Volume radius of a random polytope in a convex body

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Abstract

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. We choose N points $x_1, \dots, x_N$ independently and uniformly from K, and write $C(x_1, \dots, x_N)$ for their convex hull. If $n(\log n)^2 \leq N \leq \exp(c_1 n)$, we show that the expected volume radius

$$\mathbb{E}_{1/n}(K,N) = \int_K \cdots \int_K |C(x_1, \dots, x_N)|^{1/n}\, dx_N \cdots dx_1$$

of this random N-tope can be estimated by

$$c_1 \frac{\sqrt{\log(N/n)}}{\sqrt{n}} \leq \mathbb{E}_{1/n}(K,N) \leq c_2 L_K \frac{\log(N/n)}{\sqrt{n}},$$

where c_1, c_2 are absolute positive constants and L_K is the isotropic constant of K.

1 Introduction

We work in $\mathbb{R}^n$ which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $|\cdot|$ and fix an orthonormal basis $\{e_1, \dots, e_n\}$.

Let K be a compact subset of $\mathbb{R}^n$ with volume $|K| = 1$. The *matrix of inertia* $M(K)$ of K is the $n \times n$ matrix with entries

$$\big(M(K)\big)_{ij} = \int_K \langle x, e_i \rangle \langle x, e_j \rangle \, dx.$$

The determinant of $M(K)$ can be expressed in the form

$$n! \det\big(M(K)\big) = \int_K \cdots \int_K [\det(x_1, \dots, x_n)]^2 \, dx_n \cdots dx_1. \tag{1.1}$$

This identity goes back to Blaschke and can be easily verified by expanding the determinant inside the integral. Since the right hand side is invariant under $SL(n)$, we see that $\det\big(M(K)\big) = \det\big(M(TK)\big)$ for every $T \in SL(n)$. On the other hand, there exists $T_1 \in SL(n)$ for which $M(T_1K)$ is a multiple of the identity. Equivalently,

$$\int_{T_1K} \langle x, e_i\rangle\langle x, e_j\rangle\, dx = L_K^2 \delta_{ij} \tag{1.2}$$

where L_K is the *isotropic constant* of K (we refer to [5] for the isotropic position of a convex body). By (1), L_K is well defined and satisfies

$$n! L_K^{2n} = \int_K \cdots \int_K [\det(x_1, \ldots, x_n)]^2\, dx_n \cdots dx_1. \tag{1.3}$$

Assume now that K is a convex body with centroid at the origin. We fix $N \geq n+1$ and choose points $x_1, \ldots, x_N$ independently and uniformly from K. Let $C(x_1, \ldots, x_N)$ be their convex hull. For every $p > 0$ we consider the quantity

$$\mathbb{E}_p(K, N) = \left(\int_K \cdots \int_K |C(x_1, \ldots, x_N)|^p\, dx_N \cdots dx_1\right)^{1/pn}. \tag{1.4}$$

When $N = n+1$, these quantities are exact functions of the isotropic constant of K. To see this, note that

$$A_n(K) := \int_K \cdots \int_K |\mathrm{co}(0, x_1, \ldots, x_n)|^2\, dx_n \cdots dx_1$$

satisfies the identity $L_K^{2n} = n! A_n(K)$ and

$$A_n(K) \leq \mathbb{E}_2^{2n}(K, n+1) \leq (n+1)^2 A_n(K). \tag{1.5}$$

It follows that

$$c_1 \frac{L_K}{\sqrt{n}} \leq \mathbb{E}_2(K, n+1) \leq c_2 \frac{L_K}{\sqrt{n}} \tag{1.6}$$

where $c_1, c_2 > 0$ are absolute constants. Moreover, using Khintchine type inequalities for linear functionals on convex bodies (see the next section) one can show that $\mathbb{E}_p(K, n+1) \geq cL_K/\sqrt{n}$ for every $p > 0$, where $c > 0$ is an absolute constant.

In this paper, we give estimates for the volume radius $\mathbb{E}_{1/n}(K, N)$ of a random N-tope $C(x_1, \ldots, x_N)$ in K. It turns out that a generalization of the upper bound in (6) is possible.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. For every $N \geq n+1$, we have*

$$\mathbb{E}_{1/n}(K, N) \leq cL_K \frac{\log(2N/n)}{\sqrt{n}}$$

where $c > 0$ is an absolute constant.

The proof of this fact is presented in Section 2. We can also obtain upper estimates for $\mathbb{E}_p(K, N)$, $p > 1/n$, but the dependence on p does not seem to be optimal. Our method shows that if K is a ψ_2-*body* then one has the stronger estimate $\mathbb{E}_{1/n}(K, N) \leq cL_K\sqrt{\log(2N/n)}/\sqrt{n}$. This is optimal and might be the right dependence for every convex body K in $\mathbb{R}^n$.

Our lower bound is based on an extension of a result of Groemer [3].

Theorem 1.2. *Let K be a convex body of volume 1 and B be a ball of the same volume. Then,*

$$\mathbb{E}_p(K,N) \geq \mathbb{E}_p(B,N)$$

for every $p > 0$. In particular, the expected volume radius $\mathbb{E}_{1/n}(K,N)$ of a random N-tope in K is minimal when $K = B$.

Groemer proved the same fact for $p \geq 1$. Schöpf [7] proved Theorem 1.2 in the case $N = n+1$. Our argument is along the same lines: we show that Steiner symmetrization decreases $\mathbb{E}_p(K,N)$. A different extremal problem concerning the p-th moment of the volume of Minkowski sums of intervals defined by random points from a convex body was solved in [2]: the solution is again given by the Euclidean ball for every $p > 0$.

In the case of the ball, one can give a lower bound for $\mathbb{E}_{1/n}(K,N)$.

Theorem 1.3. *Let B be a ball of volume 1 in $\mathbb{R}^n$. If $n(\log n)^2 \leq N \leq \exp(cn)$, then*

$$\mathbb{E}_{1/n}(B,N) \geq c\frac{\sqrt{\log(N/n)}}{\sqrt{n}}$$

where $c > 0$ is an absolute constant.

Theorems 1.2 and 1.3 are proved in Section 3. It follows that $\mathbb{E}_{1/n}(K,N) \geq c\sqrt{\log(N/n)}/\sqrt{n}$ for every convex body K of volume 1. This bound is optimal. However, an interesting question is to give lower bounds for $\mathbb{E}_{1/n}(K,N)$ in terms of L_K. Since $\mathbb{E}_{1/n}(K,N) \leq 1$, this would lead to non-trivial upper estimates for the isotropic constant.

For standard notation and definitions we refer to [5] and [6]. We use c, c_1, c' etc. for absolute positive constants which are not necessarily the same in all their occurences.

2 The upper bound

Let $\alpha \in [1,2]$. We say that a convex body K in $\mathbb{R}^n$ is a *ψ_α-body with constant b_α* if

$$\left(\int_K |\langle x,y\rangle|^p\,dx\right)^{1/p} \leq b_\alpha p^{1/\alpha}\int_K |\langle x,y\rangle|\,dx \tag{2.1}$$

for every $y \in \mathbb{R}^n$ and $p \geq 1$. It is clear by the definition that if K is a ψ_α-body then the same is true for TK, $T \in GL(n)$ (with the same constant b_α). By Borell's lemma (see [6], Appendix III), every convex body K is a ψ_1-body with constant $b_1 = c$, where $c > 0$ is an absolute constant.

Assume that K has volume 1 and satisfies the isotropic condition

$$\int_K \langle x,y\rangle^2\,dx = L_K^2$$

for every $y \in S^{n-1}$. Then, the fact that K is a ψ_α-body with constant b_α is equivalent to the inequality

$$\left(\int_K |\langle x,y\rangle|^p dx\right)^{1/p} \leq b_\alpha p^{1/\alpha} L_K$$

for every $p \geq 1$ and $y \in S^{n-1}$. We shall prove the following.

Theorem 2.1. *Let K be a convex body in $\mathbb{R}^n$ with volume 1. Assume that K is a ψ_α-body with constant b_α. Then, for every $N \geq n+1$*

$$\mathbb{E}_{1/n}(K, N) \leq c b_\alpha L_K \frac{\big(\log(2N/n)\big)^{1/\alpha}}{\sqrt{n}}.$$

This implies Theorem 1.1. For the proof, we will use a result of Ball and Pajor [1] on the volume of symmetric convex bodies which are intersections of symmetric strips in $\mathbb{R}^n$.

Lemma 2.2. *Let $x_1, \dots, x_N \in \mathbb{R}^n$ and $1 \leq q < \infty$. If $W = \{z \in \mathbb{R}^n : |\langle z, x_j \rangle| \leq 1,\ j = 1, \dots, N\}$, then*

$$|W|^{1/n} \geq 2 \left(\frac{n+q}{n} \sum_{j=1}^N \frac{1}{|B_q^n|} \int_{B_q^n} |\langle z, x_j \rangle|^q dz \right)^{-1/q} . \square$$

Proof of Theorem 2.1: We may assume that K is isotropic. Write K_N for the absolute convex hull $\mathrm{co}\{\pm x_1, \dots, \pm x_N\}$ of N random points from K. By the Blaschke-Santaló inequality,

$$\mathbb{E}_{1/n}(K, N) \leq \mathbb{E}|K_N|^{1/n} \leq \omega_n^{2/n} \cdot \mathbb{E}|K_N^\circ|^{-1/n}$$

where K_N° is the polar body of K_N. Lemma 2.2 shows that

$$|K_N^\circ|^{-1/n} \leq \frac{1}{2} \left(\frac{n+q}{n} \sum_{j=1}^N \frac{1}{|B_q^n|} \int_{B_q^n} |\langle z, x_j \rangle|^q dz \right)^{1/q} \tag{2.2}$$

for every $q \geq 1$. Consider the convex body $W = K \times \cdots \times K$ (N times) in $\mathbb{R}^{Nn}$. We apply Hölder's inequality, change the order of integration and use the ψ_α-property of K:

$$\begin{aligned}
\mathbb{E}|K_N^\circ|^{-1/n} &\leq \int_W \frac{1}{2} \left(\frac{n+q}{n} \sum_{j=1}^N \frac{1}{|B_q^n|} \int_{B_q^n} |\langle z, x_j \rangle|^q \, dz \right)^{1/q} dx_N \cdots dx_1 \\
&\leq \frac{1}{2} \left(\frac{n+q}{n} \sum_{j=1}^N \frac{1}{|B_q^n|} \int_{B_q^n} \int_W |\langle z, x_j \rangle|^q \, dx_N \cdots dx_1 \, dz \right)^{1/q} \\
&\leq \frac{1}{2} \left(\frac{n+q}{n} \left(q^{1/\alpha} b_\alpha L_K \right)^q N \frac{1}{|B_q^n|} \int_{B_q^n} |z|^q \, dz \right)^{1/q} .
\end{aligned}$$

Since $\omega_n^{2/n} \leq c_1/n$ and $|z| \leq n^{1/2-1/q}$ for all $z \in B_q^n$, we get

$$\mathbb{E}_{1/n}(K, N) \leq \frac{c}{\sqrt{n}} b_\alpha L_K q^{1/\alpha} \left(\frac{N}{n} \right)^{1/q} \left(\frac{n+q}{n} \right)^{1/q} .$$

Choosing $q = \log(2N/n)$ we complete the proof. $\square$

Remark The proof shows that $\mathbb{E}_{p_0}(K, N) \leq c b_\alpha L_K \big(\log(2N/n)\big)^{1/\alpha}/\sqrt{n}$, where $p_0 = \log(2N/n)/n$. Since

$$f(x_1, \dots, x_N) = \left(\sum_{j=1}^N \int_{B_q^n} |\langle z, x_j \rangle|^q dz \right)^{1/q}$$

is a norm on $\mathbb{R}^{Nn}$, one can estimate $\mathbb{E}_p(K,N)$ for larger values of p by a standard application of Borell's lemma (see [6], Appendix III). When p is close to 1, the right dependence of $\mathbb{E}_p(K,N)$ on p is not clear.

3 The lower bound

Let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. We identify H with $\mathbb{R}^{n-1}$ and write $x=(y,t)$, $y\in H$, $t\in\mathbb{R}$ for a point $x\in\mathbb{R}^n$. If K is a convex body in $\mathbb{R}^n$ with $|K|=1$ and $P(K)$ is the orthogonal projection of K onto H, then

$$\mathbb{E}_p^{pn}(K,N)=\int_{P(K)}\cdots\int_{P(K)} M_{p,K}(y_1,\dots,y_N)\,dy_N\cdots dy_1 \tag{3.1}$$

where

$$M_{p,K}(y_1,\dots,y_N)=\int\limits_{\ell(K,y_1)}\cdots\int\limits_{\ell(K,y_N)}\left|C\big((y_1,t_1),\dots,(y_N,t_N)\big)\right|^p dt_N\cdots dt_1 \tag{3.2}$$

and $\ell(K,y)=\{t\in\mathbb{R}:(y,t)\in K\}$.

We fix $y_1,\dots,y_N\in H$ and consider the function $F_Y:\mathbb{R}^N\to\mathbb{R}$ defined by

$$F_Y(t_1,\dots,t_N)=\left|C\big((y_1,t_1),\dots,(y_N,t_N)\big)\right|, \tag{3.3}$$

where $Y=(y_1,\dots,y_N)$. The key observation in [3] is the following:

Lemma 3.1. *For any $y_1,\dots,y_N\in H$, the function F_Y is convex.* □

We now also fix $r_1,\dots,r_N>0$ and define $Q=\{U=(u_1,\dots,u_N):|u_i|\le r_i,\ i=1,\dots,N\}$. For every N-tuple $W=(w_1,\dots,w_N)\in\mathbb{R}^N$ we set

$$G_W(u_1,\dots,u_N)=F_Y(w_1+u_1,\dots,w_N+u_N), \tag{3.4}$$

and write

$$G_W(U)=F_Y(W+U).$$

This is the volume of the polytope which is generated by the points (y_i,w_i+u_i). Finally, for every $W\in\mathbb{R}^N$ and $\alpha>0$, we define

$$A(W,\alpha)=\{U\in Q:G_W(U)\le\alpha\}. \tag{3.5}$$

With this notation, we have

Lemma 3.2. *Let $\alpha>0$ and $\lambda\in(0,1)$. If $W,W'\in\mathbb{R}^N$, then*

$$|A(\lambda W+(1-\lambda)W',\alpha)|\ge|A(W,\alpha)|^\lambda|A(W',\alpha)|^{1-\lambda}. \tag{3.6}$$

Proof: Let $U\in A(W,\alpha)$ and $U'\in A(W',\alpha)$. Then, using the convexity of F_Y we see that

$$\begin{aligned}G_{\lambda W+(1-\lambda)W'}\big(\lambda U+(1-\lambda)U'\big)&=F_Y\big(\lambda(W+U)+(1-\lambda)(W'+U')\big)\\&\le\lambda F_Y(W+U)+(1-\lambda)F_Y(W'+U')\\&=\lambda G_W(U)+(1-\lambda)G_{W'}(U')\\&\le\alpha.\end{aligned}$$

Therefore,

$$A\big(\lambda W + (1-\lambda)W'\big) \supseteq \lambda A(W,\alpha) + (1-\lambda)A(W',\alpha)$$

and the result follows from the Brunn-Minkowski inequality. $\square$

Observe that the polytopes $C\big((y_i, w_i + u_i)_{i\le N}\big)$ and $C\big((y_i, -w_i - u_i)_{i\le N}\big)$ have the same volume since they are reflections of each other with respect to H. It follows that

$$A(-W,\alpha) = -A(W,\alpha) \tag{3.7}$$

for every $\alpha > 0$. Taking $W' = -W$ and $\lambda = 1/2$ in Lemma 3.2, we obtain the following:

Lemma 3.3. *Let $y_1, \dots, y_N \in H$. For every $W \in \mathbb{R}^N$ and every $\alpha > 0$,*

$$|A(O,\alpha)| \ge |A(W,\alpha)|, \tag{3.8}$$

where O is the origin in $\mathbb{R}^N$. $\square$

For every $y \in P(K)$, we denote by $w(y)$ the midpoint and by $2r(y)$ the length of $\ell(K,y)$. Let $S(K)$ be the Steiner symmetral of K. By definition, $P(S(K)) = P(K) = P$ and for every $y \in P$ the midpoint and length of $\ell(S(K),y)$ are $w'(y) = 0$ and $2r'(y) = 2r(y)$ respectively.

Lemma 3.4. *Let $y_1, \dots, y_N \in P(K) = P(S(K))$. Then,*

$$M_{p,K}(y_1, \dots, y_N) \ge M_{p,S(K)}(y_1, \dots, y_N)$$

for every $p > 0$.

Proof: In the notation of the previous lemmata, we have

$$\begin{aligned} M_{p,K}(y_1, \dots, y_N) &= \int_Q \left(G_W(U)\right)^p dU \\ &= \int_0^\infty |\{U \in Q : G_W(U) \ge t^{1/p}\}|\, dt \\ &= \int_0^\infty \Big(|Q| - |A(W, t^{1/p})|\Big)\, dt. \end{aligned}$$

By the definition of $S(K)$,

$$M_{p,K}(y_1, \dots, y_N) = \int_Q \left(G_O(U)\right)^p dU = \int_0^\infty \Big(|Q| - |A(O, t^{1/p})|\Big)\, dt,$$

and the result follows from Lemma 3.3. $\square$

It is now clear that $\mathbb{E}_p(K,N)$ decreases under Steiner symmetrization.

Theorem 3.5. *Let K be a convex body with volume $|K| = 1$ and let H be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. If $S_H(K)$ is the Steiner symmetral of K with respect to H, then*

$$\mathbb{E}_p\left(S_H(K), N\right) \le \mathbb{E}_p(K,N)$$

for every $p > 0$.

Proof: We may assume that $H = \mathbb{R}^{n-1}$. Since $P(S_H(K)) = P(K)$, Lemma 3.4 and (9) show that

$$
\begin{aligned}
\mathbb{E}_p^{pn}(K,N) &= \int_{P(K)} \cdots \int_{P(K)} M_{p,K}(y_1,\dots,y_N)\,dy_N \cdots dy_1 \\
&\geq \int\limits_{P(S_H(K))} \cdots \int\limits_{P(S_H(K))} M_{p,S_H(K)}(y_1,\dots,y_N)\,dy_N \cdots dy_1 \\
&= \mathbb{E}_p^{pn}(S_H(K),N),
\end{aligned}
$$

completing the proof. □

Proof of Theorem 1.2: Since the ball B of volume 1 is the Hausdorff limit of a sequence of successive Steiner symmetrizations of K, Theorem 3.1 shows that the expected volume radius is minimal in the case of B. □

Remark The argument shows that a more general fact holds true.

Theorem 3.6. *Let K be a convex body of volume 1 and let B be a ball of the same volume. Then,*

$$
\begin{aligned}
&\int_K \cdots \int_K f\big(|C(x_1,\dots,x_N)|\big)\,dx_N \cdots dx_1 \\
&\qquad\qquad \geq \int_B \cdots \int_B f\big(|C(x_1,\dots,x_N)|\big)\,dx_N \cdots dx_1
\end{aligned}
$$

for every increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$. □

Next, we give a lower bound for $\mathbb{E}_{1/n}(B,N)$. We will actually prove that the convex hull of N random points from $K = B$ contains a ball of radius $c\sqrt{\log(2N/n)}/\sqrt{n}$ (an analogous fact was proved in [4] for an arbitrary convex body K, but the fact that we are dealing with a ball leads to the much better estimate which is needed).

Lemma 3.7. *Let $B = rD_n$ be the centered ball of volume 1 in $\mathbb{R}^n$. If $\theta \in S^{n-1}$, then*

$$
\text{Prob}\,(x \in B : \langle x,\theta\rangle \geq \varepsilon r) \geq \exp(-4\varepsilon^2 n)
$$

for every $\varepsilon \in (c_1/\sqrt{n}, 1/4)$, where $c_1 > 0$ is an absolute constant.

Proof: A simple calculation shows that

$$
\begin{aligned}
\text{Prob}\,(x \in B : \langle x,\theta\rangle \geq \varepsilon r) &= \omega_{n-1} r^n \int_\varepsilon^1 (1-t^2)^{(n-1)/2} dt \\
&\geq \frac{\omega_{n-1}}{\omega_n}\varepsilon(1-4\varepsilon^2)^{(n-1)/2} \\
&\geq \exp\big(-4(n-1)\varepsilon^2\big) \\
&\geq \exp(-4\varepsilon^2 n)
\end{aligned}
$$

since $\sqrt{n}\omega_n \leq c_1\omega_{n-1}$ for some absolute constant $c_1 > 0$. □

Lemma 3.8. *There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $n(\log n)^2 \leq N \leq \exp(cn)$, then*

$$
C(x_1,\dots,x_N) \supseteq \frac{\sqrt{\log(N/n)}}{6\sqrt{n}} B
$$

with probability greater than $1 - \exp(-n)$.

Proof: By Lemma 3.7, for every $\theta \in S^{n-1}$ we have

$$\begin{aligned}\mathrm{Prob}\Big((x_1,\dots,x_N) : \max_{j\le N}\langle x_j,\theta\rangle \le \varepsilon r\Big) &\le \big(1-\exp(-4\varepsilon^2 n)\big)^N \\ &\le \exp\big(-N\exp(-4\varepsilon^2 n)\big)\end{aligned}$$

for every $\varepsilon \in (c_1/\sqrt{n}, 1/4)$. Let $\mathcal{N}$ be a ρ-net for S^{n-1} with cardinality $|\mathcal{N}| \le \exp\big(\log(1+2/\rho)n\big)$. If

$$\exp\big(n\log(1+2/\rho) - N\exp(-4\varepsilon^2 n)\big) \le \exp(-n), \tag{3.9}$$

we have $\max_{j\le N}\langle x_j,\theta\rangle > \varepsilon r$ for all $\theta \in \mathcal{N}$ with probability greater than $1-\exp(-n)$. For every $u \in S^{n-1}$ we find $\theta \in \mathcal{N}$ with $|\theta - u| < \rho$. Then,

$$\max_{j\le N}\langle x_j,u\rangle \ge \max_{j\le N}\langle x_j,\theta\rangle - \max_{j\le N}\langle x_j,\theta-u\rangle \ge (\varepsilon-\rho)r.$$

We choose $\varepsilon = 2a\big((\log(N/n)/n\big)^{1/2}$ ($a>0$ is an absolute constant to be determined) and $\rho = \varepsilon/2$. Then,

$$n\log(1+2/\rho) + n \le 2n\log(3/\rho) \le n\log\left(\frac{9n}{a^2\log(N/n)}\right) \le n\log n,$$

if $a^2 \ge 9/\log(N/n)$. Therefore, (17) will be a consequence of

$$\exp\big(16a^2\log(N/n)\big) \le \frac{N}{n\log n}, \tag{3.10}$$

which can be written equivalently in the form

$$\left(\frac{N}{n}\right)^{1-16a^2} \ge \log n. \tag{3.11}$$

If $N \ge n(\log n)^2$ and $a = 1/6$, then (19) is clearly satisfied. The restriction we had posed on a was $a^2 \ge 9/2\log\log n$, which is also satisfied when $n \ge n_0$, for a suitable (absolute) $n_0 \in \mathbb{N}$. □

Theorem 3.9. *Let B be the ball of volume 1 in $\mathbb{R}^n$. If $n(\log n)^2 \le N \le \exp(cn)$, then*

$$\mathbb{E}_{1/n}(B,N) \ge c\frac{\sqrt{\log(N/n)}}{\sqrt{n}}$$

where $c>0$ is an absolute constant.

Proof: Let $f(N,n) = \sqrt{\log(N/n)}/6\sqrt{n}$ and

$$A = \{(x_1,\dots,x_N) \,:\, C(x_1,\dots,x_n) \supseteq f(N,n)B\}.$$

By Lemma 3.8, $\mathrm{Prob}(A) \ge 1-\exp(-n)$, and hence

$$\begin{aligned}\mathbb{E}(B,N) &\ge \int_A |C(x_1,\dots,x_N)|\,dx_1\cdots dx_N \\ &\ge \big(1-\exp(-n)\big)f(N,n)|B| \\ &\ge f(N,n)/2,\end{aligned}$$

completing the proof. $\square$

This completes the proof of Theorem 1.3.

Remarks (a) The argument shows that if $\delta > 0$ and $c_1 n(\log n)^{1+\delta} \leq N \leq \exp(c_2 n)$, then

$$\mathbb{E}_{1/n}(B,N) \geq c\sqrt{\delta}\,\frac{\sqrt{\log(2N/n)}}{\sqrt{n}}. \tag{3.12}$$

(b) The estimate of Lemma 3.8 implies that

$$\mathbb{E}_p(B,N) \geq c\,\frac{\sqrt{\log(2N/n)}}{\sqrt{n}} \tag{3.13}$$

for every $p \geq \exp(-n)$. However, it is not clear if (21) holds true for every $p > 0$.

References

[1] K. Ball and A. Pajor, *Convex bodies with few faces*, Proc. Amer. Math. Soc. **110** (1990), 225–231.

[2] J. Bourgain, M. Meyer, V. Milman and A. Pajor *On a geometric inequality*, Lecture Notes in Mathematics **1317**, Springer, Berlin (1986–87), 271–282.

[3] H. Groemer, *On the mean value of the volume of a random polytope in a convex set*, Arch. Math. **25** (1974), 86–90.

[4] A.A. Giannopoulos and V.D. Milman, *Concentration property on probability spaces*, Advances in Math. **156** (2000), 77–106.

[5] V.D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Lecture Notes in Mathematics **1376**, Springer, Berlin (1989), 64–104.

[6] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics **1200** (1986), Springer, Berlin.

[7] P. Schöpf, *Gewichtete Volumsmittelwerte von Simplices, welche zufällig in einemen konvexen Körper des $\mathbb{R}^n$ gewählt werden*, Monatsch. Math. **83** (1977), 331–337.

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