Matrix Range Characterizations of Operator System Properties

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Overview

Talk is based on a paper of the same title with B. Passer.
Given an operator system $\mathcal{S}$ there are two sequences of operator systems that one can build from $\mathcal{S}$, denoted $OMAX_k(\mathcal{S})$ and $OMIN_k(\mathcal{S})$, $k \in \mathbb{N}$.
As unital *-vector spaces, these are just $\mathcal{S}$ but with possibly new matrix orders. These matrix orders are uniquely characterized by the following universal properties:

$$id : OMAX_k(\mathcal{S}) \to \mathcal{S} \text{ and } id : \mathcal{S} \to OMIN_k(\mathcal{S}) \text{ are CP,}$$

$$id : \mathcal{S} \to OMAX_k(\mathcal{S}) \text{ and } id : OMIN_k(\mathcal{S}) \to \mathcal{S} \text{ are k-positive,}$$

$$\forall \mathcal{T},\ \phi : \mathcal{S} \to \mathcal{T} \text{ is k-positive } \iff \phi : OMAX_k(\mathcal{S}) \to \mathcal{T} \text{ is CP}$$

$$\forall \mathcal{T},\ \psi : \mathcal{T} \to \mathcal{S} \text{ is k-positive } \iff \psi : \mathcal{T} \to OMIN_k(\mathcal{S}) \text{ is CP.}$$

Since a map is CP if and only if it is k-positive for all k, it is natural to wonder if as $k \to +\infty$ whether or not $OMAX_k(\mathcal{S})$ and $OMIN_k(\mathcal{S})$ "converge" to $\mathcal{S}$ in some sense. The answer is "not always" and the obstruction to convergence is related to whether or not $\mathcal{S}$ possesses two important properties: the *operator system local lifting property(OSLLP)* and *exactness*.
In the special case that we start with a d-tuple $\mathrm{T} = (T_1, ..., T_d),\ T_i \in B(\mathcal{H})$ of operators and

$$\mathcal{S}_{\mathrm{T}} = span\{I_{\mathcal{H}}, T_1, ..., T_d, T_1^*, ..., T_d^*\},$$

then we prove that these two properties are characterized by "geometric" properties of the *joint matrix ranges* $\{W^n(\mathrm{T}) : n \in \mathbb{N}\}$ which are the prototypical *matrix convex sets*.

Operator Systems, briefly

Recall that a *concrete* operator system is just a subspace $\mathcal{S} \subseteq B(\mathcal{H})$ containing $I_{\mathcal{H}}$ and with the property that $X \in \mathcal{S} \implies X^* \in \mathcal{S}$ together with their family of *matrix cones* given by the identification:

$$M_n(\mathcal{S}) \subseteq B(\mathbb{C}^n \otimes \mathcal{H}) \equiv M_n(B(\mathcal{H})),\ M_n(\mathcal{S})^+ =: M_n(\mathcal{S}) \cap B(\mathbb{C}^n \otimes \mathcal{H})^+.$$

These were given an abstract characterization by Choi-Effros as *-vector spaces, along with a set of cones $M_n(\mathcal{S})^+, n \in \mathbb{N}$ that define the positive elements and an *Archimedean matrix order unit, 1*.

$OMIN_k(\mathcal{S})$ and $OMAX_k(\mathcal{S})$

Given an operator system $\mathcal{S}$, **B. Xhabli** introduced new operator systems, keeping the unit and *-vector space the same but defining new matrix cones by:

$$M_n(OMIN_k(\mathcal{S}))^+ = \{(x_{i,j}) \in M_n(\mathcal{S}) : (\phi(x_{i,j})) \in M_n(M_k)^+ \equiv M_{nk}^+, \forall \phi : \mathcal{S} \to M_k \text{ CP}\},$$

$$\begin{aligned} M_n(OMAX_k(\mathcal{S}))^+ = \{(x_{i,j}) \in M_n(\mathcal{S}) :& \\ \forall \epsilon > 0,\ \exists m,\ & P_t \in M_k(\mathcal{S})^+, 1 \le t \le m, \\ & A \in M_{n,mk}(\mathbb{C}), \epsilon 1_n + (x_{i,j}) = A\big(\oplus_{t=1}^m P_t\big)A^*\} \end{aligned}$$

Xhabli proved that these have the universal properties mentioned earlier.

Note that $\forall n, k$,

$$M_n(OMAX_k(\mathcal{S}))^+ \subseteq M_n(OMAX_{k+1}(\mathcal{S}))^+ \subseteq M_n(\mathcal{S})^+ \\ \subseteq M_n(OMIN_{k+1}(\mathcal{S}))^+ \subseteq M_n(OMIN_k(\mathcal{S}))^+,$$

which implies that the identity maps

$$OMAX_k(\mathcal{S}) \to OMAX_{k+1}(\mathcal{S}) \to \mathcal{S} \to OMIN_{k+1}(\mathcal{S}) \to OMIN_k(\mathcal{S}),$$

are UCP.
Also,

$$\cup_k M_n(OMAX_k(\mathcal{S}))^+ = M_n(\mathcal{S})^+ = \cap_k M_n(OMIN_k(\mathcal{S}))^+, \ \forall n.$$

Is

$$\lim_k \|id\|_{CB(\mathcal{S}, OMAX_k(\mathcal{S}))} \stackrel{?}{=} 1 \text{ and/or } \lim_k \|id\|_{CB(OMIN_k(\mathcal{S}), \mathcal{S})} \stackrel{?}{=} 1.$$

We will see that there are underlying properties of $\mathcal{S}$ that can be obstructions to these stronger notions of convergence.

The OSLLP

Let $\mathcal{A}, \mathcal{B}$ be unital C*-algebras. By a **quotient** map $\pi : \mathcal{A} \to \mathcal{B}$ we mean a unital, onto *-homomorphism. We say that a UCP map $\phi : \mathcal{S} \to \mathcal{B}$ **lifts** provided that there is a UCP map $\psi : \mathcal{S} \to \mathcal{A}$ with $\phi = \pi \circ \psi$. We say that $\mathcal{S}$ has the **lifting property(LP)** provided that every UCP map into any quotient lifts. We say that $\mathcal{S}$ has the **operator system local lifting property(OSLLP)** provided that for every UCP $\phi : \mathcal{S} \to \mathcal{B}$ and for every *finite dimensional* operator subsystem $\mathcal{S}_0 \subseteq \mathcal{S}$ the map $\phi|_{\mathcal{S}_0} : \mathcal{S}_0 \to \mathcal{B}$ lifts.

Kavruk-P-Todorov-Tomforde: $\mathcal{S}$ has the OSLLP
$\iff \mathcal{S} \otimes_{min} B(\mathcal{H}) = \mathcal{S} \otimes_{max} B(\mathcal{H})$, i.e., the identity map is a complete order isomorphism.
The minimal tensor product is the same as the spatial tensor product. The maximal tensor product is the maximal tensor product in the category of operator systems. For C*-algebras it agrees with the maximal C*-tensor product. It is not the same as the maximal operator space tensor product $\hat{}$, in fact,
$M_2 \hat{\otimes} M_2 \neq M_4$.

Corollary(Junge-Pisier): $B(\mathcal{H}) \otimes_{min} B(\mathcal{H}) \neq B(\mathcal{H}) \otimes_{max} B(\mathcal{H})$.
Or else $B(\mathcal{H})$ would have the OSLLP $\implies$ every finite dimensional operator system has the LP.

Kavruk:

- For LP and OSLLP it is enough to consider $\mathcal{A} = B(\mathcal{H})$ and $\mathcal{B} = \mathcal{Q}(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}(\mathcal{H})$, i.e., the Calkin algebra.
- If $\mathcal{S}$ is finite dimensional, then every k-positive map into a quotient has a k-positive lifting.
- Consequently, if $\mathcal{S}$ is finite dimensional, then $OMAX_k(\mathcal{S})$ has the LP and $OMAX_k(\mathcal{S}) \otimes_{min} B(\mathcal{H}) = OMAX_k(\mathcal{S}) \otimes_{max} B(\mathcal{H})$.

Proposition(new): Assume that $\mathcal{S}$ is finite dimensional. If $\lim_k \|id\|_{CB(\mathcal{S}, OMAX_k(\mathcal{S}))} = 1$, then $\mathcal{S}$ has the LP.

$$\begin{array}{ccc}
\mathcal{S} \otimes_{min} B(\mathcal{H}) & \xrightarrow{\gamma = id \otimes id} & \mathcal{S} \otimes_{max} B(\mathcal{H}) \\
{\scriptstyle id\otimes id, cb}\Big\downarrow & & \Big\uparrow{\scriptstyle id\otimes id, cp} \\
OMAX_k(\mathcal{S}) \otimes_{min} B(\mathcal{H}) & = & OMAX_k(\mathcal{S}) \otimes_{max} B(\mathcal{H})
\end{array}$$

we see that γ is unital and completely contractive, hence UCP. The work of Passer-P shows that the converse holds in finite dimensional case.

Problem

- *Does $OMAX_k(\mathcal{S})$ always have the OSLLP?*
- *How far apart can $\mathcal{T}$ and $OMAX_k(\mathcal{T})$ be for $\mathcal{T} \subseteq OMAX_k(\mathcal{S})$ finite dimensional? See Oikhberg for operator space versions for k=1.*

Some Intriguing Examples

Let $\mathcal{S}_n = \{1, u_1, ..., u_n, u_1^*, ..., u_n^*\} \subseteq C^*(\mathbb{F}_n)$ where $\mathbb{F}_n$ is the free group on n generators and u_i are the generating unitaries. This has the LP, because contractions lift.

Let $\mathcal{U}_n = span\{I_{2n}, E_{k,k+1}, E_{k+1,k} : k \text{ odd } \} \subseteq M_{2n}$.
Farenick-Kavruk-P-Todorov proved that $\mathcal{U}_n$ is the operator system dual of $\mathcal{S}_n$ and fails the LP. However, no concrete example of a UCP map into $\mathcal{Q}(\mathcal{H})$ that fails to lift has been given.

Let $\mathcal{W}_{3,2} = \{(a_1, ..., a_6) : a_1 + a_2 = a_3 + a_4 = a_5 + a_6\} \subseteq \ell_6^\infty$.
Kavruk proved that a C*-algebra is nuclear iff $\mathcal{W}_{3,2} \otimes_{min} \mathcal{A} = \mathcal{W}_{3,2} \otimes_{max} \mathcal{A}$. Because $B(\mathcal{H})$ is not nuclear, $\mathcal{W}_{3,2}$ does not have the LP.
However, no concrete example of a UCP map $\phi : \mathcal{W}_{3,2} \to \mathcal{Q}(\mathcal{H})$ that fails to lift has been given.

Note that if ϕ extends to ℓ_6^∞ then you are asking if six positive elements $p_i \in \mathcal{Q}(\mathcal{H})$ that satisfy $p_1 + ... + p_6 = 1$, $p_1 + p_2 = p_3 + p_4 = p_5 + p_6$ lift to six positive elements satisfying the same relations. This can be done! So extendable maps lift. However, since $\mathcal{Q}(\mathcal{H})$ is not injective ϕ need not extend. In fact, one can show that ϕ lifts iff ϕ extends. More generally, **Kavruk** defined spaces $\mathcal{W}_{n,k} \subseteq \ell_{nk}^\infty$ and proved that these are all *nuclearity detectors* for all $n, k \geq 2$, $(n, k) \neq (2, 2)$ and hence fail the LP for the same reason. Kavruk showed that $\mathcal{W}_{2,2}$ is nuclear, so has the LP and maps from $\mathcal{W}_{2,2}$ into $\mathcal{Q}(\mathcal{H})$ extend to ℓ_4^∞.

Exactness

An element p of an operator system $\mathcal{R}$ is *strictly positive* if $\exists \delta > 0$ such that $p - \delta 1 \geq 0$.
An onto UCP map $\pi : \mathcal{R} \to \mathcal{T}$ is called a *quotient map* if $\forall n$ every strictly positive element of $M_n(\mathcal{T})$ has a strictly positive preimage.
An operator system $\mathcal{S}$ is called **exact** provided that for every quotient map $\pi : \mathcal{A} \to \mathcal{B}$ the map $id \otimes \pi : \mathcal{S} \otimes_{min} \mathcal{A} \to \mathcal{S} \otimes_{min} \mathcal{B}$ is a quotient map and $ker(id \otimes \pi) = \mathcal{S} \otimes \ker(\pi)$.
We define $\mathcal{S} \otimes_{el} \mathcal{T} \subseteq_{coi} I(\mathcal{S}) \otimes_{max} \mathcal{T}$.

Kavruk-P-Tomforde-Todorov: $\mathcal{S}$ is exact iff $\mathcal{S} \otimes_{min} \mathcal{T} = \mathcal{S} \otimes_{el} \mathcal{T}, \forall \mathcal{T}$.
The operator systems, $\mathcal{U}_n, \mathcal{W}_{n,k}, OMIN_k(\mathcal{S})$ are exact, $\mathcal{S}_n$ is not exact.

Proposition(new): If $\lim_k \|id\|_{CB(OMIN_k(\mathcal{S}),\mathcal{S})} = 1$, then $\mathcal{S}$ is exact.

$$\begin{array}{ccc}
\mathcal{S} \otimes_{min} \mathcal{T} & \xrightarrow{\gamma} & \mathcal{S} \otimes_{el} \mathcal{T} \\
{\scriptstyle id\otimes id}\downarrow & & \uparrow{\scriptstyle \delta = id\otimes id} \\
OMIN_k(\mathcal{S}) \otimes_{min} \mathcal{T} & = & OMIN_k(\mathcal{S}) \otimes_{el} \mathcal{T}
\end{array}$$

Tricky bit is showing that $\|\delta\|_{cb} = \|id\|_{CB(OMIN_k(\mathcal{S}),\mathcal{S})}$.

Matrix Ranges

Given a d-tuple of operators, $\mathrm{T} = (\mathrm{T}_1, ..., \mathrm{T}_\mathrm{d}),\ \mathrm{T}_\mathrm{i} \in \mathrm{B}(\mathcal{H})$, we set

$$\mathcal{S}_\mathrm{T} = span\{I_\mathcal{H}, T_1, ... T_d, T_1^*, ..., T_d^*\}.$$

The *n-th matrix range* is the set

$$W^n(\mathrm{T}) =: \{(\phi(\mathrm{T}_1), ..., \phi(\mathrm{T}_\mathrm{d})) : \phi \in \mathrm{UCP}(\mathcal{S}_\mathrm{T}, \mathrm{M}_\mathrm{n})\} \subseteq \mathrm{M}_\mathrm{n} \oplus \cdots \oplus \mathrm{M}_\mathrm{n},$$

and the *matrix range* is the collection $\mathcal{W}(\mathrm{T}) = \{\mathrm{W}^\mathrm{n}(\mathrm{T}) : \mathrm{n} \in \mathbb{N}\}$. Note that $W^1(\mathrm{T}) \subseteq \mathbb{C}^\mathrm{d}$ is a closed bounded convex subset. The family $\mathcal{W}(\mathrm{T})$ is the prototypical example of a *matrix convex set*. Also the map $\phi \in UCP(\mathcal{S}_\mathrm{T}, M_n) \to (\phi(T_1), ..., \phi(T_n)) \in W^n(\mathrm{T})$ is a one-to-one affine map. So the matrix ranges are just a geometric representation of these spaces of UCP maps.

For $\mathcal{S}_d = span\{1, u_1, ..., u_d, u_1^*, ..., u_d^*\} \subseteq C^*(\mathbb{F}_d)$ we have *-homorphisms sending u_i to any d-tuple of unitaries. Using the Halmos dilation of a contraction to a unitary it follows that

$$W^n(u_1, ..., u_d) = \{(A_1, ..., A_d) : A_i \in M_n, \|A_i\| \leq 1\}.$$

For

$$\mathcal{U}_2 = span\{I_4, E_{1,2}, E_{3,4}, E_{1,2}^*, E_{3,4}^*\} = \{ \begin{pmatrix} a & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & e \\ 0 & 0 & f & a \end{pmatrix} | a, b, c, d, e \in \mathbb{C}\}$$

we have that

$$W^n(E_{1,2}, E_{3,4}) \subsetneq W^n(u_1, u_2), \exists n,$$

or else $\mathcal{U}_2 = \mathcal{S}_2$ as operator systems, which would imply that $C^*(\mathbb{F}_2) = C_e^*(\mathcal{S}_2) = C_e^*(\mathcal{U}_2) = M_2 \oplus M_2$!!

Given $\mathcal{S}_{\mathrm{T}}$, we write T_i^{k-max} for the image of T_i in $OMAX_k(\mathcal{S}_{\mathrm{T}})$ and set $\mathrm{T}^{\mathrm{k-max}} = (\mathrm{T}_1^{\mathrm{k-max}}, ..., \mathrm{T}_{\mathrm{d}}^{\mathrm{k-max}})$, with similar definition for T_i^{k-min} and $\mathrm{T}^{\mathrm{k-min}}$.
Since the identity maps $OMAX_k(\mathcal{S}_{\mathrm{T}}) \to \mathcal{S}_{\mathrm{T}}$ and $\mathcal{S}_{\mathrm{T}} \to OMIN_k(\mathcal{S}_{\mathrm{T}})$ are UCP, we have

$$W^n(\mathrm{T}^{\mathrm{k-min}}) \subseteq \mathrm{W}^{\mathrm{n}}(\mathrm{T}) \subseteq \mathrm{W}^{\mathrm{n}}(\mathrm{T}^{\mathrm{k-max}}).$$

These sets have several characterizations:

$$W^n(\mathrm{T}^{\mathrm{k-max}}) =$$
$$\{(\phi(T_1), ..., \phi(T_d)) : \phi : \mathcal{S}_{\mathrm{T}} \to M_n, \text{ unital and k-positive } \} =$$
$$\{(A_1, ..., A_d) : \forall V \in M_{k,n}, V^*V = I_n, (V^*A_1V, ..., V^*A_dV) \in W^k(\mathrm{T})\},$$

$$W^n(\mathrm{T}^{\mathrm{k-min}}) = \{(\mathrm{V^*B_1V, ..., V^*B_dV})\} \text{ where}$$
$$B_i = V^*\big(\oplus_{j=1}^m A_{i,j}\big)V, \text{ with } (A_{1,j}, ..., A_{d,j}) \in W^k(\mathrm{T}), \forall \mathrm{j}$$
$$\text{and } V \in M_{n,km}, V^*V = I_n$$

Studying these two operations on matrix convex sets has become popular. They are dual to Xhabli's constructions.

Recall that for two sets X, Y in a metric space, their *Hausdorff distance* is

$$d_H(X,Y) = \max\{\sup_{y\in Y}\inf_{x\in X} d(x,y), \ \sup_{x\in X}\inf_{y\in Y} d(x,y)\}.$$

Passer-P: Let $\mathcal{S} = \mathcal{S}_{\mathrm{T}}$ be a finite dimensional operator system. Then the following are equivalent:

1. $\mathcal{S}$ has the LP,
2. $\lim_k sup_n\ d_H(W^n(\mathrm{T}), \mathrm{W}^{\mathrm{n}}(\mathrm{T}^{\mathrm{k-max}})) = 0,$
3. (new) $\lim_k \|id\|_{CB(\mathcal{S}, OMAX_k(\mathcal{S}))} = 1.$

Passer-P: Let $\mathcal{S} = \mathcal{S}_{\mathrm{T}}$ be a finite dimensional operator system. Then the following are equivalent:

1. $\mathcal{S}$ is exact,
2. $\lim_k sup_n d_H(W^n(\mathrm{T}^{\mathrm{k-min}}), \mathrm{W}^{\mathrm{n}}(\mathrm{T})) = 0,$
3. (new) $\lim_k \|id\|_{CB(OMIN_k(\mathcal{S}),\mathcal{S})} = 1.$

For the operator systems, $\mathcal{S}_n, \mathcal{U}_n, \mathcal{W}_{n,k}$, no idea what the values of these limits are for the cases where they are not 0 or 1.

Problem

Let $\mathcal{S}$ be an operator system.

- *Does $\mathcal{S}$ have the OSLLP iff $\lim_k \|id\|_{CB(\mathcal{S},OMAX_k(\mathcal{S}))} = 1$?*
- *Does $\mathcal{S}$ exact imply $\lim_k \|id\|_{CB(OMIN_k(\mathcal{S}),\mathcal{S})} = 1$?*

I suspect that all three implications are false.

Dilation/Scaling Constants

These are studied quite a bit by researchers on matrix convex sets. We set

- $\alpha_k(\mathrm{T}) = \inf\{\mathrm{r} : \mathrm{W}^{\mathrm{n}}(\mathrm{T}^{\mathrm{k-max}}) \subseteq \mathrm{r}\mathrm{W}^{\mathrm{n}}(\mathrm{T}), \forall \mathrm{n}\}$,
- $\beta_k(\mathrm{T}) = \inf\{\mathrm{r} : \mathrm{W}^{\mathrm{n}}(\mathrm{T}) \subseteq \mathrm{r}\mathrm{W}^{\mathrm{n}}(\mathrm{T}^{\mathrm{k-min}}), \forall \mathrm{n}\}$,
- $\gamma_k(\mathrm{T}) = \inf\{\mathrm{r} : \mathrm{W}^{\mathrm{n}}(\mathrm{T}^{\mathrm{k-max}}) \subseteq \mathrm{r}\mathrm{W}^{\mathrm{k}}(\mathrm{T}^{\mathrm{k-min}}), \forall \mathrm{n}\}$.

It is easily seen that these parameters are non-increasing and we set $\alpha(\mathrm{T}) = \lim_{\mathrm{k}} \alpha_{\mathrm{k}}(\mathrm{T})$, $\beta(\mathrm{T}) = \lim_{\mathrm{k}} \beta_{\mathrm{k}}(\mathrm{T})$, $\gamma(\mathrm{T}) = \lim_{\mathrm{k}} \gamma_{\mathrm{k}}(\mathrm{T})$.

Passer-P: Assume that the interior of $W^1(\mathrm{T})$ is non-empty. Then:
$\mathcal{S}_{\mathrm{T}}$ has the LP iff $\alpha(\mathrm{T}) = 1$,
$\mathcal{S}_{\mathrm{T}}$ is exact iff $\beta(\mathrm{T}) = 1$,
$\mathcal{S}_{\mathrm{T}}$ is exact and has the LP iff $\gamma(\mathrm{T}) = 1$.
So $\alpha((u_1, ..., u_n)) = 1$, $\alpha((E_{1,2}, E_{3,4})) \neq 1$, $\alpha(\mathcal{W}_{n,k}) \neq 1$, while $\beta((u_1, ..., u_n)) \neq 1$, $\beta((E_{1,2}, E_{3,4})) = 1$, $\beta(\mathcal{W}_{n,k}) = 1$, and γ is not 1 for all.

The Smith-Ward Problem(1979)

Given $T \in B(\mathcal{H})$ we let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators, $\mathcal{Q}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and let $\pi : B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ denote the quotient map.

The **essential matrix ranges** are defined to be $W_e^n(T) = W^n(\pi(T))$. The Smith-Ward problem asks if there exists $K \in \mathcal{K}(\mathcal{H})$ such that

$$\forall n, W^n(T+K) = W_e^n(T).$$

Smith-Ward proved that if we fix k, then there exists $K \in \mathcal{K}(\mathcal{H})$ such that

$$\forall 1 \leq n \leq k, W^n(T+K) = W_e^n(T).$$

In 1982, I gave a new proof of this using Arveson's quasicentral approximate units.

I also proved that for $\lambda(g_i) \in C^*_\lambda(\mathbb{F}_2) \subseteq B(\ell^2(\mathbb{F}_2))$ there is no pair of compacts K_1, K_2 so that

$$\forall n, W^n\big((\pi(\lambda(g_1)), \pi(\lambda(g_2)))\big) = W^n\big((\lambda(g_1) + K_1, \lambda(g_2) + K_2)\big),$$

so the two variable version of Smith-Ward fails.

Kavruk, Passer-P: The following are equivalent:

1. The Smith-Ward Problem has an affirmative answer,
2. every three dimensional operator system has the LP,
3. every three dimensional operator system is exact,
4. every three dimensional operator system is exact and has the LP,
5. $\alpha(T) = 1, \forall T \in B(\mathcal{H})$,
6. $\beta(T) = 1, \forall T \in B(\mathcal{H})$,
7. $\gamma(T) = 1, \forall T \in B(\mathcal{H})$.

$1 \iff 7$ is in my 1982 paper, with slightly different notation.

$E\upsilon\chi\alpha\rho\iota\sigma\tau\omega$!