$

Therefore, if the ratio $\overline{L}(n)/\underline{L}(n)$ is not bounded, β_n cannot be bounded too (and hence, the isotropic position is not an M-position). We will use this construction to show that the minimal surface area position and the minimal mean width position are not M-positions uniformly in the dimension.

4 Minimal surface area position

We know that $\overline{\partial}(n) = \max_{|K|=1} \partial_K \simeq n$ and $\underline{\partial}(n) = \min_{|K|=1} \partial_K \simeq \sqrt{n}$. In other words,

$$\overline{\partial}(n) \geqslant c\sqrt{n}\underline{\partial}(n), \tag{4.1}$$

where $c > 0$ is an absolute constant. The next lemma will allow us to describe the minimal surface area position of the product of two convex bodies of minimal surface area in a case which is enough for our general construction.

Lemma 4.1. *Let P be a polytope of volume 1 in $\mathbb{R}^n$. Let $F_1, \ldots, F_N$ be the facets of P and let $u_1, \ldots, u_N$ be corresponding normal vectors. Let $a, b > 0$ with $a^n b^m = 1$ and define $Q := aP \times bC_m$, where $C_m = \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ is the unit cube in $\mathbb{R}^m$. Then,*
(i) *Q has $N + 2m$ facets, $G_1, \ldots, G_{N+2m}$, with corresponding normals $v_i = u_i$ if $1 \leqslant i \leqslant N$, $v_{N+i} = e_{n+i}$ and $v_{N+m+i} = -e_{n+i}$ if $1 \leqslant i \leqslant m$.*
(ii) *$|G_i| = \frac{|F_i|}{a}$ if $1 \leqslant i \leqslant N$ and $|G_i| = \frac{1}{b}$ if $N + 1 \leqslant i \leqslant N + 2m$.*
(iii) *The surface area of Q is given by*

$$\partial(Q) = \frac{\partial(P)}{a} + \frac{2m}{b}. \tag{4.2}$$

Proof. For each $k = 1, \ldots, m$ we define $Q^{(k)} = aP \times bC_k$. Note that if

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1, 1 \leqslant i \leqslant N\}, \tag{4.3}$$

then we can write

$$Q^{(1)} = \{(x, t) \in \mathbb{R}^{n+1} : \langle x, u_i \rangle \leqslant 1, 1 \leqslant i \leqslant N \text{ and } |t| \leqslant b/2\}. \tag{4.4}$$

So, if $v_i := u_i$ for $1 \leqslant i \leqslant N$ and $v_{N+1} = e_{n+1}$, $v_{N+2} = -e_{n+1}$, then v_j, $1 \leqslant j \leqslant N+2$, are the normals of $Q^{(1)}$. Moreover, we can easily compute the volume of the facets $G_i^{(1)}$ corresponding to v_i: If $1 \leqslant i \leqslant N$, we have

$$|G_i^{(1)}| = a^{n-1}b|F_i|, \tag{4.5}$$

and for $i = N+1, N+2$ we have that

$$|G_i^{(1)}| = a^n|P|. \tag{4.6}$$

Using induction we see that $Q^{(k)}$ has N facets of volume $a^{n-1}b^k|F_i|$, $1 \leqslant i \leqslant N$, and $2k$ facets of volume $a^n b^{k-1}|P|$. The volume of $Q^{(k)}$ is equal to $a^n b^k |P|$.

Setting $k = m$ we see that the first N facets of $Q = Q^{(m)}$ have volume

$$|G_i| = |G_i^{(m)}| = a^{n-1}b^m|F_i| = \frac{|F_i|}{a}, \tag{4.7}$$

because $a^n b^m = 1$, while Q has $2m$ additional facets, with normals $\pm e_i$, $n+1 \leqslant i \leqslant n+m$ and volume

$$|G_i| = a^n b^{m-1}|P| = \frac{1}{b}, \qquad N+1 \leqslant i \leqslant N+2m. \tag{4.8}$$

Using the above, we easily check that

$$\partial(Q) = \frac{1}{a}\sum_{i=1}^{N}|F_i| + \frac{2m}{b} = \frac{\partial(P)}{a} + \frac{2m}{b}. \tag{4.9}$$

This proves the Lemma. □

Lemma 4.2. *Let P and Q be as in Lemma* 4.1. *If P is in minimal surface area position, then Q is in minimal surface area position if and only if*

$$a = \left(\frac{\partial_P}{2n}\right)^{\frac{m}{n+m}} \qquad \textit{and} \qquad b = \left(\frac{2n}{\partial_P}\right)^{\frac{n}{n+m}}. \tag{4.10}$$

Moreover,

$$\partial_Q = \frac{n+m}{n}\partial_P^{\frac{n}{n+m}}(2n)^{\frac{m}{n+m}}. \tag{4.11}$$

Proof. Since P is in minimal surface position, for every $j,k = 1,\dots,n$ we have

$$\sum_{i=1}^{N}\langle u_i, e_j\rangle\langle u_i, e_k\rangle|F_i| = \frac{\partial_P}{n}\delta_{j,k} \tag{4.12}$$

from Lemma 2.1. Then, for every $j,k = 1,\dots,n$,

$$\sum_{i=1}^{N+2m}\langle v_i, e_j\rangle\langle v_i, e_k\rangle|G_i| = \sum_{i=1}^{N}\langle u_i, e_j\rangle\langle u_i, e_k\rangle\frac{|F_i|}{a} = \frac{\partial_P}{an}\delta_{j,k}. \tag{4.13}$$

If $n+1 \leqslant j,k \leqslant n+m$, then

$$\sum_{i=1}^{N+2m}\langle v_i, e_j\rangle\langle v_i, e_k\rangle|G_i| = \frac{1}{b}\sum_{s=n+1}^{n+m}\langle e_s, e_j\rangle\langle e_s, e_k\rangle = \frac{2\delta_{j,k}}{b}. \tag{4.14}$$

Finally, if $1 \leqslant j \leqslant n < k \leqslant n+m$, we easily check that

$$\sum_{i=1}^{N+2m} \langle v_i, e_j\rangle\langle v_i, e_k\rangle |G_i| = 0. \tag{4.15}$$

From Lemma 2.1 we conclude that Q will be in minimal surface area position, provided that

$$\frac{\partial_P}{an} = \frac{2}{b}. \tag{4.16}$$

Since $a^n b^m = 1$, this gives $a = \left(\frac{\partial_P}{2n}\right)^{\frac{m}{n+m}}$. Then, we can solve the equation $a^n b^m = 1$ to find b and, substituting into (4.9) we complete the proof. □

Lemma 4.3. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\partial(C) \leqslant n|C|/r(C). \tag{4.17}$$

In particular, if K is an isotropic symmetric convex body in $\mathbb{R}^n$ then

$$\partial_K \leqslant \partial(K) \leqslant n/L_K. \tag{4.18}$$

Proof. Using the monotonicity of mixed volumes (see [21]) we have

$$\partial(C) = nV(C, \dots, C, B_2^n) \leqslant nV\left(C, \dots, C, \tfrac{1}{r(C)}C\right) = \frac{n|C|}{r(C)}. \tag{4.19}$$

Then, (4.18) follows from (4.17) and the fact that, in the isotropic position, $h_K^2(\theta) \geqslant \int_K \langle x, \theta\rangle^2 dx = L_K^2$, and hence, $r(K) \geqslant L_K$. □

We are now ready to give our example. We choose $P = \overline{B}_1^n$, the unit ball of ℓ_1^n, normalized so that it will have volume 1; in general, we set $\overline{A} = |A|^{-1/n}A$ for a compact set of positive Lebesgue measure.

Theorem 4.4. *Let $a, b > 0$ so that $K = a\overline{B}_1^n \times bC_n$ is in minimal surface area position. Then,*

$$|K + D_{2n}|^{\frac{1}{2n}} \geqslant c\sqrt[8]{n} \tag{4.20}$$

where $c > 0$ is an absolute constant.

Proof. Since $\overline{B}_1^n$ is 1-symmetric, Lemma 2.1 implies that it is in minimal surface area position. We know that $B_1^n \supseteq \frac{1}{\sqrt{n}}B_2^n$ because $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$ for all $x \in \mathbb{R}^n$. Observe that $\overline{B}_1^n \simeq nB_1^n$, and hence,

$$\overline{B}_1^n \supseteq c_1\sqrt{n}B_2^n. \tag{4.21}$$

Then, Lemma 4.3 shows that

$$\partial(\overline{B}_1^n) \leqslant n/r(\overline{B}_1^n) \leqslant c_2\sqrt{n}, \tag{4.22}$$

where $c_2 = c_1^{-1}$. From the isoperimetric inequality we see that $\partial(\overline{B}_1^n) \geqslant \partial(D_n) \geqslant c_3\sqrt{n}$, and this shows that

$$\partial(\overline{B}_1^n) \simeq \sqrt{n}. \tag{4.23}$$

From Lemma 4.2 we have $a = \sqrt{\frac{\partial(\overline{B}_1^n)}{2n}}$, $b = \sqrt{\frac{2n}{\partial(\overline{B}_1^n)}}$ and

$$\partial_K = 2\sqrt{2n\partial(\overline{B}_1^n)} \simeq n^{\frac{3}{4}}. \tag{4.24}$$

We apply the reasoning of Lemma 3.2: If E is the subspace spanned by the first n standard unit vectors in $\mathbb{R}^{2n}$ and $F = E^{\perp}$ then,

$$|K + D_{2n}|^{\frac{1}{2n}} \geqslant c\sqrt{b} \geqslant c\left(\frac{2n}{\partial(\overline{B}_1^n)}\right)^{\frac{1}{4}} \simeq \sqrt[8]{n}, \tag{4.25}$$

because of (4.23). □

Our next result shows that, at least in the symmetric case, the estimate of Theorem 4.4 is optimal up to the value of the isotropic constant of the body.

Theorem 4.5. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$|K + D_n|^{1/n} \leqslant C\sqrt[8]{n}L_K, \tag{4.26}$$

where $C > 0$ is an absolute constant.

We need the following observation about the surface area of an isotropic convex body.

Proposition 4.6. *Let K be a symmetric convex body which has minimal surface area and let $T \in SL(n)$. Then*

$$\frac{\mathrm{tr}(T)}{n}\partial_K \leqslant \partial(T^{-*}(K)) \leqslant \frac{\|T\|_{\mathrm{HS}}}{\sqrt{n}}\partial_K, \tag{4.27}$$

where $\|T\|_{\mathrm{HS}}^2 = \mathrm{tr}(T^*T)$ is the Hilbert-Schmidt norm of T.

Proof. Using Lemma 2.1 for the isotropic measure σ_K, we have that

$$\partial_K\frac{\mathrm{tr}(T)}{n} = \int_{S^{n-1}} \langle\theta, T\theta\rangle\, d\sigma_K(\theta) \leqslant \int_{S^{n-1}} \|T\theta\|_2 d\sigma_K(\theta). \tag{4.28}$$

Since $\|T\theta\|_2 = h_{T^*(B_2^n)}(\theta)$, by the integral representation of the mixed volumes and the fact that, for every affine transformation A of $\mathbb{R}^n$ and any n-tuple $K_1,\dots,K_n$ of convex bodies we have

$$V(A(K_1),\dots,A(K_n)) = |\det A|V(K_1,\dots,K_n), \tag{4.29}$$

(see [21]) we get

$$\begin{aligned}\partial_K \frac{\mathrm{tr}(T)}{n} &\leqslant \int_{S^{n-1}} h_{T^*(B_2^n)}(\theta) d\sigma_K(\theta) = nV(K,\dots,K,T^*(B_2^n)) \\ &= nV(T^*(T^{-*}(K)),\dots,T^*(T^{-*}(K)),T^*(B_2^n)) \\ &= n\,|\det T^*|V(T^{-*}(K),\dots,T^{-*}(K),B_2^n) \\ &= \partial(T^{-*}(K)).\end{aligned}$$

This proves the left-hand side inequality. On the other hand, if we set $S := T^*T$ then, using Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned}\partial_K \frac{\|T\|_{\mathrm{HS}}^2}{n} &= \partial_K \frac{\mathrm{tr}(S)}{n} = \int_{S^{n-1}} \langle \theta, S\theta\rangle d\sigma_K(\theta) = \int_{S^{n-1}} \|T\theta\|_2^2 d\sigma_K(\theta) \\ &\geqslant \frac{1}{\partial_K}\left(\int_{S^{n-1}} \|T\theta\|_2 d\sigma_K(\theta)\right)^2,\end{aligned}$$

which shows that

$$\partial_K \frac{\|T\|_2}{\sqrt{n}} \geqslant \int_{S^{n-1}} \|T\theta\|_2 d\sigma_K(\theta) = \partial(T^{-*}(K)) \tag{4.30}$$

(for the last equality see [10]). This proves the right hand side inequality. □

Remark. Note that the estimates in the previous proposition are sharp (at least up to a universal constant) as the examples of the cube and the Euclidean ball show.

We are now able to prove Theorem 4.5.

Proof of Theorem 4.5. Since both the minimal surface area position and the isotropic position are preserved by orthogonal transformations, we may assume that there exists a diagonal positive definite operator $T = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ in $SL(n)$ such that $K = T(\tilde{K})$ and $\tilde{K}$ is isotropic. We may assume that $\lambda_1,\dots,\lambda_m \geqslant 1$ and $0 < \lambda_{m+1},\dots,\lambda_n < 1$ for some $1 \leqslant m \leqslant n-1$. Observe that

$$|C_n + T(C_n)| = \prod_{i=1}^n \left(\tfrac{1+\lambda_i}{2}\right) \leqslant \prod_{i=1}^m \lambda_i. \tag{4.31}$$

Since $T \in SL(n)$ and $\tilde{K}$ is isotropic, for every $i = 1,\dots,n$ we have (see [16])

$$L_K \simeq \int_{\tilde{K}} |\langle x, e_i\rangle|\, dx = \frac{1}{\lambda_i}\int_K |\langle x, e_i\rangle|\, dx \simeq \frac{1}{\lambda_i}\frac{1}{|K\cap e_i^\perp|}. \tag{4.32}$$

It follows that

$$\frac{1}{\lambda_i} \simeq L_K|K\cap e_i^\perp| \leqslant |P_{e_i^\perp}(K)|L_K \leqslant \frac{\partial_K L_K}{2\sqrt{n}}, \tag{4.33}$$

for every $i = 1, \dots, n$. Moreover, using Proposition 4.6 we see that $\partial(\tilde{K}) = \partial(T^{-1}(K)) \geqslant \frac{\mathrm{tr}(T)}{n}\partial_K$. From Lemma 4.3 it follows that

$$1 \leqslant \frac{\mathrm{tr}(T)}{n} \leqslant \frac{\partial(\tilde{K})}{\partial_K} \leqslant \frac{n}{\partial_K L_K}. \tag{4.34}$$

Claim. We have

$$\left(\prod_{i=1}^{m} \lambda_i\right)^{1/n} \leqslant \sqrt[8]{n}. \tag{4.35}$$

Proof of the Claim. We write $\partial_K L_K = n^{\frac{1}{2}+\kappa}$ for some $0 \leqslant \kappa \leqslant 1/2$ and we distinguish two cases. First, assume that $8\big(\frac{1}{2} - \kappa\big)m \leq n$. Then,

$$\begin{aligned}\left(\prod_{i=1}^{m} \lambda_i\right)^{\frac{1}{n}} &\leqslant \left(\frac{\sum_{i=1}^{m} \lambda_i}{m}\right)^{\frac{m}{n}} \leqslant \left(\frac{n}{m}\right)^{\frac{m}{n}} \left(\frac{\mathrm{tr}(T)}{n}\right)^{\frac{m}{n}} \\ &\leqslant c_1 \left(\frac{n}{\partial_K L_K}\right)^{\frac{m}{n}} = c_1 (n^{\frac{1}{2}-\kappa})^{\frac{m}{n}} \leqslant c_1 \sqrt[8]{n}.\end{aligned}$$

Next, assume that $8\big(\frac{1}{2} - \kappa\big)m > n$. This implies that $n - m < \frac{3-8\kappa}{4-8\kappa}n$. Then, (4.33) shows that

$$\prod_{i=1}^{m} \lambda_i = \prod_{i=m+1}^{n} \frac{1}{\lambda_i} \leqslant \left(\frac{c_3 \partial_K L_K}{\sqrt{n}}\right)^{n-m} = (c_3 n^{\kappa})^{n-m} = c_3^{n-m} n^{\kappa(n-m)}, \tag{4.36}$$

and hence,

$$\left(\prod_{i=1}^{m} \lambda_i\right)^{1/n} \leqslant c_4 n^{\frac{\kappa(n-m)}{n}} \leqslant c_4 n^{g(\kappa)}, \tag{4.37}$$

where $g : [0, 1/2] \to \mathbb{R}$ is the function defined by $g(\kappa) = \frac{3\kappa - 8\kappa^2}{4-8\kappa}$. Since g attains its maximum at $\kappa = 1/4$, the claim follows. □

We now use the fact (see, for example, [20]) that if A, B, C are symmetric convex bodies of volume 1 in $\mathbb{R}^n$ then

$$|A + B|^{1/n} \leqslant c_5 |A + C|^{1/n} |B + C|^{1/n}. \tag{4.38}$$

Then, we can write

$$\begin{aligned}|K + D_n|^{1/n} &\leqslant c_5 |T(\tilde{K}) + C_n|^{1/n} |C_n + D_n|^{1/n} \\ &\leqslant c_5^2 |T(\tilde{K}) + T(C_n)|^{1/n} |T(C_n) + C_n|^{1/n} |C_n + D_n|^{1/n}.\end{aligned}$$

Observe that

$$|T(\tilde{K}) + T(C_n)|^{1/n} = |\tilde{K} + C_n|^{1/n} \leqslant c_6 L_K, \tag{4.39}$$

because $|\tilde{K}+C_n|^{1/n} \leqslant c_5[N(\tilde{K},L_KD_n)]^{1/n}|L_KD_n+C_n|^{1/n}$ and $N(\tilde{K},L_KD_n) \leqslant \exp(cn)$ (see e.g. [7, Theorem 1.6.4]) while $|C_n+D_n|^{1/n} \leqslant c_7$, and hence, $|L_KD_n+C_n|^{1/n} \leqslant c_8L_K$. It follows that

$$|K+D_n|^{1/n} \leqslant c_9L_K|T(C_n)+C_n|^{1/n} \leqslant c_9L_K\left(\prod_{i=1}^{m}\lambda_i\right)^{1/n}, \tag{4.40}$$

from (4.31). Using the Claim we conclude the proof. □

Remark. One can extend Theorem 4.5 to the not necessarily symmetric case, in the following sense. Let K be a convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Consider the Blaschke sum ∇K of K and $-K$: this is the convex body whose surface area measure is $\sigma_K+\sigma_{-K}$. Then, ∇K is a symmetric convex body and Lemma 2.1 shows that it has minimal surface area. Moreover, one can check that $|\nabla K| \simeq 1$ and $|K+D_n|^{1/n} \leqslant c_1|\overline{\nabla K}+D_n|^{1/n}$ (we omit the details). Applying Theorem 4.5 to $\overline{\nabla K}$, we conclude that

$$|K+D_n|^{1/n} \leqslant c_2\sqrt[8]{n}L_{\nabla K}. \tag{4.41}$$

5 Minimal mean width position

We know that $\overline{w}(n)=\max_{|K|=1}\partial_K \geqslant c_1\sqrt{n\log n}$ and $\underline{w}(n)=\min_{|K|=1}w_K \simeq \sqrt{n}$. In other words,

$$\overline{w}(n) \geqslant c\sqrt{\log n}\,\underline{w}(n), \tag{5.1}$$

where $c>0$ is an absolute constant. The next lemma will allow us to describe the minimal mean width position of the product of two convex bodies of minimal mean width.

Lemma 5.1. *Let K and P be two convex bodies in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $a,b>0$ and define $Q:=aK\times bP$. Then,*

$$\sqrt{n+m}\,w(Q) \simeq a\sqrt{n}w(K)+b\sqrt{m}w(P). \tag{5.2}$$

Proof. We will use the fact that, for every convex body V in $\mathbb{R}^k$,

$$\sqrt{k}w(V)=c_k\int_{\mathbb{R}^k}h_V(z)\,d\gamma_k(z), \tag{5.3}$$

where $c_k \simeq 1$ is a positive constant depending only on the dimension and γ_k is the standard k-dimensional Gaussian measure; this is easily verified by integration in polar coordinates. Observe that, if $z=(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$, then

$$\begin{aligned} h_Q(z) &= h_{aK\times bP}(x,y)=\sup\{\langle(au,bv),(x,y)\rangle : u\in K, v\in P\} \\ &= a\sup_{u\in K}\langle u,x\rangle + b\sup_{v\in P}\langle v,y\rangle = ah_K(x)+bh_P(y). \end{aligned}$$

Therefore,

$$\begin{aligned}
\sqrt{n+m}w(Q) &= c_{n+m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\big(ah_K(x)+bh_P(y)\big)\,d\gamma_n(x)\,d\gamma_m(y)\\
&= c_{n+m}a\int_{\mathbb{R}^n}h_K(x)\,d\gamma_n(x)+c_{n+m}b\int_{\mathbb{R}^m}h_P(y)\,d\gamma_m(y)\\
&= \frac{c_{n+m}}{c_n}a\sqrt{n}w(K)+\frac{c_{n+m}}{c_m}b\sqrt{m}w(P)\\
&\simeq a\sqrt{n}w(K)+b\sqrt{m}w(P),
\end{aligned}$$

because $c_{n+m}\simeq c_n\simeq c_m\simeq 1$. □

Lemma 5.2. *Let K and P be two 1-symmetric convex bodies of volume 1 in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume that K and P have minimal mean width. If $a,b>0$ and $a^nb^m=1$ are chosen so that $Q=aK\times bP$ will have minimal mean width, then*

$$a\simeq\left(\frac{n}{m}\right)^{\frac{m}{2(n+m)}}\left(\frac{w_P}{w_K}\right)^{\frac{m}{n+m}}\quad\text{and}\quad b\simeq\left(\frac{m}{n}\right)^{\frac{n}{2(n+m)}}\left(\frac{w_K}{w_P}\right)^{\frac{n}{n+m}}. \tag{5.4}$$

Moreover,

$$w_Q\simeq\frac{(n+m)^{\frac12}}{m^{\frac{m}{2(n+m)}}n^{\frac{n}{2(n+m)}}}w_K^{\frac{n}{n+m}}w_P^{\frac{m}{n+m}}. \tag{5.5}$$

Proof. Let E be the subspace spanned by the first n standard unit vectors in $\mathbb{R}^{n+m}$. Since K and P are 1-symmetric, there exists a diagonal volume preserving transformation of the form $aI_E\times bI_{E^\perp}$ which brings $K\times T$ to the minimal mean width position. Let $Q=aK\times bP$. Then, we must have

$$c_{n+m}\int_{\mathbb{R}^{n+m}}h_Q(z)\langle z,e_i\rangle^2d\gamma_{n+m}(z)=\sqrt{n+m}\frac{w(Q)}{n+m}=\frac{w(Q)}{\sqrt{n+m}} \tag{5.6}$$

for all $i=1,\dots,2n$. Working as in the proof of Lemma 4.2 we see that

$$\frac{aw(K)}{\sqrt{n}}\simeq\frac{bw(P)}{\sqrt{m}}\simeq\frac{w(Q)}{\sqrt{n+m}}. \tag{5.7}$$

From the condition $a^nb^m=1$ and (5.7) we see that

$$a\simeq\left(\frac{n}{m}\right)^{\frac{m}{2(n+m)}}\left(\frac{w_P}{w_K}\right)^{\frac{m}{n+m}}. \tag{5.8}$$

Then, we can solve the equation $a^nb^m=1$ to find b and substituting into (5.2) we complete the proof. □

We are now able to give our example. We choose $K=\overline{B}_1^n$ and $P=C_n$. Note that

$$w_{C_n}\simeq\sqrt{n}\qquad\text{and}\qquad w_{\overline{B}_1^n}\simeq\sqrt{n\log n}. \tag{5.9}$$

Theorem 5.3. *Let $a, b > 0$ so that $Q = a\overline{B}_1^n \times bC_n$ is in minimal mean width position. Then,*

$$|K + D_{2n}|^{\frac{1}{2n}} \geqslant c\sqrt[8]{\log n}, \tag{5.10}$$

where $c > 0$ is an absolute constant.

Proof. From Lemma 5.2 we have $a \simeq \sqrt{\frac{w_{C_n}}{w_{\overline{B}_1^n}}} \simeq \frac{1}{\sqrt[4]{\log n}}$, $b \simeq \sqrt{\frac{w_{\overline{B}_1^n}}{w_{C_n}}} \simeq \sqrt[4]{\log n}$ and

$$w_Q \simeq \sqrt{w_{\overline{B}_1^n} w_{C_n}} \simeq \sqrt{n}\sqrt[4]{\log n}. \tag{5.11}$$

We apply the reasoning of Lemma 3.2: If E is the subspace spanned by the first n standard unit vectors in $\mathbb{R}^{2n}$ and $F = E^{\perp}$ then,

$$|K + D_{2n}|^{\frac{1}{2n}} \geqslant c\sqrt{b} \simeq \sqrt[8]{\log n}, \tag{5.12}$$

because $b \simeq \sqrt[4]{\log n}$. □

Remark. From Urysohn's inequality it is clear that if K is a convex body of volume 1 in $\mathbb{R}^n$ which has minimal mean width, then

$$|K + D_n|^{1/n} \leqslant c_1 \frac{w(K + D_n)}{\sqrt{n}} = c_1 \frac{w(K) + w(D_n)}{\sqrt{n}} \leqslant c_2 \log n, \tag{5.13}$$

because $w(D_n) \leqslant w(K) \leqslant \overline{w}(n) \leqslant c\sqrt{n}\log n$.

6 Hyperplane projections in the minimal surface area position

Let K be a convex body in $\mathbb{R}^n$. The projection body ΠK of K is the symmetric convex body whose support function is defined by $h_{\Pi K}(\theta) = |P_{\theta^{\perp}}(K)|$, $\theta \in S^{n-1}$. We write $\Pi^* K$ for the polar projection body. It was proved in [10] that the volume radius of ΠK and $\Pi^* K$ are determined by the minimal surface area parameter ∂_K: If $|K| = 1$, then

$$|\Pi^* K|^{1/n} \simeq \frac{1}{\partial_K} \quad \text{and} \quad |\Pi K|^{1/n} \simeq \frac{\partial_K}{n} \tag{6.1}$$

Actually, the upper and lower estimates given in [10] are sharp; they become equalities when K is either the cube or the Euclidean unit ball.

Recall that, from Cauchy's formula, the area of the $(n-1)$–dimensional projection $P_{\theta^{\perp}}(K)$ of K, $\theta \in S^{n-1}$, can be written in the form

$$|P_{\theta^{\perp}}(K)| = \frac{1}{2}\int_{S^{n-1}} |\langle u, \theta\rangle| d\sigma_K(u). \tag{6.2}$$

Assume that K is in minimal surface area position. From the Cauchy–Schwarz inequality we have

$$\begin{aligned} |P_{\theta^\perp}(K)| &= \frac{1}{2}\int_{S^{n-1}} |\langle u,\theta\rangle|\, d\sigma_K(u) \leqslant \frac{1}{2}\left(\int_{S^{n-1}} |\langle u,\theta\rangle|^2 d\sigma_K(u)\right)^{1/2} \sqrt{\partial_K} \\ &= \frac{\partial_K}{2\sqrt{n}} \end{aligned}$$

for all $\theta \in S^{n-1}$. On the other hand,

$$|P_{\theta^\perp}(K)| \geqslant \frac{1}{2}\int_{S^{n-1}} \langle u,\theta\rangle^2 d\sigma_K(u) = \frac{\partial_K}{n}. \tag{6.3}$$

Writing the volume of $\Pi^*(K)$ in polar coordinates and using (6.1) we get

$$\int_{S^{n-1}} \frac{1}{|P_{\theta^\perp}(K)|^n}\sigma(d\theta) = \frac{|\Pi^* K|}{\omega_n} \leqslant \left(\frac{C\sqrt{n}}{\partial_K}\right)^n, \tag{6.4}$$

and Markov's inequality shows that

$$\frac{c\partial_K}{\sqrt{n}} \leqslant |P_\theta(K)| \leqslant \frac{\partial_K}{2\sqrt{n}} \tag{6.5}$$

for all θ in a subset of S^{n-1} of measure greater than $1-2^{-n}$, where $c>0$ is an absolute constant. Since $\partial_K \geqslant c_1\sqrt{n}$ by the isoperimetric inequality, this implies that with high probability the projections of a convex body with minimal surface area satisfy

$$|P_{\theta^\perp}(K)| \geqslant c. \tag{6.6}$$

It was asked in [10] if (6.6) holds true *for every* $\theta \in S^{n-1}$. We will show that the answer to this question is negative.

Theorem 6.1. *There exists an unconditional convex body K of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies*

$$\min_{\theta\in S^{n-1}} |P_{\theta^\perp}(K)| \leqslant \frac{C}{\sqrt{n}}, \tag{6.7}$$

where $C>0$ is an absolute constant.

Proof. Let $k,m \in \mathbb{N}$ with $k+m=n$ and $a,b>0$ and define $K = a\overline{B}_1^k \times bC_m$. From Lemma 4.2 we know that K has minimal surface area if

$$a = \left(\frac{\partial_{\overline{B}_1^k}}{2k}\right)^{\frac{m}{k+m}} \text{ and } b = \left(\frac{2k}{\partial_{\overline{B}_1^k}}\right)^{\frac{k}{k+m}}. \tag{6.8}$$

Moreover,

$$\partial_K := \frac{k+m}{k}\partial_{\overline{B}_1^k}^{\frac{k}{k+m}}(2k)^{\frac{m}{k+m}}. \tag{6.9}$$

We choose $m \simeq \frac{k}{\log k}$. Note that $k \leqslant n \leqslant 2k$. Then, since $\partial_{\overline{B}_1^k} \simeq \sqrt{k}$, we check that

$$a \simeq 1,\ b \simeq \sqrt{k} \simeq \sqrt{n} \text{ and } \partial_K \simeq \sqrt{k} \simeq \sqrt{n}. \tag{6.10}$$

Write $z = (x, y)$ for a point in $\mathbb{R}^k \times \mathbb{R}^m$. It is easily checked that

$$\int_K \langle z, e_i\rangle^2 dz = \int_{a\overline{B}_1^k} \int_{bC_m} y_i^2 dy\, dx = a^k \int_{bC_m} y_i^2 dy = a^k b^{m+2} \int_{C_m} u_i^2 du \simeq b^2 \tag{6.11}$$

for all $i = k+1, \dots, n$, and similarly,

$$\int_K \langle z, e_i\rangle^2 dx \simeq a^2 \tag{6.12}$$

for all $i = 1, \dots, k$. We also know that

$$\int_K \langle z, e_i\rangle^2 dz \simeq \frac{1}{|K \cap e_i^\perp|^2} = \frac{1}{|P_{e_i^\perp}(K)|^2}. \tag{6.13}$$

For the first assertion, see [16]; the second equality is clear, because K is unconditional. Combining (6.11) and (6.13) we see that

$$|P_{e_i^\perp}(K)| \simeq \frac{1}{b} \simeq \frac{1}{\sqrt{n}} \tag{6.14}$$

for all $i = k+1, \dots, n$. In other words,

$$\beta := \min\{|P_{\theta^\perp}(K)| : \theta \in S^{n-1}\} \leqslant \frac{C}{\sqrt{n}}, \tag{6.15}$$

for some absolute constant $C > 0$. □

Remark. The estimate of Theorem 6.1 is clearly optimal. From (6.3) we know that if K has volume 1 and minimal surface area then, for every $\theta \in S^{n-1}$,

$$|P_{\theta^\perp}(K)| \geqslant \frac{\partial_K}{2n} \geqslant \frac{\partial_{D_n}}{2n} = \frac{\omega_n^{1/n}}{2} \geqslant \frac{c}{\sqrt{n}}. \tag{6.16}$$

7 Mean width in the minimal surface area position

In this Section we give an upper bound for the mean width of a symmetric convex body K of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Our bound is "close to the minimal order" $\sqrt{n}$ when the minimal surface parameter ∂_K of K is "large".

Theorem 7.1. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$w(K) \leqslant C\frac{n^{3/2}}{\partial_K}, \tag{7.1}$$

where $C > 0$ is an absolute constant.

Proof. It is proved in [10] that every surface isotropic convex body is the limit of a sequence of surface isotropic polytopes in the Hausdorff metric. Therefore, we may assume that K is a polytope with facets F_j and normals u_j, $j = 1, \dots, m$, which has isotropic surface measure. Then, the isotropic condition for σ_K is equivalent to the representation of the identity

$$I = \sum_{j=1}^{m} \frac{n|F_j|}{\partial_K} u_j \otimes u_j. \tag{7.2}$$

The fact that $|K| = 1$ can be expressed in the form

$$\sum_{j=1}^{m} h_K(u_j)|F_j| = n. \tag{7.3}$$

For every $\theta \in \mathbb{R}^n$ we have

$$\theta = \frac{n}{\partial_K} \sum_{j=1}^{m} |F_j| \langle \theta, u_j \rangle u_j, \tag{7.4}$$

and hence,

$$h_K(\theta) \leqslant \frac{n}{\partial_K} \sum_{j=1}^{m} |F_j| \, |\langle \theta, u_j \rangle| \, h_K(u_j). \tag{7.5}$$

We integrate (7.5) over the sphere. Using (7.3) and the fact that, for some constant $c_n \simeq 1$, we have

$$\int_{S^{n-1}} |\langle \theta, u \rangle| \, d\sigma(\theta) = \frac{c_n}{\sqrt{n}} \tag{7.6}$$

for every $u \in S^{n-1}$, we write

$$\begin{aligned} w(K) &= \int_{S^{n-1}} h_K(\theta) \, d\sigma(\theta) \leqslant \frac{n}{\partial_K} \sum_{j=1}^{m} |F_j| \, h_K(u_j) \int_{S^{n-1}} |\langle \theta, u_j \rangle| \, d\sigma(\theta) \\ &= \frac{c_n \sqrt{n}}{\partial_K} \sum_{j=1}^{m} |F_j| \, h_K(u_j) = c_n \sqrt{n} \frac{n}{\partial_K}. \end{aligned}$$

This proves our claim. □

Corollary 7.2. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$w(K) \leqslant Cn, \tag{7.7}$$

where $C > 0$ is an absolute constant.

Proof. It is an immediate consequence of the isoperimetric inequality and Theorem 7.1: we know that $\partial_K \geqslant \partial_{D_n} \geqslant c\sqrt{n}$. □

Remark. It is well-known that if K is a symmetric convex body of volume 1 in $\mathbb{R}^n$, then

$$|K \cap \theta^{\perp}|\, h_K(\theta) \leqslant \frac{n}{2} \tag{7.8}$$

for all $\theta \in S^{n-1}$. Using the idea of the proof of Proposition 7.1, we can check that the expectation of $|P_{\theta^\perp}(K)|\, h_K(\theta)$ satisfies the same bound when K has minimal surface area.

Proposition 7.3. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$\int_{S^{n-1}} |P_{\theta^\perp}(K)|\, h_K(\theta)\, d\sigma(\theta) \leqslant \frac{n}{2}. \tag{7.9}$$

Proof. We write

$$\begin{aligned}\int_{S^{n-1}} |P_{\theta^\perp}(K)|\, h_K(\theta)\, d\sigma(\theta) &= \frac{1}{2}\int_{S^{n-1}}\int_{S^{n-1}} h_K(\theta)|\langle\theta, x\rangle|\, d\sigma_K(x)\, d\sigma(\theta)\\ &= \frac{1}{2}\int_{S^{n-1}}\left(\int_{S^{n-1}} h_K(\theta)|\langle\theta, x\rangle|\, d\sigma(\theta)\right) d\sigma_K(x),\end{aligned}$$

and using (7.5) we get

$$\int_{S^{n-1}} h_K(\theta)|\langle\theta, x\rangle|\, d\sigma(\theta) \leqslant \frac{n}{\partial_K}\sum_{j=1}^{m} |F_j| h_K(u_j)\int_{S^{n-1}} |\langle\theta, x\rangle|\, |\langle\theta, u_j\rangle|\, d\sigma(\theta). \tag{7.10}$$

A simple application of the Cauchy–Schwarz inequality shows that

$$\int_{S^{n-1}} |\langle\theta, x\rangle|\, |\langle\theta, u_j\rangle|\, d\sigma(\theta) \leqslant \left(\int_{S^{n-1}} \langle\theta, x\rangle^2 d\sigma(\theta)\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} \langle\theta, u_j\rangle^2 d\sigma(\theta)\right)^{\frac{1}{2}} = \frac{1}{n}. \tag{7.11}$$

Therefore, using (7.3) we can write

$$\int_{S^{n-1}} h_K(\theta)|\langle\theta, x\rangle|\, d\sigma(\theta) \leqslant \frac{n}{\partial_K}\sum_{j=1}^{m} \frac{|F_j| h_K(u_j)}{n} = \frac{n}{\partial_K}. \tag{7.12}$$

Going back, we see that

$$\int_{S^{n-1}} |P_{\theta^\perp}(K)|\, h_K(\theta)\, d\sigma(\theta) \leqslant \frac{n}{2\partial_K}\int_{S^{n-1}} d\sigma_K(x) = \frac{n}{2}, \tag{7.13}$$

and the result follows. □

Using the results of Sections 4 and 5 we can give an example of a convex body K of volume 1 in $\mathbb{R}^n$ which has minimal surface area and mean width as large as $n/\log n$. In other words, Corollary 7.2 is almost optimal.

Theorem 7.4. *There exists an unconditional convex body Q of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies*

$$w(Q) \geqslant \frac{cn}{\log n}, \tag{7.14}$$

where $c > 0$ is an absolute constant.

Proof. Let $k, m \in \mathbb{N}$ with $k + m = n$ and $a, b > 0$ and define $Q := a\overline{B}_1^k \times bC_m$. Working as in Lemma 4.1 and Lemma 4.2 we check that Q has minimal surface area if

$$a = \left(\frac{\partial_{\overline{B}_1^k}}{2k}\right)^{\frac{m}{k+m}} \text{ and } b = \left(\frac{2k}{\partial_{\overline{B}_1^k}}\right)^{\frac{k}{k+m}}. \tag{7.15}$$

Moreover,

$$\partial_Q := \frac{k+m}{k}\partial_{\overline{B}_1^k}^{\frac{k}{k+m}}(2k)^{\frac{m}{k+m}}. \tag{7.16}$$

We choose $m \simeq \frac{k}{\log k}$. Note that $k \leqslant n \leqslant 2k$. Then, since $\partial_{\overline{B}_1^k} \simeq \sqrt{k}$, we get that

$$a \simeq 1,\ b \simeq \sqrt{k} \simeq \sqrt{n} \text{ and } \partial_Q \simeq \sqrt{k} \simeq \sqrt{n}. \tag{7.17}$$

It is well-known (see e.g. [17]) that, for every symmetric convex body V in $\mathbb{R}^n$ and every $F \in G_{n,m}$,

$$w(V) \geqslant c\sqrt{m/n}\, w(P_F(V)). \tag{7.18}$$

So, we choose $F := \mathbb{R}^m$ and we conclude that

$$w(Q) \geqslant c\sqrt{m/n}\, w(P_F(Q)) \geqslant \frac{c}{\sqrt{\log n}} w(bC_m) \geqslant \frac{c\sqrt{n}}{\sqrt{\log n}} w(C_m). \tag{7.19}$$

Since $w(C_m) \geqslant c\sqrt{m}$, we see that $w(Q) \geqslant \frac{cn}{\log n}$. $\square$

Using Theorem 7.1 we can also complement Theorems 4.4 and 4.5 by showing that, at least in the symmetric case, one has the upper bound $|K + D_n|^{1/n} \leqslant c\sqrt[4]{n}$ for a convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area (note that in in this estimate, the parameter L_K does not appear).

Proposition 7.5. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,*

$$|K + D_n|^{1/n} \leqslant C\sqrt[4]{n}, \tag{7.20}$$

where $C > 0$ is an absolute constant.

Proof. We know that $c_1\sqrt{n} \leqslant \partial(K) = \partial_K \leqslant c_2 n$. We distinguish two cases.
Case 1. Assume that $\partial(K) \leqslant n^{3/4}$. Then, using the fact that $R(D_n) \simeq \sqrt{n}$, we write

$$\begin{aligned} V(K+D_n, K, \dots, K) &= |K| + V(D_n, K, \dots, K) \\ &= 1 + \frac{1}{n}\int_{S^{n-1}} h_{D_n}(x)\, d\sigma_K(x) \\ &\leqslant 1 + \frac{1}{n}R(D_n)\partial(K) \leqslant 1 + \frac{c_3}{\sqrt{n}}\partial(K) \\ &\leqslant c_4\sqrt[4]{n}, \end{aligned}$$

where $c_4 > 0$ is an absolute constant.
Case 2. Assume that $n^{3/4} \leqslant \partial(K) \leqslant c_2 n$. Urysohn's inequality shows that

$$|K+D_n|^{1/n} \leqslant c_1 \frac{w(K+D_n)}{\sqrt{n}} = c_1 \frac{w(K)+w(D_n)}{\sqrt{n}}. \tag{7.21}$$

We know that $w(D_n) \leqslant c_2\sqrt{n}$ and Theorem 7.1 shows that

$$w(K) \leqslant \frac{c_3 n^{3/2}}{\partial_K} \leqslant c_3 n^{3/4}, \tag{7.22}$$

by our assumption on ∂_K. It follows that

$$|K+D_n|^{1/n} \leqslant c_1 \frac{c_3 n^{3/4} + c_2\sqrt{n}}{\sqrt{n}} \leqslant c_4\sqrt[4]{n} \tag{7.23}$$

in this case as well. □

8 John and Löwner position

We say that a symmetric convex body K is in John's position if B_2^n is the ellipsoid of maximal volume inscribed in K. John's theorem [11] states that this holds true if and only if $B_2^n \subseteq K$ and there exist $u_1, \dots, u_N \in \mathrm{bd}(K) \cap S^{n-1}$ and positive real numbers $c_1, \dots, c_N$ such that the identity operator can be decomposed in the form

$$I = \sum_{j=1}^{N} c_j u_j \otimes u_j, \tag{8.1}$$

where $(u_j \otimes u_j)(y) = \langle u_j, y\rangle u_j$. From this representation of the identity we get

$$\sum_{j=1}^{N} c_j \langle u_j, \theta\rangle^2 = 1 \tag{8.2}$$

for all $\theta \in S^{n-1}$. Therefore, if we consider the measure μ on S^{n-1} which is supported by $\{u_1, \dots, u_N\}$ and gives mass c_j to $\{u_j\}$, $j = 1, \dots, N$, then μ is isotropic.

We will give an example of a zonoid, which is unconditional and its John's position fails to be an M-position. See also [22] for a comparison of John's position with ℓ-position, which plays an important role in the asymptotic theory of finite dimensional normed spaces.

Lemma 8.1. *Let K be a symmetric convex body in $\mathbb{R}^m$ which is in John's position. Let $Q_k = [-1,1]^k$ be the cube in $\mathbb{R}^k$ which is also in John's position. Then $K \times Q_k$ is also in John's position.*

Proof. We will use induction on k. Note that it is enough to show that $K_1 := K \times [-1,1]$ is in John's position. To this end, first note that $B_2^{m+1} \subseteq B_2^m \times [-1,1] \subseteq K_1$. Moreover, for every $x = (y,t) \in \mathbb{R}^{m+1}$ we have that

$$x = y + te_{m+1} = \sum_{j=1}^{N} c_j \langle x, u_j \rangle u_j + \langle x, e_{m+1} \rangle e_{m+1}, \tag{8.3}$$

using the decomposition of identity (8.1) of K. Note that e_{m+1} is also a contact point for K_1. So, the proof is complete by John's theorem. □

It will be convenient to say that K is in the normalized John's position if $|K| = 1$ and there exists $\lambda > 0$ such that λK is in John's position.

Proposition 8.2. *There exists an unconditional convex body K in $\mathbb{R}^n$ which is in the normalized John's position, such that*

$$|K + D_n|^{1/n} \geqslant c\sqrt[8]{n}, \tag{8.4}$$

where $c > 0$ is an absolute constant.

Proof. Let $K := |B_2^m \times Q_k|^{-\frac{1}{n}} (B_2^m \times Q_k)$, where $m + k = n$. Lemma 8.1 shows that K is in normalized John's position; it is also clear that K is an unconditional body. Note that

$$\alpha := |B_2^m \times Q_k|^{1/n} = \left(2^k \omega_m\right)^{1/n}. \tag{8.5}$$

Let $F := \mathbb{R}^k$. Then,

$$\begin{aligned} N(K, D_n) &\geqslant N\left(P_F(K), P_F(D_n)\right) = N\left(\tfrac{1}{\alpha} Q_k, c_1\sqrt{n} B_2^k\right) \\ &= N\left(Q_k, \tfrac{\alpha\sqrt{n}}{\sqrt{k}} (c_1 \sqrt{k} B_2^k)\right) \geqslant \left(\frac{c_2\sqrt{k}}{\alpha\sqrt{n}}\right)^k. \end{aligned}$$

It follows that

$$|K + D_n|^{\frac{1}{n}} \geqslant N(K, D_n)^{1/n} \geqslant \left(\frac{c_2\sqrt{k}}{\alpha\sqrt{n}}\right)^{\frac{k}{n}} \geqslant c_3 \left(\sqrt{m}\right)^{\frac{mk}{n^2}} \geqslant c_4\sqrt[8]{n}, \tag{8.6}$$

if we choose $m = k = n/2$. □

We say that a symmetric convex body K is in Löwner's position if B_2^n is the ellipsoid of minimal volume which contains K. We also say that K is in the normalized Löwner's position if $|K| = 1$ and there exists $\lambda > 0$ such that λK is in Löwner's position.

Corollary 8.3. *There exists an unconditional convex body W in $\mathbb{R}^n$ which is in the normalized Löwner's position and satisfies*

$$|W + D_n|^{1/n} \geqslant c\sqrt[8]{n}, \tag{8.7}$$

where $c > 0$ is an absolute constant.

Proof. Let K be the unconditional body from Proposition 8.2. We choose $W := |K^\circ|^{-1/n}K^\circ$. Then, K is in the normalized Löwner's position. Since $K^\circ = |K^\circ|^{1/n}W \simeq \frac{1}{n}W$, using the Blaschke-Santaló inequality and its reverse together with some elementary entropy estimates, we have that

$$\begin{aligned} |W + D_n|^{1/n} &\simeq \frac{1}{|W \cap D_n|^{1/n}} \simeq \frac{1}{|[\mathrm{conv}(W^\circ, D_n^\circ)]^\circ|^{1/n}} \\ &\simeq n|\mathrm{conv}(W^\circ, D_n^\circ)|^{1/n} \simeq n\left|\mathrm{conv}\left(\tfrac{1}{n}K, \tfrac{1}{n}D_n\right)\right|^{1/n} \\ &\simeq |\mathrm{conv}(K, D_n)|^{1/n} \simeq |K + D_n|^{1/n} \\ &\geqslant c\sqrt[8]{n}, \end{aligned}$$

by (8.4). □

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