An introduction to tensor products of operator algebras

Part II

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Previously..

Let $E \subseteq \mathcal{B}(H)$ and $F \subseteq \mathcal{B}(K)$ be two operator spaces. Recall that their minimal tensor product, is the vector space $E \otimes F$ equipped with the operator space structure, inherited by its inclusion

$$E \otimes F \subseteq \mathcal{B}(H \hat{\otimes}_{hs} K)$$

and denoted by $E \otimes_{min} F$.
However, as we will see shortly, the minimal tensor norm on $E \otimes F$ really depends on the "abstract" operator space structure of E and F rather than their particular embeddings.

Consider a Hilbert space H and $H_n \subseteq H$, an n-dimensional subspace with $P_{H_n} : H \to H_n$ being the orthogonal projection.

Using an orthonormal basis we can identify H_n with the n-dimensional Hilbert space $\ell^2([n])$ and consequently $\mathcal{B}(H_n) = M_n$ and recall that $M_n \otimes_{min} F = M_n(F)$.

Let $v_n : \mathcal{B}(H) \to \mathcal{B}(H_n) = M_n$ be the map $a \mapsto P_{H_n} a|_{H_n}$ and $\mathcal{C}_n$ the collection of all such mappings with H_n arbitrary n-dimensional.

If $E \subseteq \mathcal{B}(H)$, $F \subseteq \mathcal{B}(K)$ are operator spaces. Then, for any $x = \sum_i a_i \otimes b_i \in E \otimes F$ we have

$$\|x\|_{E \otimes_{min} F} = \sup_{n \in \mathbb{N}, v_n \in \mathcal{C}_n} \left\| \sum v_n(a_i) \otimes b_i \right\|_{M_n(F)}. \tag{1}$$

Indeed, we may write

$$\|x\|_{min} = \sup\Big\{|\langle xs, t\rangle| : s, t \in \mathrm{Ball}(\mathcal{H} \otimes_2 \mathcal{K})\Big\}.$$

For some finite dimensional subspace $\mathcal{H}_n \subseteq \mathcal{H}$, we have $s, t \in \mathcal{H}_n \otimes \mathcal{K}$. Hence, if v_n is the map defined above, we write

$$\langle xs, t\rangle = \langle (\sum_i a_i \otimes b_i)s, t\rangle = \langle (\sum_i v_n(a_i) \otimes b_i)s, t\rangle$$

thus we have that

$$\|x\|_{E\otimes_{min}F} \leq \sup_{n\in\mathbb{N}, v_n \in \mathcal{C}_n} \Big\|\sum v_n(a_i) \otimes b_i\Big\|_{M_n(F)}.$$

The reverse inequality is obvious thus we obtain the relation (1).

Proposition

Let E_i, F_i, $i = 1, 2$ be operator spaces and $u_i : E_i \to F_i$ be completely bounded maps. Then $u_1 \otimes u_2$ continuously extends by density to a completely bounded map

$$u_1 \otimes u_2 : E_1 \otimes_{min} E_2 \to F_1 \otimes_{min} F_2.$$

Moreover, we have

$$\left\| u_1 \otimes u_2 \right\|_{cb} = \left\| u_1 \right\|_{cb} \left\| u_2 \right\|_{cb}.$$

One can actually show the more general form:

Proposition

For any $x = \sum a_i \otimes b_i \in E \otimes F$ *we have*

$$\|x\|_{min} = \sup \left\{ \left\| \sum \phi(a_i) \otimes \psi(b_i) \right\|_{M_{nm}} \right\} \tag{2}$$

where the supremum runs over $n, m \geq 1$ *and all pairs of* $\phi : E \to M_n$ *and* $\psi : F \to M_m$, *with* $\|\phi\|_{cb} \leq 1$ *and* $\|\psi\|_{cb} \leq 1$.

Contents

*-algebra tensor product

Let A and B be two unital C^*-algebras. Recall that we can turn their tensor product $A \otimes B$ to a *-algebra by defining:

- $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$
- $(a \otimes b)^* = a^* \otimes b^*$

for all $a_i, a \in A$ and $b_i, b \in B$, and extending linearly.

We can also define a unit in $A \otimes B$ by $1_{A \otimes B} := 1_A \otimes 1_B$.

Hence, $A \otimes B$ becomes a unital *-algebra.

C*-norms

Let A and B be two unital C^*-algebras. A norm $\|\cdot\|_\gamma$ on $A \otimes B$ is called a **C*-norm** if

- $\|xy\|_\gamma \leq \|x\|_\gamma \|y\|_\gamma$
- $\|x\|_\gamma = \|x^*\|_\gamma$
- $\|x^*x\|_\gamma = \|x\|_\gamma^2$

for all $x, y \in A \otimes B$, and a **C*-cross-norm** if further

- $\|a \otimes b\|_\gamma = \|a\| \cdot \|b\|$

The completion of the *-algebra $A \otimes B$ with respect to a C^*-norm $\|\cdot\|_\gamma$, becomes a C^*-algebra which we denote by $A \otimes_\gamma B$.

In general, there may be many C^*-norms on a tensor product.

The spatial tensor product

If $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ are two C^*-algebras, then we can form their spatial tensor product by the inclusion,

$$A \otimes B \subseteq \mathcal{B}(H) \otimes_{sp} \mathcal{B}(K) \subseteq \mathcal{B}(H \hat{\otimes}_{hs} K)$$

Indeed, the closure of the *-algebra $A \otimes B$ w.r.t the norm $\|\cdot\|_{sp}$ defines a C^*-algebra denoted by $A \otimes_{sp} B$.

The spatial tensor norm is a C^*-cross-norm!

Representations

Let A, B, C be $*$-algebras and $\pi_A : A \to C$, $\pi_B : B \to C$ are two *-homomorphisms such that

$$\pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$$

for all $a \in A$ and $b \in B$, i.e., with commuting ranges. Then, there exists a unique *-homomorphism $\pi : A \otimes B \to C$ with

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b).$$

Proposition

Let $\pi_1 : A_1 \to \mathcal{B}(H_1)$ *and* $\pi_2 : A_2 \to \mathcal{B}(H_2)$ *be unital *-homomorphisms of the* C^**-algebras* A_1 *and* A_2 *respectively. Then, there exists a unique unital *-homomorphism* $\pi : A_1 \otimes A_2 \to \mathcal{B}(H_1 \hat{\otimes}_{hs} H_2)$ *with*

$$\pi(a \otimes b) = \pi_1(a) \otimes_{sp} \pi_2(b).$$

Moreover, if π_i *are injective, then so is* π*.*

Proof. We define

$$\begin{aligned} \rho_1 : & A_1 \to \mathcal{B}(H_1 \hat{\otimes}_{hs} H_2) \\ & a \mapsto \pi_1(a) \otimes_{sp} \mathrm{Id}_{H_2} \end{aligned}$$

and

$$\begin{aligned} \rho_2 : & A_2 \to \mathcal{B}(H_1 \hat{\otimes}_{hs} H_2) \\ & b \mapsto \mathrm{Id}_{H_1} \otimes_{sp} \pi_2(b) \end{aligned}$$

and note that they are unital *-homomorphisms with commuting ranges.

So, there exists a unique unital *-homomorphism
$\pi : A \otimes B \to \mathcal{B}(H_1 \hat{\otimes}_{hs} H_2)$ with

$$\pi(a \otimes b) = \rho_1(a)\rho_2(b) = \pi_1(a) \otimes_{sp} \pi_2(b).$$

Injectivity is straightforward.

We denote the above *-homomorphism by $\pi := \pi_1 \otimes \pi_2$

Consider two C^*-algebras A_1 and A_2. By the Gelfand-Naimark theorem, there exist injective representations $\pi_1 : A_1 \to \mathcal{B}(H_1)$ and $\pi_2 : A_2 \to \mathcal{B}(H_2)$. So, by the previous proposition there exists an injective *-homomorphism $\pi : A_1 \otimes A_2 \to B(H_1 \hat{\otimes}_{hs} H_2)$. Hence, the tensor product inherits the spatial tensor norm by setting

$$\|a \otimes b\|_{sp} := \|\pi(a \otimes b)\|_{sp} .$$

Again, we may take the completion with respect to this norm and end up with a C^*-algebra denoted by $A \otimes_{sp} B$.

Conversely,

Proposition

*Let A_1 and A_2 be two unital C^*-algebras and let $\|\cdot\|_\gamma$ denote a C^*-norm on $A_1 \otimes A_2$. Then, for every unital *-homomorphism $\pi : A_1 \otimes_\gamma A_2 \to \mathcal{B}(H)$ there exist two unital *-homomorphisms $\pi_i : A_i \to \mathcal{B}(H)$ with commuting ranges, such that*

$$\pi(a \otimes b) = \pi_1(a)\pi_2(b).$$

Proof. Simply define

$$\begin{aligned} \pi_1 : &A_1 \to \mathcal{B}(H) \\ &a \mapsto \pi(a \otimes 1_{A_2}) \end{aligned}$$

and similarly for π_2.

Minimal C*-norm

We may now define,

Definition

Let A_1 and A_2 be two unital C^*-algebras. We define the **minimal $\mathbf{C}^*$-norm** on $A_1 \otimes A_2$ to be

$$\|x\|_{min} := \sup\{\|\pi_1 \otimes \pi_2(x)\| : \pi_i : A_i \to \mathcal{B}(H_i) \text{ unital *-homomorphism}\}$$

This is indeed a C^*-cross-norm on $A_1 \otimes A_2$. The completion with respect to this norm is a C^*-algebra denoted by $A_1 \otimes_{min} A_2$.

Minimality of the minimal C*-norm

The following deep result explains the name of this particular C^*-norm:

Theorem (Takesaki)

Let A_1 and A_2 be two unital C^*-algebras. Then, the minimal C^*-norm is the smallest of all possible C^*-norms on $A_1 \otimes A_2$, i.e., if $\|\cdot\|_\gamma$ is another C^*-norm, then

$$\|x\|_{min} \leq \|x\|_\gamma$$

for all $x \in A_1 \otimes A_2$.

Minimal=spatial

Corollary

*Let A_1 and A_2 be two unital C^*-algebras. If $\pi_i : A_i \to \mathcal{B}(H_i)$ injective, unital *-homomorphisms, then*

$$\|x\|_{min} = \|\pi_1 \otimes \pi_2(x)\|$$

for all $x \in A_1 \otimes A_2$.

Proof.

Indeed, just note that $\|x\|_\gamma := \|\pi_1 \otimes \pi_2(x)\|$ defines a C^*-norm less that the minimal one. $\square$

Hence, when $A_i \subseteq \mathcal{B}(H_i)$ are concrete C^*-algebras, the minimal and spatial C^*-norms coincide!

All C*-norms are cross-norms

Corollary

Let A_1, A_2 be two unital C^*-algebras. Then, every C^*-norm on $A_1 \otimes A_2$ is a cross-norm.

Proof. Let $\| \cdot \|_\gamma$ be a C^*-norm on $A_1 \otimes A_2$. Consider the universal representation of $A_1 \otimes_\gamma A_2$, $\pi : A_1 \otimes_\gamma A_2 \to \mathcal{B}(H)$. By a previous proposition, there exist unital *-homomorphisms $\pi_i : A_i \to \mathcal{B}(H)$ with commuting ranges s.t. $\pi(a \otimes b) = \pi_1(a)\pi_2(b)$. Hence.

$$\|a\| \, \|b\| = \|a \otimes b\|_{min} \leq \|a \otimes b\|_\gamma = \|\pi(a \otimes b)\| = \|\pi_1(a)\pi_2(b)\| \leq \|a\| \, \|b\|$$

Injectivity

Suppose that B_i are unital C^*-algebras and $A_i \subseteq B_i$ are unital C^*-subalgebras for $i = 1, 2$. We have two ways of forming the minimal C^*-norm $A_1 \otimes_{min} A_2$:

1. The completion of $A_1 \otimes A_2$ w.r.t. the norm
$$\|x\|_{min} := \sup\{\|\pi_1 \otimes \pi_2(x)\| : \pi_i : A_i \to \mathcal{B}(H_i) \text{ unital *-hom}\}$$
2. $\overline{A_1 \otimes A_2}^{\|\cdot\|_{min}} \subseteq B_1 \otimes_{min} B_2$

However, Takesaki's theorem asserts that they coincide. Indeed, consider
$$j : A_1 \otimes A_2 \to B_1 \otimes B_2$$
to be the inclusion map.

Then, if $\pi_i : B_i \to \mathcal{B}(H_i)$ are unital, injective *-homomorphisms, then so are their restrictions on A_i. So, by Takesaki's theorem,

$$\|j(x)\|_{B_1 \otimes_{min} B_2} = \|x\|_{B_1 \otimes_{min} B_2} = \|\pi_1 \otimes \pi_2(x)\| = \|x\|_{A_1 \otimes_{min} A_2}$$

Thus, the inclusion map extends to a *-isomorphism between $A_1 \otimes_{min} A_2$ and the closure of $A_1 \otimes A_2$ in $B_1 \otimes_{min} B_2$, i.e.,

$$A_1 \otimes_{min} A_2 \subseteq B_1 \otimes_{min} B_2.$$

Hence, the minimal C^*-norm on two C^*-algebras A_1, A_2 coincides with the spatial tensor norm and the minimal operator space norm when viewed as operator spaces.

Let X be a compact Hausdorff space and A be a unital C^*-algebra. Let also $C(X;A)$ denote the C^*-algebra of continuous functions from X into A with the norm $\|F\| = \sup\{\|F(x)\| : x \in X\}$. Then,

Example

$C(X;A) = C(X) \otimes_{min} A$, *-isomorphically.

Proof.

- The subspace $D := \mathrm{span}\{f \cdot a : f \in C(X), a \in A\}$, where $f \cdot a : x \mapsto f(x)a \in A$ is dense in $C(X;A)$. This is done by a partition of unity argument.
- Thus, the map Φ is an injective *-homomorphism

$$\begin{aligned} \Phi : & C(X) \otimes A \to C(X;A) \\ & \sum_{i=1}^{n} f_i \otimes a_i \mapsto \sum_{i=1}^{n} f_i \cdot a_i \end{aligned}$$

So, we may define a norm on $C(X) \otimes A$ by

$$\left\| \sum_{i=1}^{n} f_i \otimes a_i \right\|_{\gamma} := \left\| \sum_{i=1}^{n} f_i \cdot a_i \right\|$$

Note that this is a C^*-norm, hence the map Φ, extends to a *-isomorphism

$$C(X) \otimes_{\gamma} A = C(X; A).$$

By Takesaki's theorem,

$$\left\| \sum_{i=1}^{n} f_i \otimes a_i \right\|_{min} \leq \left\| \sum_{i=1}^{n} f_i \otimes a_i \right\|_{\gamma} = \sup_x \left\| \sum_{i=1}^{n} f_i(x) a_i \right\| \tag{3}$$

- Then, fix an $x \in X$, and define the map

$$\begin{aligned} \phi_x : & C(X) \to \mathbb{C} \\ & f \mapsto f(x) \end{aligned}$$

which is a linear contraction. Hence, there is a bounded map

$$\begin{aligned} \phi_x \otimes_{min} \mathrm{Id} : & C(X) \otimes_{min} A \to A \\ & \sum_{i=1}^{n} f_i \otimes a_i \mapsto \sum_{i=1}^{n} f_i(x) a_i \end{aligned}$$

with $\|\phi_x \otimes_{min} \mathrm{Id}\| = \|\phi_x\| \leq 1$. That is,

$$\left\| \sum_{i=1}^{n} f_i(x) a_i \right\| \leq \left\| \sum_{i=1}^{n} f_i \otimes a_i \right\|_{min}$$

- Finally, since $x \in X$ was arbitrary,

$$\sup_x \left\| \sum_{i=1}^n f_i(x) a_i \right\| \leq \left\| \sum_{i=1}^n f_i \otimes a_i \right\|_{min} \tag{4}$$

and hence, by the converse inequality (3)

$$C(X) \otimes_{min} A = C(X; A).$$

As a corollary,

Example

For compact Hausdorff spaces X and Y, $C(X) \otimes_{min} C(Y) = C(X \times Y)$.

Minimal - Injective norm

If we recall the exact same fact about the injective norm, i.e., $C(X)\hat{\otimes}_{\varepsilon}C(Y) = C(X \times Y)$, we conclude that

Remark

Let A, B be two unital C^*-algebras with one of them being abelian. Then,

$$A \otimes_{min} B = A\hat{\otimes}_{\varepsilon}B$$

**-isomorphically.*

Maximal C*-norm

Let A and B be two unital C^*-algebras. Recall that whenever we have two unital *-homomorphisms $\pi_1 : A \to \mathcal{B}(H)$ and $\pi_2 : B \to \mathcal{B}(H)$ with commuting ranges, there exists a unital *-homomorphism $\pi : A \otimes B \to \mathcal{B}(H)$ with $\pi(a \otimes b) = \pi_1(a)\pi_2(b)$. And conversely, whenever $\pi : A \otimes B \to \mathcal{B}(H)$ is a unital *-homomorphism, there exists such a pair (π_1, π_2) with commuting ranges s.t. $\pi(a \otimes b) = \pi_1(a)\pi_2(b)$. So, it makes sense to define

Definition

Let A, B be unital C^*-algebras. We define the **maximal C*-norm** as

$$\|x\|_{max} := \sup\{\|\pi(x)\| : \pi : A \otimes B \to \mathcal{B}(H) \text{ unital *-hom.}\}$$

This is indeed a C^*-norm, the completion of $A \otimes B$ with respect to which we denote by $A \otimes_{max} B$ and call it the *maximal tensor product*.

Maximality

Proposition

Let A, B be unital C^*-algebras. The maximal C^*-norm on $A \otimes B$ is the greatest among all C^*-norms.

Proof. Indeed, let $\|\cdot\|_\gamma$ be another C^*-norm on $A \otimes B$. By the Gelfand-Naimark theorem, the exists a unital injective *-homomorphism $\pi : A \otimes_\gamma B \to \mathcal{B}(H)$ such that $\|x\|_\gamma = \|\pi(x)\|$. Hence, by definition,

$$\|x\|_\gamma = \|\pi(x)\| \leq \|x\|_{max} .$$

Now let C^*-algebras $A_i \subseteq B_i$, $i = 1, 2$. The natural inclusion

$$A_1 \otimes A_2 \subseteq B_1 \otimes B_2 \subseteq B_1 \otimes_{max} B_2$$

induces a C*-norm on $A_1 \otimes A_2$. Let's call it $\|\cdot\|_\gamma$. So,
$A_1 \otimes_\gamma A_2 = \overline{A_1 \otimes A_2}^{\|\cdot\|_{max}} \subseteq B_1 \otimes_{max} B_2$.
Now, the inclusion map $j : A_1 \otimes A_2 \to B_1 \otimes B_2$, induces a *-homomorphism

$$j : A_1 \otimes_{max} A_2 \to A_1 \otimes_\gamma A_2 \subseteq B_1 \otimes_{max} B_2$$

since

$$\|j(x)\|_\gamma = \|x\|_\gamma \leq \|x\|_{max}$$

But this may be norm decreasing!

Moreover, the maximal C*-norm does not respect completely bounded maps!

Example (Huruya)

There exists a completely bounded map $L : A_1 \to A_2$ and a C*-algebra B such that the map $L \otimes \mathrm{Id} : A_1 \otimes B \to A_2 \otimes B$, does not even extend to a bounded map on from $A_1 \otimes_{max} B$ to $A_2 \otimes_{max} B$.

Now let A be a C*-algebra. Recall that $M_n(A)$ is a C*-algebra and,

$$M_n \otimes A = M_n(A)$$

isomorphically as *-algebras. Hence, via the norm $\|\cdot\|$, induced by $M_n(A)$ (which is in fact the minimal one) $M_n \otimes A$ is a C*-algebra. Thus, there is only one C*-norm on $M_n \otimes A$, for if $\|\cdot\|_\gamma$ is another one and $M_n \otimes_\gamma A$ denotes the completion, then the inclusion

$$j : (M_n \otimes A, \|\cdot\|) \to M_n \otimes_\gamma A$$

is an injective *-homomorphism between C*-algebras, thus it is isometric.

Thus, for any C*-algebra A, there is only one C*-norm on $M_n \otimes A$. In particular, $M_n \otimes_{min} A = M_n \otimes_{max} A$ for every C*-algebra A.

Definition

A C*-algebra A with the property that, $A \otimes_{min} B = A \otimes_{max} B$ for every C*-algebra B, is called **nuclear**.

And since, for every C*-norm $\|\cdot\|_\gamma$ on $A \otimes B$

$$\|x\|_{min} \leq \|x\|_\gamma \leq \|x\|_{max}$$

this means that there would be a unique C*-norm on $A \otimes B$ for every B.

Examples

Examples

1. M_n, $n \in \mathbb{N}$ is nuclear.
2. Every finite dimensional C*-algebra A, is nuclear.
3. $C(X)$, where X is compact Hausdorff space, is nuclear.
4. $\mathcal{K}(H)$, the space of compact operators on H, is nuclear.
5. $C^*(G)$, where G is a discrete group, is nuclear if and only if G is amenable.

So, for instance $C^*(\mathbb{F}_2)$, the full group C*-algebra of the free group on two generators, is not nuclear.

Let also G_1, G_2 be two discrete groups.

Example

- $C^*(G_1) \otimes_{max} C^*(G_2) = C^*(G_1 \times G_2)$
- $C^*_\lambda(G_1) \otimes_{min} C^*_\lambda(G_2) = C^*_\lambda(G_1 \times G_2)$

Complete positivity

Now as we will see, the *min* and *max* both get along well with completely positive maps.

Proposition

Let A_i, B_i, $i = 1, 2$ be unital C-algebras and let also $\phi_i : A_i \to B_i$, $i = 1, 2$ be completely positive maps. Then, there exists a completely positive map*

$$\begin{aligned} \phi_1 \otimes_{min} \phi_2 : & A_1 \otimes_{min} A_2 \to B_1 \otimes_{min} B_2 \\ & a_1 \otimes a_2 \mapsto \phi_1(a_1) \otimes \phi_2(a_2) \end{aligned}$$

such that $\|\phi_1 \otimes_{min} \phi_2\|_{cb} = \|\phi_1\|_{cb} \|\phi_2\|_{cb}$

Proposition

Let A_1, A_2 and B be unital C-algebras, and let $\theta_i : A_i \to B$, $i = 1, 2$ be completely positive maps with commuting ranges. Then, there exists a completely positive map*

$$\begin{aligned} \theta_1 \otimes_{max} \theta_2 : & A_1 \otimes_{max} A_2 \to B \\ & a_1 \otimes a_2 \mapsto \theta_1(a_1)\theta_2(a_2) \end{aligned}$$

Contents

Tensor products of operator systems

Recall that an operator system is a unital selfadjoint subspace $S \subseteq \mathcal{B}(H)$ on some Hilbert space H. Abstractly, an operator system is a triple $(S, \{C_n\}_{n\in\mathbb{N}}, e)$, where S is a *-vector space, $\{C_n\}_{n\in\mathbb{N}}$ is a matrix ordering and e is a Archimedean matrix-order unit.
So, if $S_i \subseteq A_i$, $i = 1, 2$ are operator systems in the unital C*-algebras A_i, then $S_1 \otimes S_2$ inherits a natural operator system structure by its inclusion

$$S_1 \otimes S_2 \subseteq A_1 \otimes_{min} A_2.$$

We denote this operator system by $S_1 \otimes_{min} S_2$.

However, the situation with the C*-algebra $A_1 \otimes_{max} A_2$ is different, because of the non-injectivity of the max C*-norm.

Abstractly

Given an pair of operator systems $(S, \{P_n\}_{n\in\mathbb{N}}, e_1)$ and $(T, \{Q_n\}_{n\in\mathbb{N}}, e_2)$, an **operator system structure on** $S \otimes T$ is defined as a family of cones $\tau = \{C_n\}_{n\in\mathbb{N}}$, with $C_n \subseteq M_n(S \otimes T)$ such that

1. $(S \otimes T, \{C_n\}_{n\in\mathbb{N}}, e_1 \otimes e_2)$ is an operator system denoted by $S \otimes_\tau T$
2. $P_n \otimes Q_m \subseteq C_{nm}$, for all $n, m \in \mathbb{N}$
3. If $\phi : S \to M_n$ and $\psi : T \to M_m$ are unital completely positive maps, then $\phi \otimes \psi : S \otimes_\tau T \to M_{nm}$ is unital completely positive.

We shall denote the cones $C_n := M_n(S \otimes_\tau T)^+$. Given two operator system structures τ_1 and τ_2 on $S \otimes T$, we will say that τ_1 **is greater than** τ_2, provided that

$$M_n(S \otimes_{\tau_1} T)^+ \subseteq M_n(S \otimes_{\tau_2} T)^+.$$

Equivalently, the $\mathrm{Id} : S \otimes_{\tau_1} T \to S \otimes_{\tau_2} T$ is completely positive.

To explain this rather confusing definition, note that *"larger norms imply smaller cones"* in the following sense.
Consider a *-vector space S and C*-algebras A, B, so that S is equipped with two operator system structures via its embeddings $S \subseteq (A, \|\cdot\|_t)$ and $S \subseteq (B, \|\cdot\|_l)$ with induced cones denoted by $\{C_n^t\}_{n=1}^\infty$ and $\{C_n^l\}_{n=1}^\infty$ resp. Then,

$$\|\cdot\|_t^{(n)} \leq \|\cdot\|_l^{(n)}, \forall n \in \mathbb{N} \iff C_n^t \supseteq C_n^l, \forall n \in \mathbb{N}\,.$$

Indeed, consider the identity map

$$\mathrm{Id} : (S, \|\cdot\|_l) \to (S, \|\cdot\|_t)$$

and recall that for any map ϕ between operator systems

$$\phi \text{ is unital complete contraction} \iff \phi \text{ is unital completely positive.}$$

The minimal operator system structure

Let S be an operator system. We define

$$\mathcal{S}_k(S) := \{\phi : S \to M_k : \phi \text{ is ucp } \}.$$

Let S, T be operator systems, and

$$C_n^{min} := \{[p_{i,j}] \in M_n(S \otimes T) : [(\phi \otimes \psi)(p_{i,j})]_{i,j} \in M_{nkm}^+, \\ \forall \phi \in \mathcal{S}_k(S), \forall \psi \in \mathcal{S}_m(T)\}$$

Theorem

Let S, T be operator systems and $i_S : S \to \mathcal{B}(H)$ and $i_T : T \to \mathcal{B}(K)$ be unital complete order embeddings. The family $\{C_n^{min}\}_{n=1}^{\infty}$ is the operator system structure on $S \otimes T$ arising from the embedding $i_S \otimes i_T : S \otimes T \to \mathcal{B}(H \hat{\otimes}_{hs} K)$.

Let's denote both the units of S and T by 1.

Definition

Let S, T be two operator systems. We call the operator system $(S \otimes T, \{C_n^{min}\}_{n=1}^{\infty}, 1 \otimes 1)$, the **minimal tensor product** of S and T and denote it by $S \otimes_{min} T$.

Hence, for $S \subseteq \mathcal{B}(H)$, $T \subseteq B(K)$ concrete operator systems, the "abstract" minimal tensor product $S \otimes_{min} T$, coincides with the "concrete" one, i.e., the one $S \otimes T$ inherits from its inclusion into $\mathcal{B}(H) \otimes_{min} B(K) = \mathcal{B}(H) \otimes_{sp} B(K)$.

Remark

Let τ be an operator system structure on $S \otimes T$. Then, τ is larger than min.

This follows by the 3rd property of an operator system structure on $S \otimes T$.

Maximal operator system structure

Let S, T be two operator systems. Define

$$D_n^{max} := \{a(P \otimes Q)a^* : P \in M_k(S)^+, Q \in M_m(T)^+, a \in M_{n,km}, k, m \in \mathbb{N}\}$$

Proposition

The family $\{D_n^{max}\}_{n=1}^{\infty}$, is a matrix ordering on $S \otimes T$, with matrix order unit $1 \otimes 1$.

However, $1 \otimes 1$, may fail to be Archimedean for $\{D_n^{max}\}_{n=1}^{\infty}$!

*Recall that an order unit $1 \in S$ is called Archimedean if $\varepsilon 1 + s \geq 0$ for every $\varepsilon > 0$ implies $s \geq 0$.

Archimedeanization

Definition

Let $\{C_n^{max}\}_{n=1}^{\infty}$ be the **Archimedeanization** of the matrix ordering $\{D_n^{max}\}_{n=1}^{\infty}$, that is,

$$[R_{i,j}] \in C_n^{max} \iff [R_{i,j}] + \varepsilon(I_n \otimes 1 \otimes 1) \in D_n^{max}, \forall \varepsilon > 0.$$

We call the operator system

$$(S \otimes T, \{C_n^{max}\}_{n=1}^{\infty}, 1 \otimes 1)$$

the **maximal tensor product** of S and T and denote is by $S \otimes_{max} T$.

We used the symbol, $I_n \otimes e = \begin{bmatrix} e & \dots & 0 \\ & \ddots & \\ 0 & \dots & e \end{bmatrix}.$

Remark

Let S, T be two operator systems, then max is the larger operator system structure on $S \otimes T$.

Now let A, B be unital C*-algebras. The tensor product $A \otimes B$ obtains a natural operator system structure by its inclusion in the C*-algebraic maximal tensor product $A \otimes_{C^* max} B$. In fact,

Theorem

Let A and B be unital C-algebras. Then, the operator system $A \otimes_{max} B$ is completely order isomorphic to the image of $A \otimes B$ inside the maximal C*-algebraic tensor product of A and B.*

Commuting tensor product

Let S, T be two operator systems and let $CP(S, \mathcal{B}(H))$ denote the collection of all completely positive maps from S into $\mathcal{B}(H)$. We define

$$cp(S, T) := \{(\phi, \psi) : \phi \in CP(S, \mathcal{B}(H)), \psi \in CP(T, \mathcal{B}(H)), \\ \text{with commuting ranges}\}.$$

And for each a pair $(\phi, \psi) \in cp(S, T)$, we define a map $\phi \cdot \psi : S \otimes T \to \mathcal{B}(H)$, with $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$
Now, for each $n \in \mathbb{N}$, we define a cone $P_n \subseteq M_n(S \otimes T)$, by

$$P_n := \{u \in M_n(S \otimes T) : (\phi \cdot \psi)^{(n)}(u) \geq 0, \text{ for all } (\phi, \psi) \in cp(S, T)\}$$

Proposition

The family $\{P_n\}_{n=1}^{\infty}$ is a matrix ordering on $S \otimes T$ with Archimedean matrix order unit $1 \otimes 1$.

Definition

We denote by $S \otimes_c T$ the operator system $(S \otimes T, \{P_n\}_{n=1}^{\infty}, 1 \otimes 1)$ and call it the **commuting tensor product**.

This tensor product bears similarities to the maximal C*-algebra tensor product because of the commutativity. Actually,

Theorem

Let A and B be unital C-algebras. Then $A \otimes_c B = A \otimes_{max} B$.*

Final remarks

Finally, one can also talk define nuclearity in the operator system category which we will not be discussing here. However,

Proposition

Let A be a unital C-algebra. Then, A is nuclear if and only if $A \otimes_{min} S = A \otimes_{max} S$ for every operator system S.*

In fact, Kavruk proved the following

Theorem

There exists a finite dimensional operator system $\mathcal{W}$ such that for every unital C-algebra, the following are equivalent,*

- *A is a nuclear C*-algebra*
- $A \otimes_{min} \mathcal{W} = A \otimes_{max} \mathcal{W}$

Such an operator system is called "nuclearity detector".

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