ARTICLES

The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha

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1. Introduction

The formula for π mentioned in the title of this article is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{1}$$

One simple and well-known modern proof goes as follows:

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt.\end{aligned}$$

The last integral tends to zero if $|x| \leqslant 1$, for

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \leqslant \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3} \to 0 \quad \text{as } n \to \infty.$$

Thus, arctan x has an infinite series representation for $|x| \leqslant 1$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots . \tag{2}$$

The series for $\pi/4$ is obtained by setting $x = 1$ in (2). The series (2) was obtained independently by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician of the fourteenth century or probably the fifteenth century whose identity is not definitely known. Usually ascribed to Nilakantha, the Indian proof of (2) appears to date from the mid-fifteenth century and was a consequence of an effort to rectify the circle. The details of the circumstances and ideas leading to the discovery of the series by Leibniz and Gregory are known. It is interesting to go into these details for several reasons. The infinite series began to play a role in mathematics only in the second half of the seventeenth century. Prior to that, particular cases of the infinite geometric series were the only ones to be used. The arctan series was obtained by Leibniz and Gregory early in their study of infinite series and, in fact, before the methods and algorithms of calculus were fully developed. The history of the arctan series is, therefore, important because it reveals early ideas on series and their relationship with quadrature or the process of finding the area under a curve. In the case of Leibniz, it is possible to see how he used and transformed older ideas on quadrature to develop his methods. Leibniz's work, in fact, was primarily concerned with quadrature; the $\pi/4$ series resulted (in 1673) when he applied his method to the circle. Gregory, by comparison, was interested in finding an infinite series representation of any given function and discovered the relationship between this and the successive derivatives of the given function. Gregory's discovery, made in 1671, is none other than the Taylor series; note that Taylor was not born until 1685. The ideas of calculus, such as integration by parts, change of variables, and higher derivatives, were not completely understood in the early 1670s. Some particular cases were known, usually garbed in geometric language. For example, the fundamental theorem of calculus was stated as a geometric theorem in a work of Gregory's written in 1668. Similar examples can also be seen in a book by Isaac Barrow, Newton's mentor, published in 1670. Of course, very soon after this transitional period, Leibniz began to create the techniques, algorithms and notations of calculus as they are now known. He had been preceded by Newton, at least as far as the techniques go, but Newton did not publish anything until considerably later. It is, therefore, possible to see how the work on arctan was at once dependent on earlier concepts and a transitional step toward later ideas.

Finally, although the proofs of (2) by Leibniz, Gregory and Nilakantha are very different in approach and motivation, they all bear a relation to the modern proof given above.

2. Gottfried Wilhelm Leibniz (1646–1716)

Leibniz's mathematical background[1] at the time he found the $\pi/4$ formula can be quickly described. He had earned a doctor's degree in law in February 1667, but had studied mathematics on his own. In 1672, he was a mere amateur in mathematics. That year, he visited Paris and met Christiaan Huygens (1629–1695), the foremost physicist and mathematician in continental Europe. Leibniz told the story of this meeting in a 1679 letter to the mathematician Tschirnhaus, "at that time... I did not know the correct definition of the center of gravity. For, when by chance I spoke of it to Huygens, I let him know that I thought that a straight line drawn through the center of gravity always cut a figure into two equal parts,... Huygens laughed when he heard this, and told me that nothing was further from the truth. So I, excited by this stimulus, began to apply myself to the study of the more intricate geometry, although as a matter of fact I had not at that time really studied the *Elements* [Euclid]... Huygens, who thought me a better geometer than I was, gave me to read the letters of Pascal, published under the name of Dettonville; and from these I gathered the method of indivisibles and centers of gravity, that is to say the well-known methods of Cavalieri and Guldinus."[2]

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[1]For further information about Leibniz's mathematical development, the reader may consult: J. E. Hofmann, *Leibniz in Paris 1672–1676* (Cambridge: The Cambridge University Press, 1974) and its review by A. Weil, *Collected Papers* Vol. 3 (New York: Springer-Verlag, 1979). An English translation of Leibniz's own account, *Historia et origo calculi differentialis*, can be found in J. M. Child, *The Early Mathematical Manuscripts of Leibniz* (Chicago: Open Court, 1920). An easily available synopsis of Leibniz's work in calculus is given in C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).

[2]*The Early Mathematical Manuscripts*, p. 215.

Bonaventura Cavalieri (1598–1647) published his *Geometria Indivisibilibus* in 1635. This book was very influential in the development of calculus. Cavalieri's work indicated that

The study of Pascal played an important role in Leibniz's development as a mathematician. It was from Pascal that he learned the ideas of the "characteristic triangle" and "transmutation." In order to understand the concept of transmutation, suppose A and B are two areas (or volumes) which have been divided up into indivisibles usually taken to be infinitesimal rectangles (or prisms). If there is a one-to-one correspondence between the indivisibles of A and B and if these indivisibles have equal areas (or volumes), then B is said to be obtained from A by transmutation and it follows that A and B have equal areas (or volumes). Pascal had also considered infinitesimal triangles and shown their use in finding, among other things, the area of the surface of a sphere. Leibniz was struck by the idea of an infinitesimal triangle and its possibilities. He was able to derive an interesting transmutation formula, which he then applied to the quadrature of a circle and thereby discovered the series for π. To obtain the transmutation formula, consider two neighboring points $P(x, y)$, and $Q(x+dx, y+dy)$ on a curve $y=f(x)$. First Leibniz shows that area $(\Delta OPQ) = (1/2)$ area (rectangle $(ABCD)$). See FIGURE 1. Here PT is tangent to $y=f(x)$ at P and OS is perpendicular to PT. Let p denote the length of OS and z that of $AC = BD =$ ordinate of T.

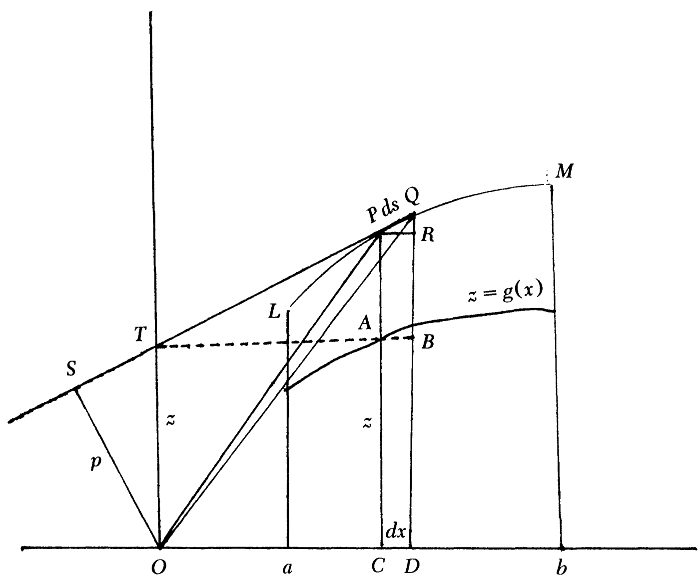

FIGURE 1

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

when n is a positive integer.

Blaise Pascal (1623–1662) made important and fundamental contributions to projective geometry, probability theory and the development of calculus. The work to which Leibniz refers was published in 1658 and contains the first statement and proof of

$$\int_{\theta_0}^{\theta} \sin\phi\,d\phi = \cos\theta_0 - \cos\theta.$$

This proof is presented in D. J. Struik's *A Source Book in Mathematics 1200–1800* (Cambridge: Harvard University Press, 1969), p. 239.

Paul Guldin (1577–1643), a Swiss mathematician of considerable note, contributed to the development of calculus, and his methods were generally more rigorous than those of Cavalieri.

Since ΔOST is similar to the characteristic ΔPQR,

$$\frac{dx}{p}=\frac{ds}{z},$$

where ds is the length of PQ. Thus,

$$\text{area}\,(OPQ)=\frac{1}{2}pds=\frac{1}{2}zdx. \tag{3}$$

Now, observe that for each point P on $y=f(x)$ there is a corresponding point A. Thus, as P moves from L to M, the points A form a curve, say $Z=g(x)$. If sector OLM denotes the closed region formed by $y=f(x)$ and the straight lines OL and OM, then (3) implies that

$$\text{area}\,(\text{sector } OLM)=\frac{1}{2}\int_a^b g(x)\,dx. \tag{4}$$

This is the transmutation formula of Leibniz. From (4), it follows that the area under $y=f(x)$ is

$$\begin{aligned}\int_a^b y\,dx&=\frac{b}{2}f(b)-\frac{a}{2}f(a)+\text{area}\,(\text{sector } OLM)\\ &=\frac{1}{2}\left([xy]_a^b+\int_a^b z\,dx\right).\end{aligned} \tag{5}$$

This is none other than a particular case of the formula for integration by parts. For it is easily seen from FIGURE 1 that

$$z=y-x\frac{dy}{dx}. \tag{6}$$

Substituting this value of z in (5), it follows that

$$\int_a^b y\,dx=[xy]_a^b-\int_{f(a)}^{f(b)}x\,dy,$$

which is what one gets on integration by parts.

Now consider a circle of radius 1 and center (1,0). Its equation is $y^2=2x-x^2$. In this case, (6) implies that

$$z=\sqrt{2x-x^2}-\frac{x(1-x)}{\sqrt{2x-x^2}}=\sqrt{\frac{x}{2-x}}=\frac{x}{y}, \tag{7}$$

so that

$$x=\frac{2z^2}{1+z^2}. \tag{8}$$

In FIGURE 2, let $A\hat{O}B=2\theta$. Then the area of the sector $AOB=\theta$ and

$$\theta=\text{area}\,(\triangle\,AOB)+\text{area}\,(\text{region between arc } AB \text{ and line } AB). \tag{9}$$

By the transmutation formula (4), the second area is $\frac{1}{2}\int_0^x z\,dt$ where z is given by (7). Now, from FIGURE 3 below it is seen that

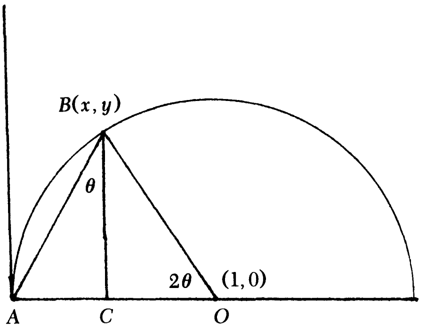

FIGURE 2

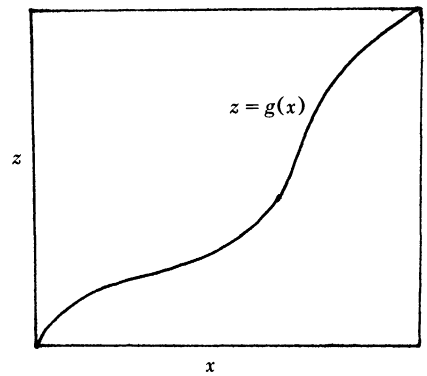

FIGURE 3

$$\frac{1}{2}\int_0^x z\,dt = \frac{1}{2}\left(xz - \int_0^z x\,du\right). \tag{10}$$

Using (8) and (10), it is now possible to rewrite (9) as

$$\begin{aligned}\theta &= \tfrac{1}{2}y + \tfrac{1}{2}xz - \int_0^z \frac{t^2}{1+t^2}\,dt \\ &= \tfrac{1}{2}[z(2-x) + xz] - \int_0^z \frac{t^2}{1+t^2}\,dt \quad (\text{since } y = z(2-x)) \\ &= z - \int_0^z \frac{t^2}{1+t^2}\,dt.\end{aligned}$$

At this point, Leibniz was able to use a technique employed by Nicolaus Mercator (1620–1687). The latter had considered the problem of the quadrature of the hyperbola $y(1+x)=1$. Since it was already known that

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1},$$

he solved the problem by expanding $1/(1+x)$ as an infinite series and integrating term by term. He simultaneously had the expansion for $\log(1+x)$. Mercator published this result in 1668, though he probably had obtained it a few years earlier. A year later, John Wallis (1616–1703) determined the values of x for which the series is valid. Thus, Leibniz found that

$$\theta = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots. \tag{11}$$

In FIGURE 2, $A\hat{B}C = \theta$ and $z = x/y = \tan\theta$. Therefore, (11) is the series for $\arctan z$.

Of course, Leibniz did not invent the notation for the integral and differential used above until 1675, and his description of the procedures is geometrical but the ideas are the same.

The discovery of the infinite series for π was Leibniz's first great achievement. He communicated his result to Huygens, who congratulated him, saying that this remarkable property of the circle will be celebrated among mathematicians forever. Even Isaac Newton (1642–1727) praised Leibniz's discovery. In a letter of October 24, 1676, to Henry Oldenburg, secretary of the Royal Society of London, he writes, "Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else."[3] Of course, for Leibniz this was only a first step to greater things as he himself says in his "Historia et origo calculi differentialis."

3. James Gregory (1638–1675)

Leibniz had been anticipated in the discovery of the series for arctan by the Scottish mathematician, James Gregory, though the latter did not note the particular case for $\pi/4$.[4] Since Gregory did not publish most of his work on infinite series and also because he died early and worked in isolation during the last seven years of his life, his work did not have the influence it deserved. Gregory's early scientific interest was in optics about which he wrote a masterly book at the age of twenty-four. His book, the *Optica Promota*, contains the earliest description of a reflecting telescope. It was in the hope, which ultimately remained unfulfilled, of constructing such an instrument that he travelled to London in 1663 and made the acquaintance of John Collins (1624–1683), an accountant and amateur mathematician. This friendship with Collins was to prove very important for Gregory when the latter was working alone at St. Andrews University in Scotland. Collins kept him abreast of the work of the English mathematicians such as Isaac Newton, John Pell (1611–1685) and others with whom Collins was in contact.[5]

Gregory spent the years 1664–1668 in Italy and came under the influence of the Italian school of geometry founded by Cavalieri. It was from Stefano degli Angeli (1623–1697), a student of Cavalieri, that Gregory learned about the work of Pierre de Fermat (1601–1665), Cavalieri, Evangelista Torricelli (1608–1647) and others. While in Italy, he wrote two books: *Vera Circuli et Hyperbolae Quadratura* in 1667, and *Geometriae Pars Universalis* in 1668. The first book contains some highly original ideas. Gregory attempted to show that the area of a general sector of an ellipse, circle or hyperbola could not be expressed in terms of the areas of the inscribed and circumscribed triangle and quadrilateral using arithmetical operations and root extraction. The attempt failed but Gregory introduced a number of important ideas such as convergence and algebraic and transcendental functions. The second book contains the first published statement and proof of the fundamental theorem of calculus in geometrical form. It is known that Newton had discovered this result not later than 1666, although he did not make it public until later.

Gregory returned to London in the summer of 1668; Collins then informed him of the latest discoveries of mathematicians working in England, including Mercator's recently published proof of

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots.$$

[3]See H. W. Turnbull (ed.), *The Correspondence of Isaac Newton* (Cambridge: The University Press, 1960), Vol. 2, p. 130.

[4]Peter Beckmann has persuasively argued that Gregory must have known the series for $\pi/4$ as well. See Beckmann's *A History of Pi* (Boulder, Colorado: The Golem Press, 1977), p. 133.

[5]The reader might find it of interest to consult: H. W. Turnbull (ed.), *James Gregory Tercentenary Memorial Volume* (London: G. Bell, 1939). This volume contains Gregory's scientific correspondence with John Collins and a discussion of the former's life and work.

Meditation on these discoveries led Gregory to publish his next book, *Exercitationes Geometricae*, in the winter of 1668. This is a sequel to the *Pars Universalis* and is mainly about the logarithmic function and its applications. It contains, for example, the first evaluations of the indefinite integrals of $\sec x$ and $\tan x$.[6] The results are stated in geometric form.

In the autumn of 1668, Gregory was appointed to the chair in St. Andrews and he took up his duties in the winter of 1668/1669. He began regular correspondence with Collins soon after this, communicating to him his latest mathematical discoveries and requesting Collins to keep him informed of the latest activities of the English mathematicians. Thus, in a letter of March 24, 1670, Collins writes, "Mr. Newtone of Cambridge sent the following series for finding the Area of a Zone of a Circle to Mr. Dary, to compare with the said Dary's approaches, putting R the radius and B the parallel distance of a Chord from the Diameter the Area of the space or Zone between them is =

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{576R^7}."^{7}$$

This is all Collins writes about the series but it is, in fact, the value of the integral $2\int_0^B (R^2 - x^2)^{1/2}\,dx$ after expanding by the binomial theorem and term by term integration. Newton had discovered the binomial expansion for fractional exponents in the winter of 1664/1665, but it was first made public in the aforementioned letter of 1676 to Oldenburg.

There is evidence that Gregory had rediscovered the binomial theorem by 1668.[8] However, it should be noted that the expansion for $(1 - x)^{1/2}$ does not necessarily

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[6]A proof of the formula

$$\int_0^\theta \sec\phi\, d\phi = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

was of considerable significance and interest to mathematicians in the 1660's due to its connection with a problem in navigation. Gerhard Mercator (1512–1594) published his engraved "Great World Map" in 1569. The construction of the map employed the famous Mercator projection. Edward Wright, a Cambridge professor of mathematics, noted that the ordinate on the Mercator map corresponding to a latitude of θ^0 on the globe is given by $c\int_0^\theta \sec\phi\, d\phi$, where c is suitably chosen according to the size of the map. In 1599, Wright published this result in his *Certaine Errors in Navigation Corrected*, which also contained a table of latitudes computed by the continued addition of the secants of $1', 2', 3'$, etc. This approximation to $\int_0^\theta \sec\phi\, d\phi$ was sufficiently exact for the mariner's use. In the early 1640's, Henry Bond observed that the values in Wright's table could be obtained by taking the logarithm of $\tan(\pi/4 + \theta/2)$. This observation was published in 1645 in Richard Norwood's *Epitome of Navigation*. A theoretical proof of this observation was very desirable and Nicolaus Mercator had offered a sum of money for its demonstration in 1666. John Collins, who had himself written a book on navigation, drew Gregory's attention to this problem and, as we noted, Gregory supplied a proof. For more details, one may consult the following two articles by F. Cajori: "On an Integration ante-dating the Integral Calculus," *Bibliotheca Mathematica* Vol. 14 (1913/14), pp. 312–19, and "Algebra In Napier's Day and Alleged Prior Invention of Logarithms," in C. G. Knott (ed.), *Napier Memorial Volume* (London: Longmans, Green & Co., 1915), pp. 93–106. More recently, J. Lohne has established that Thomas Harriot (1560–1621) had evaluated the integral $\int_0^\theta \sec\phi\, d\phi$ in 1594 by a stereographic projection of a spherical loxodrome from the south pole into a logarithmic spiral. This work was unpublished and remained unknown until Lohne brought it to light. See J. A. Lohne, "Thomas Harriot als Mathematiker," *Centaurus*, Vol. 11, 1965–66, pp. 19–45. Thus it happened that, although $\int \sec\theta\, d\theta$ is a relatively difficult trigonometric integral, it was the first one to be discovered.

[7]*James Gregory*, p. 89.

[8]See *The Correspondence of Isaac Newton*, Vol. 1, p. 52, note 1.

imply a knowledge of the binomial theorem. Newton himself had proved the expansion by applying the well-known method for finding square roots of numbers to the algebraic expression $1-x$. Moreover, it appears that the expansion of $(1-x)^{1/2}$ was discovered by Henry Briggs (1556–1630) in the 1620's, while he was constructing the log tables.[9] But there is no indication that Gregory or Newton knew of this. In any case, for reasons unknown, Gregory was unable to make anything of the series—as evidenced by his reply of April 20, "I cannot understand the series you sent me of the circle, if this be the original, I take it to be no series."[10] However, by September 5, 1670, he had discovered the general interpolation formula, now called the Gregory-Newton interpolation formula, and had made from it a number of remarkable deductions. He now knew how "to find the sinus having the arc and to find the number having the logarithm." The latter result is precisely the binomial expansion for arbitrary exponents. For, if we take x as the logarithm of y to the base $1+d$, then $y=(1+d)^x$ and Gregory gives the solution as

$$(1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots .$$ [11]

It is possible that Newton's series in Collins' letter had set Gregory off on the course of these discoveries, but he did not even at this point see that he could deduce Newton's result. Soon after, he did observe this and wrote on December 19, 1670, "I admire much my own dullness, that in such a considerable time I had not taken notice of this."[12] All this time, he was very eager to learn about Newton's results on series and particularly the methods he had used. Finally on December 24, 1670, Collins sent him Newton's series for $\sin x$, $\cos x$, $\sin^{-1}x$ and $x\cot x$, adding that Newton had a universal method which could be applied to any function. Gregory then made a concentrated effort to discover a general method for himself. He succeeded. In a famous letter of February 15, 1671 to Collins he writes, "As for Mr. Newton's universal method, I imagine I have some knowledge of it, both as to geometrick and mechanick curves, however I thank you for the series ye sent me and send you these following in requital."[13] Gregory then gives the series for $\arctan x$, $\tan x$, $\sec x$, $\log\sec x$, $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, $\operatorname{arcsec}(\sqrt{2}\,e^x)$, and $2\arctan\tanh x/2$. However, what he had found was not Newton's method but rather the Taylor expansion more than forty years before Brook Taylor (1685–1731). Newton's method consisted of reversion of series, expansion by the binomial theorem, long division by series and term by term integration.[14] Thinking that he had rediscovered Newton's method, Gregory did not

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[9] See D. T. Whiteside, "Henry Briggs: The Binomial theorem Anticipated," *The Mathematical Gazette*, Vol. 15, (1962), p. 9. Whiteside shows how the expansion of $(1+x)^{1/2}$ arose out of Brigg's work on logarithms.

[10] *James Gregory*, p. 92.

[11] In their review of the Gregory Memorial Volume, M. Dehn and E. Hellinger explain how the binomial expansion comes out of the interpolation formula. See *The American Mathematical Monthly*, Vol. 50, (1943), p. 149.

[12] *James Gregory*, p. 148.

[13] *Ibid.*, p. 170.

[14] It should be mentioned that Newton himself discovered the Taylor series around 1691. See D. T. Whiteside (ed.), *The Mathematical Papers of Isaac Newton*, Vol. VII (Cambridge: The Cambridge University Press, 1976), p. 19. In fact, Taylor was anticipated by at least five mathematicians. However, the

publish his results. It is only from notes that he made on the back of a letter from Gedeon Shaw, an Edinburgh stationer, dated January 29, 1671, that it is possible to conclude that Gregory had the idea of the Taylor series. These notes contain the successive derivatives of $\tan x$, $\sec x$, and the other functions whose expansions he sent to Collins. The following extract from the notes gives the successive derivatives of $\tan\theta$; here m is successively $y, \frac{dy}{d\theta}, \frac{d^2y}{d\theta^2}$, etc., and $q = r\tan\theta$. Gregory writes[15]:

1st $\quad m = q$

2nd $\quad m = r + \frac{q^2}{r}$

3rd $\quad m = 2q + \frac{2q^3}{r^2}$

4th $\quad m = 2r + \frac{8q^2}{r} + \frac{6q^4}{r^3}$

5th $\quad m = 16q + \frac{40q^3}{r^2} + \frac{24q^5}{r^4}$

6th $\quad m = 16r + \frac{136q^2}{r} + \frac{240q^4}{r^3} + \frac{120q^6}{r^5}$

7th

$$m = 272q + 987\frac{q^3}{r^2} + 1680\frac{q^5}{r^4} + 720\frac{q^7}{r^6}$$

8th

$$m = 272r + 3233\frac{q^2}{r} + 11361\frac{q^4}{r^3} + 13440\frac{q^6}{r^5} + 5040\frac{q^8}{r^7}.$$

It is clear from the form in which the successive derivatives are written that each one is formed by multiplying the derivative with respect to q of the preceding term by $r + \frac{q^2}{r}$. Now writing $a = r\theta$, Gregory gives the series in the letter to Collins as follows:

$$r\tan\theta = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{3233a^9}{181440r^8} + \cdots.$$

The reasons for supposing that these notes were written not much before he wrote to Collins and were used to construct the series are (i) the date of Gedeon Shaw's letter and (ii) Gregory's error in computing the coefficient of $\frac{q^3}{r^2}$ in the 7th m, which should be 1232 instead of 987 and which, in turn, leads to the error in the 8th m, where the coefficient of $\frac{q^2}{r}$ should be 3968 instead of 3233. This error is then repeated in the series showing the origin of the series. Moreover, in the early parts of the notes, Gregory is unsure about how he should write the successive derivatives. For example, he attempts to write the derivative of $\sec\theta$ as a function of $\sec\theta$ but then abandons the idea. He comes back to it later and sees that it is easier to work with m^2 instead of m since the m^2's can be expressed as polynomials in $\tan\theta$. This is, of course, sufficient to give him the series for $\sec\theta$. The series for $\log\sec\theta$ and $\log\tan(\pi/4+\theta)$ he then obtains by term by term integration of the series for $\tan\theta$ and $\sec\theta$

Taylor series is not unjustly named after Brook Taylor who published it in 1715. He saw the importance of the result and derived several interesting consequences. For a discussion of these matters see: L. Feigenbaum, "Brook Taylor and the Method of Increments," *Archive for History of Exact Sciences*, Vol. 34, (1985), pp. 1–140.

[15] *James Gregory*, p. 352.

respectively. Naturally, the 3233 error is repeated. He must have obtained the series for arc tan x from the 2nd m which can be written as

$$\frac{da}{dq} = \frac{r^2}{r^2 + q^2} = 1 - \frac{q^2}{r^2} + \frac{q^4}{r^4} - \cdots .$$

The arctan series follows after term by term integration. Clearly, Gregory had made great progress in the study of infinite series and the calculus and, had he lived longer and published his work, he might have been classed with Newton and Leibniz as a co-discoverer of the calculus. Unfortunately, he was struck by a sudden illness, accompanied with blindness, as he was showing some students the satellites of Jupiter. He did not recover and died soon after in October, 1675, at the age of thirty-seven.

4. Kerala Gargya Nilakantha (c.1450–c.1550)

Another independent discovery of the series for arctan x and other trigonometric functions was made by mathematicians in South India during the fifteenth century. The series are given in Sanskrit verse in a book by Nilakantha called *Tantrasangraha* and a commentary on this work called *Tantrasangraha-vakhya* of unknown authorship. The theorems are stated without proof but a proof of the arctan, cosine and sine series can be found in a later work called *Yuktibhasa*. This was written in Malayalam, the language spoken in Kerala, the southwest coast of India, by Jyesthadeva (c.1500–c.1610) and is also a commentary on the *Tantrasangraha*. These works were first brought to the notice of the western world by an Englishman named C. M. Whish in 1835. Unfortunately, his paper on the subject had almost no impact and went unnoticed for almost a century when C. Rajagopal[16] and his associates began publishing their findings from a study of these manuscripts. The contributions of medieval Indian mathematicians are now beginning to be recognized and discussed by authorities in the field of the history of mathematics.[17]

It appears from the astronomical data contained in the *Tantrasangraha* that it was composed around the year 1500. The *Yuktibhasa* was written about a century later. It is not completely clear who the discoverer of these series was. In the *Aryabhatiya-bhasya*, a work on astronomy, Nilakantha attributes the series for sine to Madhava. This mathematician lived between the years 1340–1425. It is not known whether

[16]Rajagopal's work may be found in the following papers: (with M. S. Rangachari) "On an Untapped Source of Medieval Keralese Mathematics," *Archive for History of Exact Sciences*, Vol. 18, (1977), pp. 89–102; "On Medieval Kerala Mathematics," *Archive for History of Exact Sciences*, Vol. 35, (1986), pp. 91–99. These papers give the Sanskrit verses of the *Tantrasangrahavakhya* which describe the series for the arctan, sine and cosine. An English translation and commentary is also provided. A commentary on the proof of arctan series given in the *Yuktibhasa* is available in the two papers: "A Neglected Chapter of Hindu Mathematics," *Scripta Magnematica*, Vol. 15, (1949), pp. 201–209; "On the Hindu Proof of Gregory's Series," *Ibid.*, Vol. 17, (1951), pp. 65–74. A commentary on the *Yuktibhasa's* proof of the sine and cosine series is contained in C. Rajagopal and A. Venkataraman, "The sine and cosine power series in Hindu mathematics," *Journal of the Royal Asiatic Society of Bengal, Science*, Vol. 15, (1949), pp. 1–13.

[17]See J. E. Hofmann, "Über eine alt indische Berechnung von π und ihre allgemeine Bedeutung," *Mathematische-Physikalische Semester Berichte*, Bd. 3, H. 3/4, Hamburg (1953). See also D. T. Whiteside, "Patterns of Mathematical Thought in the later Seventeenth Century," *Archive for History of Exact Sciences*, Vol. 1, (1960–1962), pp. 179–388. For a discussion of medieval Indian mathematicians and the *Tantrasangraha* in particular, one might consult: A. P. Jushkevich, *Geschichte der Mathematik in Mittelalter* (German translation Leipzig, 1964, of the Russian original, Moscow, 1961).

Madhava found the other series as well or whether they are somewhat later discoveries.

Little is known about these mathematicians. Madhava lived near Cochin in the very southern part of India (Kerala) and some of his astronomical work still survives. Nilakantha was a versatile genius who wrote not only on astronomy and mathematics but also on philosophy and grammar. His erudite expositions on the latter subjects were well known and studied until recently. He attracted several gifted students, including Tuncath Ramanujan Ezuthassan, an early and important figure in Kerala literature. About Jyesthadeva, nothing is known except that he was a Brahmin of the house of Parakroda.

In the *Tantrasangraha-vakhya*, the series for arctan, sine and cosine are given in verse which, when converted to mathematical symbols may be written as

$$r\arctan\frac{y}{x}=\frac{1}{1}\cdot\frac{ry}{x}-\frac{1}{3}\cdot\frac{ry^3}{x^3}+\frac{1}{5}\cdot\frac{ry^5}{x^5}-\cdots,\ \text{where } \frac{y}{x}\leqslant 1,$$

$$y=s-s\cdot\frac{s^2}{(2^2+2)r^2}+s\cdot\frac{s^2}{(2^2+2)r^2}\cdot\frac{s^2}{(4^2+4)r^2}-\cdots\quad\text{(sine)}$$

$$r-x=r\cdot\frac{s^2}{(2^2-2)r^2}-r\cdot\frac{s^2}{(2^2-2)r^2}\cdot\frac{s^2}{(4^2-4)r^2}+\cdots\quad\text{(cosine)}.$$

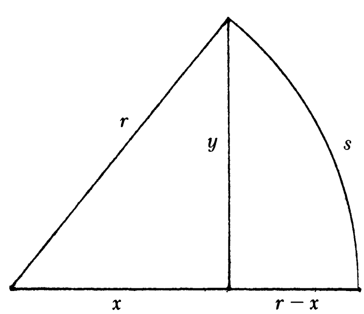

FIGURE 4

There are also some special features in the *Tantrasangraha's* treatment of the series for $\pi/4$ which were not considered by Leibniz and Gregory. Nilakantha states some rational approximations for the error incurred on taking only the first n terms of the series. The expression for the approximation is then used to transform the series for $\pi/4$ into one which converges more rapidly. The errors are given as follows:

$$\frac{\pi}{4}\simeq 1-\frac{1}{3}+\frac{1}{5}-\cdots\mp\frac{1}{n}\pm f_i(n+1)\qquad i=1,2,3,\tag{12}$$

where

$$f_1(n)=\frac{1}{2n},\ f_2(n)=\frac{n/2}{n^2+1}\ \text{and}\ f_3(n)=\frac{(n/2)^2+1}{(n^2+5)n/2}.$$

The transformed series are as follows:

$$\frac{\pi}{4}=\frac{3}{4}+\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots\tag{13}$$

and

$$\frac{\pi}{4}=\frac{4}{1^5+4\cdot 1}-\frac{4}{3^5+4\cdot 3}+\frac{4}{5^5+4\cdot 5}-\cdots.$$

Leibniz's proof of the formula for $\pi/4$ was found by the quadrature of a circle. The proof in Jyesthadeva's book is by a direct rectification of an arc of a circle. In the diagram given below, the arc AC is a quarter circle of radius one with center O and $OABC$ is a square. The side AB is divided into n equal parts of length δ so that $n\delta=1$, $P_{r-1}P_r=\delta$. EF and $P_{r-1}D$ are perpendicular to OP_r. Now, the triangles OEF and $OP_{r-1}D$ are similar, which gives

$$\frac{EF}{OE}=\frac{P_{r-1}D}{OP_{r-1}}, \quad \text{that is, } EF=\frac{P_{r-1}D}{OP_{r-1}}.$$

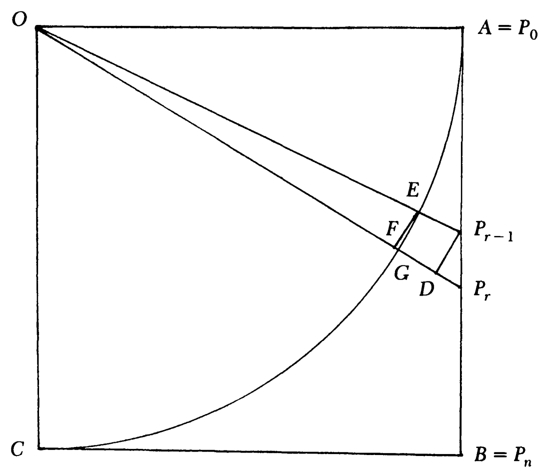

FIGURE 5

The similarity of the Δs $P_{r-1}P_rD$ and OAP_r gives

$$\frac{P_{r-1}P_r}{OP_r}=\frac{P_{r-1}D}{OA} \text{ or } P_{r-1}D=\frac{P_{r-1}P_r}{OP_r}.$$

Thus,

$$EF=\frac{P_{r-1}P_r}{OP_{r-1}OP_r}\simeq\frac{P_{r-1}P_r}{OP_r^2}=\frac{\delta}{1+AP_r^2}=\frac{\delta}{1+r^2\delta^2}.$$

Since arc $EG\simeq EF\simeq\dfrac{\delta}{1+r^2\delta_r^2}$, $\dfrac{1}{8}$ arc of circle is

$$\frac{\pi}{4}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\delta}{1+r^2\delta^2}. \tag{14}$$

Of course, a clear idea of limits did not exist at that time so that the relation was understood in an intuitive sense only. To evaluate the limit, Jyesthadeva uses two lemmas. One is the geometric series

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots.$$

Jyesthadeva says that the expansion is obtained on iterating the following procedure:

$$\frac{1}{1+x} = 1 - x\left(\frac{1}{1+x}\right) = 1 - x\left(1 - x\left(\frac{1}{1+x}\right)\right).$$

The other result is that

$$S_n^{(p)} \equiv 1^p + 2^p + \cdots + n^p \sim \frac{n^{p+1}}{p+1} \quad \text{for large } n. \tag{15}$$

A sketch of a proof is given by Jyesthadeva. He notes first that

$$nS_n^{(p-1)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)}. \tag{16}$$

This is easy to verify. Relation (16) is also contained in the work of the tenth century Arab mathematician Alhazen, who gives a geometrical proof in the Greek tradition[18]. He uses it to evaluate $S_n^{(3)}$ and $S_n^{(4)}$ which occur in a problem about the volume of a certain solid of revolution. *Yuktibhasa* shows that for $p = 2, 3$

$$S_1^{(p-1)} + S_2^{(p-1)} + \cdots + S_{n-1}^{(p-1)} \sim \frac{S_n^{(p)}}{p}, \tag{17}$$

and then suggests that by induction the result will be true for all values of p. Once this is granted, it follows that if

$$S_n^{(p-1)} \sim \frac{n^p}{p},$$

then by (16) and (17),

$$nS_n^{(p-1)} \sim S_n^{(p)} + \frac{S_n^{(p)}}{p} \quad \text{or} \quad S_n^{(p)} \sim \frac{n^{p+1}}{p+1},$$

and (15) is inductively proved.

We now note that (14) can be rewritten, after expanding $1/(1+r^2\delta^2)$ into a geometric series, as

$$\begin{aligned}\frac{\pi}{4} &= \lim_{n\to\infty}\left[\delta\sum_{r=1}^{n} 1 - \delta^3\sum_{r=1}^{n} r^2 + \delta^5\sum_{r=1}^{n} r^4 - \cdots\right]\\ &= \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\sum_{r=1}^{n} r^2 + \frac{1}{n^5}\sum_{r=1}^{n} r^4 - \cdots\right]\\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,\end{aligned}$$

where we have used relation (15) and the fact that $\delta = 1/n$. Now consider the approximation (12) and its application to the transformation of series. Suppose that

$$\sigma_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{n} \mp f(n+1),$$

where $f(n+1)$ is a rational function of n which will make σ_n a better approximation of $\pi/4$ than the nth partial sum S_n. Changing n to $n-2$ we get

[18] See *The Historical Development of the Calculus* (mentioned in footnote 1), p. 84. Alhazen is the latinized form of the name Ibn Al-Haytham (c. 965–1039).

$$\sigma_{n-2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \mp \frac{1}{n-2} \pm f(n-1).$$

Subtracting the second relation from the first,

$$\pm u_n = \sigma_n - \sigma_{n-2} = \pm \frac{1}{n} \mp f(n+1) \mp f(n-1). \tag{18}$$

Then

$$\begin{aligned}\sigma_n &= \sigma_{n-2} \pm u_n \\ &= \sigma_{n-4} \mp u_{n-2} \pm u_n \\ &= \cdots = \sigma_1 - u_3 + u_5 - u_7 + \cdots \pm u_n \\ &= 1 - f(2) - u_3 + u_5 - u_7 + \cdots u_n.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} \sigma_n = \frac{\pi}{4}$$

and therefore

$$\frac{\pi}{4} = 1 - f(2) - u_3 + u_5 - u_7 + \cdots. \tag{19}$$

Thus, we have a new series for $\pi/4$ which depends on how the function $f(n)$ is chosen. Naturally, the aim is to choose $f(n)$ in such a way that (19) is more rapidly convergent than (1). This is the idea behind the series (13). Now equation (18) implies that

$$f(n+1) + f(n-1) = \frac{1}{n} - u_n. \tag{20}$$

For (19) to be more rapidly convergent than (1), u_n should be $o(1/n)$, that is, negligible compared to $1/n$. It is reasonable to assume $f(n+1) \simeq f(n-1) \simeq f(n)$. These observations together with (20) imply that $f(n) = 1/2n$ is a possible rational approximation in equation (12). With this $f(n)$, the value of u_n is given by (20) to be

$$u_n = \frac{1}{n} - \frac{1}{2(n+1)} - \frac{1}{2(n-1)} = -\frac{1}{n^3 - n}.$$

Substituting this in (19) gives us (13), which is

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots.$$

The other series

$$\frac{\pi}{4} = \frac{4}{1^5 + 4\cdot 1} - \frac{4}{3^5 + 4\cdot 3} + \frac{4}{5^5 + 4\cdot 5} - \cdots$$

is obtained by taking $f(n) = \dfrac{n/2}{n^2+1}$ in (19).

It should be mentioned that Newton was aware of the correction $f_1(n) = 1/2n$. For in the letter to Oldenburg, referred to earlier, he says, "By the series of Leibniz also if half the term in the last place be added and some other like devices be employed, the computation can be carried to many figures." However, he says nothing about transforming the series by means of this correction.

It appears that Nilakantha was aware of the impossibility of finding a finite series of rational numbers to represent π. In the *Aryabhatiya-bhasya* he writes, "If the diameter, measured using some unit of measure, were commensurable with that unit, then the circumference would not likewise allow itself to be measured by means of the same unit; so likewise in the case where the circumference is measurable by some unit, then the diameter cannot be measured using the same unit."[19]

The *Yuktibhasa* contains a proof of the arctan series also and it is obtained in exactly the same way except that one rectifies only a part of the 1/8 circle.

It can be shown that if $\pi/4 = S_n + f(n)$, where S_n is the nth partial sum, then $f(n)$ has the continued fraction representation

$$f(n) = \frac{1}{2}\left[\frac{1}{n+}\,\frac{1^2}{n+}\,\frac{2^2}{n+}\,\frac{3^2}{n+}\,\cdots\right]. \tag{21}$$

Moreover, the first three convergents are

$$f_1(n) = \frac{1}{2n}, \qquad f_2(n) = \frac{n/2}{n^2+1} \quad \text{and} \quad f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2},$$

which are the values quoted in (13). Clearly, Nilakantha was using some procedure which gave the successive convergents of the continued fraction (21) but the text contains no suggestion that (20) was actually known to him. This continued fraction implies that

$$\frac{2}{4-\pi} = 2 + \frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+}\,\cdots,$$

which may be compared with the continued fraction of the seventeenth century English mathematician, William Brouncker (1620–1684), who gave the result

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\,\cdots.$$

The third approximation

$$f_3(n) = \frac{(n/2)^2+1}{(n^2+5)n/2}$$

is very effective in obtaining good numerical values for π without much calculation. For example

$$1 - \frac{1}{3} + \cdots - \frac{1}{19} + f_3(20)$$

gives the value of π correct up to eight decimal places.[20] Nilakantha himself gives 104348/33215 which is correct up to nine places. It is interesting that the Arab mathematician Jamshid-al-Kasi, who also lived in the fifteenth century, had obtained the same approximation by a different method.

[19] See *Geschichte der Mathematik*, p. 169.

[20] These observations concerning the continued fraction expansion of $f(n)$ and its relation to the Indian work and that of Brouncker, and concerning the decimal places in f (20), are due to D. T. Whiteside. See "On Medieval Kerala Mathematics" of footnote 13.

5. Independence of these discoveries.

The question naturally arises of the possibility of mutual influence between or among the discoverers of power series, in particular the series for the trigonometric functions. Because of the lively trade relations between the Arabs and the west coast of India over the centuries, it is generally accepted that mathematical ideas were also exchanged. However, there is no evidence in any existing mathematical works of the Arabs that they were aware of the concept of a power series. Therefore, we may grant the Indians priority in the discovery of the series for sine, cosine and arctangent. Moreover, historians of mathematics are in agreement that the European mathematicians were unaware of the Indian discovery of infinite series.[21] Thus, we may conclude that Newton, Gregory and Leibniz made their discoveries independently of the Indian work. In fact, it appears that yet another independent discovery of an infinite series giving the value of π was made by the Japanese mathematician Takebe Kenko (1664–1739) in 1722. His series is

$$\pi^2 = 4\left[1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!}\right].^{22}$$

This series was not obtained from the arctan series and its discussion is therefore not included. However, the independent discovery of the infinite series by different persons living in different environments and cultures gives us insight into the character of mathematics as a universal discipline.

Acknowledgement. I owe thanks to Phil Straffin for encouraging me to write this paper and to the referees for their suggestions.

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[21] See "Patterns of Mathematical Thought in the later Seventeenth Century" of footnote 17. See also A. Weil, "History of Mathematics: Why and How" in *Collected Papers*, Vol. 3 (New York: Springer-Verlag, 1979), p. 435.

[22] See D. E. Smith and Y. Mikami, *A History of Japanese Mathematics* (Chicago: Open Court, 1914). This series was also obtained by the French missionary Pierre Jartoux (1670–1720) in 1720. He worked in China and was in correspondence with Leibniz, but the present opinion is that Takebe's discovery was independent. Leonhard Euler (1707–1783) rediscovered the same series in 1737. A simple evaluation of it can be given using Clausen's formula for the square of a hypergeometric series.