An introduction to tensor products of operator algebras
Part I

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Tensor products of vector spaces

Let E, F be two $\mathbb{C}$-vector spaces.

Definition (Tensor product)

A tensor product of E and F is a pair (M, ϕ), where M is a vector space and $\phi : E \times F \to M$ is a bilinear map such that

1. The image of ϕ spans the whole space M.
2. Whenever $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ are linearly independent sets of E and F respectively, $\{\phi(x_i, y_j)\}_{i,j}$ is a linearly independent set of M.

We will see that such an object exists and is in fact unique up to a linear isomorphism.

Remark

Let E and F be two $\mathbb{C}$-vector spaces. Then, there exist sets I and K so that

$$E \hookrightarrow \mathbb{C}^I \quad \text{and} \quad F \hookrightarrow \mathbb{C}^K$$

where $\mathbb{C}^I$ and $\mathbb{C}^K$ denote the spaces of functions from I and K resp.

Proof. Let $u \in E$ and $X = \{x_i : i \in I\}$ be a Hamel basis for E. Then, there exist a finite subset $J \subseteq I$, linearly independent vectors $\{x_i\}_{i \in J} \subseteq X$ and scalars $u(i) \in \mathbb{C}$ such that u is written uniquely as

$$u = \sum_{i \in J} u(i) x_i.$$

Now map u to the function $f_u : I \to \mathbb{C}$ such that $f_u(i) = u(i)$ if $i \in J$ and is supported in J. This correspondence is a linear isomorphism of E (resp. F) into a subspace of $\mathbb{C}^I$ (resp. $\mathbb{C}^K$).

Existence of tensor products

Theorem (Existence)

Let E, F be two vector spaces. There exists a tensor product of E and F.

Sketch. Embed $E \hookrightarrow \mathbb{C}^I$ and $F \hookrightarrow \mathbb{C}^K$ and define

$$M := \mathrm{span}\{f \cdot g : f \in E, g \in F\} \subseteq \mathbb{C}^{I \times K}$$

where $(f \cdot g)(i, k) = f(i)g(k)$. Let also $\phi : E \times F \to M$ be the map

$$\phi : (u, v) \mapsto f_u \cdot g_v.$$

Note that ϕ is bilinear and that conditions (1) and (2) of the definition of tensor products are satisfied.
Hence, (M, ϕ) is a tensor product of E and F.

Tensor products satisfy the following universal property.

Theorem (Universal property)

Let (M, ϕ) be a tensor product of the vector spaces E and F. For every vector space G and bilinear map $b : E \times F \to G$, there exists a unique linear map $B : M \to G$ such that $B(\phi(x, y)) = b(x, y)$ for every $(x, y) \in E \times F$.

It can be said that the tensor product "linearizes" bilinear maps.

Universal property

Equivalently, the following diagramm is commutative.

$$\begin{tikzcd} & M \arrow[dr, "B"] & \\ E \times F \arrow[ur, "\phi"] \arrow[rr, "b"] & & G \end{tikzcd}$$

The universal property of the tensor product makes it unique!

Theorem (Uniqueness)

Let (M_1, ϕ) and (M_2, ϕ_2) be two tensor products of E and F. Then, there exists a linear isomorphism $\pi : M_1 \to M_2$ such that $\pi \circ \phi_1 = \phi_2$.

Notation

So, if E and F are two vector spaces, there is "only one" tensor product between them. For this tensor product (M, ϕ), we denote

$$x \otimes y := \phi(x, y) \quad \text{for every} \;\; x \in E, y \in F$$

and also

$$E \otimes F := M$$

and since, $M = \operatorname{span}\{\phi(x, y) : x \in E, y \in F\}$ we may write

$$E \otimes F = \operatorname{span}\{x \otimes y : x \in E, y \in F\}.$$

So, every element $u \in E \otimes F$ has the (not unique) representation

$$u = \sum_{i=1}^{k} x_i \otimes y_i$$

for $k \in \mathbb{N}$, $x_i \in E$ and $y_i \in F$. Elements of the form $x \otimes y$ are called **simple tensors**.

Remark

Every element $u \in E \otimes F$ can be written as

$$u = \sum_{i=1}^{r} e_i \otimes f_i$$

where $\{f_i\}_{i=1}^r \subseteq F$ are linearly independent and the elements $e_i \in E$ are uniquely determined by the f_i.

Proposition

Let $T_1 : E_1 \to G_1$ and $T_2 : E_2 \to G_2$ be linear maps. There exists a unique linear map

$$\begin{aligned} S : & E_1 \otimes E_2 \to G_1 \otimes G_2 \\ & x_1 \otimes x_2 \mapsto T_1(x_1) \otimes T_2(x_2) \end{aligned}$$

which we denote by $S := T_1 \otimes T_2$.

Examples

Let E be a vector space.

Examples

1. $E \otimes \mathbb{C} = E$ via the map $x \otimes \lambda \mapsto \lambda x$, where $x \in E$ and $\lambda \in \mathbb{C}$,
2. $E \otimes \mathbb{C}^{\mathrm{n}} = E^n$ via the map $x \otimes e_i \mapsto \begin{bmatrix} 0 \\ \vdots \\ x \\ \vdots \\ 0 \end{bmatrix}$, where $\{e_i\}$ is the canonical basis in $\mathbb{C}^{\mathrm{n}}$
3. $\mathbb{C}^{\mathrm{n}} \otimes \mathbb{C}^m = \mathbb{C}^{nm}$
4. $M_{n,m}(\mathbb{C}) \otimes E = M_{n,m}(E)$ via the map $[a_{i,j}] \otimes x \mapsto [a_{i,j}x]$ for $[a_{i,j}] \in M_{n,m}(\mathbb{C})$ and $x \in E$

Canonical shuffle

Remark

If E, F, W are vector spaces,

- $E \otimes F = F \otimes E$
- $E \otimes (F \otimes W) = (E \otimes F) \otimes W$

So, if E is a vector space, we have that,

$$M_n(M_m(E)) = M_n \otimes M_m \otimes E = M_m \otimes M_n \otimes E = M_m(M_n(E)).$$

Through this identification

$$\left[[a_{i,j,k,l}]_{k,l=1}^{m}\right]_{i,j=1}^{n} \longleftrightarrow \left[[a_{i,j,k,l}]_{i,j=1}^{n}\right]_{k,l=1}^{m}$$

which is, in fact, done via permutation matrices!

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Norms on tensor products

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We will say that a norm on $X \otimes Y$ is a:

- *subcross-norm*, if $\|x \otimes y\| \leq \|x\|_X \|y\|_Y$
- *cross-norm*, if $\|x \otimes y\| = \|x\|_X \|y\|_Y$

Definition (Projective tensor norm)

Let X, Y be two normed spaces. The *projective norm* is defined by

$$\|u\|_\pi = \inf \left\{ \sum_i \|x_i\|_X \|y_i\|_Y : u = \sum_i x_i \otimes y_i \right\}$$

for all $u \in X \otimes Y$.

Projective tensor norm

Proposition

Let X, Y be normed spaces. The projective norm $\|\cdot\|_\pi$ is a norm on $X \otimes Y$ and it satisfies

$$\|x \otimes y\|_\pi = \|x\|_X \|y\|_Y$$

for any $x \in X$ and $y \in Y$.

We denote by $X \otimes_\pi Y$ the tensor product of X and Y endowed with the projective norm $\|\cdot\|_\pi$. We denote the completion of $(X \otimes Y, \|\cdot\|_\pi)$ by $X \hat{\otimes}_\pi Y$ and call it the *projective tensor product*.
Note that, if X and Y are both infinite dimensional, $X \otimes_\pi Y$ is never complete.
The projective norm is the largest sub-cross norm on $X \otimes Y$!

Properties of the Projective tensor product

Remark

Let X, Y, Z be vector spaces. Then, the projective tensor product is,

- *(symmetric):* $X \hat{\otimes}_\pi Y = Y \hat{\otimes}_\pi X$
- *(associative):* $(X \hat{\otimes}_\pi Y) \hat{\otimes}_\pi Z = X \hat{\otimes}_\pi (Y \hat{\otimes}_\pi Z)$

However, it is not "injective".

Remark

There exist normed spaces X, Y and Z such that $X \subseteq Y$ isometrically but $X \hat{\otimes}_\pi Z$ is not contained isometrically in $Y \hat{\otimes}_\pi Z$

Examples

Let X be a Banach space.

1. $\ell^1(\mathbb{N})\hat{\otimes}_\pi X = \ell^1(X)$, where $\ell^1(X)$ is the Banach space of all sequences $x = (x_n)_n$, with $\|x\|_1 = \sum_n \|x_n\|_X < \infty$
2. $\ell^1(I)\hat{\otimes}_\pi X = \ell^1(I; X)$, for an arbitrary set I
3. $L^1(\mu)\hat{\otimes}_\pi L^1(\nu) = L^1(\mu \times \nu)$, where $\mu \times \nu$ is the product measure.

Injective tensor norm

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. There is a well defined linear injection

$$X \otimes Y \hookrightarrow \mathcal{B}(X^* \times Y^*)$$

via

$$u = \sum_i x_i \otimes y_i \mapsto \Big(B_u : (\phi, \psi) \mapsto \sum_i \phi(x_i)\psi(y_i)\Big)$$

Through this (algebraic) embedding, the tensor product inherits the norm:

Definition

The *injective norm* $\|\cdot\|_\epsilon$ on $X \otimes Y$ is defined by

$$\|u\|_\epsilon = \sup \left\{ \left| \sum_i \phi(x_i)\psi(y_i) \right| : \phi \in \mathrm{Ball}(X^*), \psi \in \mathrm{Ball}(Y^*) \right\}$$

where $u = \sum_i x_i \otimes y_i \in X \otimes Y$.

Proposition

Let X, Y be normed spaces, then the injective norm $\|\cdot\|_\varepsilon$ is indeed a norm on $X \otimes Y$ and

1. $\|u\|_\epsilon \leq \|u\|_\pi$ *for every* $u \in X \otimes Y$.
2. $\|x \otimes y\|_\epsilon = \|x\|_X \|y\|_Y$ *for every* $x \in X$, $y \in Y$.

Again we denote by $X \otimes_\epsilon Y$ the tensor product with the injective norm, and unless the spaces are finite dimensional we take the completion $X \hat{\otimes}_\epsilon Y$ which will be called the *injective tensor product*.

Properties of the Injective tensor product

Remark

Let X, Y, Z be normed spaces. Then, the injective tensor product is,

- *(symmetric):* $X\hat{\otimes}_\varepsilon Y = Y\hat{\otimes}_\varepsilon X$
- *(associative):* $(X\hat{\otimes}_\varepsilon Y)\hat{\otimes}_\varepsilon Z = X\hat{\otimes}_\varepsilon(Y\hat{\otimes}_\varepsilon Z)$

As one would expect, the injective tensor product is "injective".

Remark

If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ are normed spaces with isometric inclusions then,

$$X_1\hat{\otimes}_\varepsilon Y_1 \subseteq X_2\hat{\otimes}_\varepsilon Y_2$$

isometrically.

Examples

Let X be a Banach space and K, L be two compact Hausdorff spaces.

1. $c_0 \hat{\otimes}_\varepsilon X = c_0(X)$, where $c_0(X)$ is the Banach space of sequences in X that converge to zero with norm $\|(x_n)\| = \sup_n \|x_n\|_X$.
2. $C(K) \hat{\otimes}_\varepsilon C(L) = C(K \times L)$

• Let's briefly discuss the isomorphism $C(K)\hat{\otimes}_\varepsilon C(L) = C(K \times L)$.
Sketch. Note first that for an element $u = \sum_i x_i \otimes y_i \in X \otimes Y$, we may write its injective norm equivalently as:

$$\|u\|_\varepsilon = \sup\Big\{\Big|\sum_i \phi(x_i)\psi(y_i)\Big| : \phi \in A, \psi \in B\Big\}$$

where A and B are norming sets. [1]
The Dirac functionals $\{\delta_t\}_{t\in K}$ form a norming set for $C(K)$, so if $u = \sum_i f_i \otimes g_i \in C(K) \otimes C(L)$, then

$$\|u\|_\varepsilon = \sup\Big\{\Big|\sum_i \delta_t(f_i)\delta_s(g_i)\Big| : t \in K, s \in L\Big\}$$

equivalently,

$$\|u\|_\varepsilon = \sup_{t\in K, s\in L}\Big|\sum_i f_i(t)g_i(s)\Big|. \tag{1}$$

[1] Recall, that a subset $A \subseteq \mathrm{Ball}(X^*)$ is called a *norming set*, if we have that $\|x\| = \sup\{|\phi(x)| : \phi \in A\}$ for all $x \in X$.

Now, define

$$D := \mathrm{span}\{f \cdot g : f \in C(K), g \in C(L)\} \subseteq C(K \times L)$$

and note that by the Stone-Weierstrass theorem it is dense in $C(K \times L)$. Finally, the map

$$\begin{aligned} J :& C(K) \otimes C(L) \to C(K \times L) \\ & \sum_i f_i \otimes g_i \mapsto \sum_i f_i \cdot g_i \end{aligned}$$

is a linear bijection onto D, and an isometry (by 1). Hence, J extends to an isometric isomorphism from the completion $C(K)\hat{\otimes}_\varepsilon C(L)$, onto the closure of D, i.e. $C(K \times L)$.

However

Example 1

$\ell^\infty(\mathbb{N})\hat{\otimes}_\varepsilon\ell^\infty(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N}\times\mathbb{N})$.

Consequently, using that $\ell^\infty(\mathbb{N}) = C(\beta\,\mathbb{N})$, where $\beta\,\mathbb{N}$ is the Stone-Čech compactification of the natural numbers, and the previous result:

Example 2

$C(\beta\,\mathbb{N}\times\beta\,\mathbb{N}) \subsetneq C(\beta(\mathbb{N}\times\mathbb{N}))$, that is, $\beta\,\mathbb{N}\times\beta\,\mathbb{N} \neq \beta(\mathbb{N}\times\mathbb{N})$.

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Until now, we were able to turn the tensor product of two Banach spaces (resp. vector spaces) into a Banach space (resp. vector space) using the projective and injective norms. One would hope that "tensoring" Hilbert spaces w.r.t. these norms, would end up with a Hilbert space. However,

Examples

1. $\ell^2(\mathbb{N})\hat{\otimes}_\varepsilon\ell^2(\mathbb{N}) = \mathcal{K}(\ell^2(\mathbb{N}))$, where $\mathcal{K}(\ell^2(\mathbb{N}))$ is the space of compact operators on $\ell^2(\mathbb{N})$.
2. $\ell^2(\mathbb{N})\hat{\otimes}_\pi\ell^2(\mathbb{N}) = \mathcal{C}_1(\ell^2(\mathbb{N}))$, where $\mathcal{C}_1(\ell^2(\mathbb{N}))$ is the space of trace-class operators on $\ell^2(\mathbb{N})$.

Which are not Hilbert spaces.

And in general,

Proposition

If H and K are two Hilbert spaces,

1. $H\hat{\otimes}_\varepsilon K^* = \mathcal{K}(K, H)$
2. $H\hat{\otimes}_\pi K^* = \mathcal{C}_1(K, H)$

Sketch.

1. The map

$$\begin{aligned}\Phi : & H \otimes_\varepsilon K^* \to (\mathcal{FB}(K,H), \|\cdot\|_{\mathcal{B}(K,H)}) \\ & x \otimes y^* \mapsto xy^*\end{aligned}$$

is an algebraic isomorphism and an isometry. Thus, the completion w.r.t. $\|\cdot\|_\varepsilon$ is isomorphic to the closure of $(\mathcal{FB}(K,H), \|\cdot\|_{\mathcal{B}(K,H)})$, that is, $\mathcal{K}(K,H)$.

2. Similarly, the map

$$\begin{aligned}\Phi : & H \otimes_\pi K^* \to (\mathcal{FB}(K,H), \|\cdot\|_{tr}) \\ & x \otimes y^* \mapsto xy^*\end{aligned}$$

is an an algebraic isomorphism and an isometry. Thus, the completion w.r.t. to the norm $\|\cdot\|_\pi$ is isomorphic to the closure of $(\mathcal{FB}(K,H), \|\cdot\|_{tr})$ [2] that is, $\mathcal{C}_1(K,H)$.

[2] $\|T\|_{tr} = \mathrm{Tr}\,|T|$

Definition (Hilbert-Schmidt inner product)

Let H_1, H_2 be two Hilbert spaces. On the vector space $H_1 \otimes H_2$, we define

$$\langle \sum_i v_i \otimes w_i, \sum_j v_j^{'} \otimes w_j^{'} \rangle_{hs} := \sum_{i,j} \langle v_i, v_j^{'} \rangle_{H_1} \langle w_i, w_j^{'} \rangle_{H_2}.$$

Proposition

The map $\langle \cdot, \cdot \rangle_{hs} : (H_1 \otimes H_2) \times (H_1 \otimes H_2) \to \mathbb{C}$ *is a well defined inner product and*

$$\|h_1 \otimes h_2\|_{hs} = \|h_1\|_{H_1} \|h_2\|_{H_2}.$$

We denote by $H_1 \hat{\otimes}_{hs} H_2$ the completion of the space $(H_1 \otimes H_2, \| \cdot \|_{hs})$, where $\| \cdot \|_{hs} := \sqrt{\langle \cdot, \cdot \rangle_{hs}}$ and call it the *Hilbert space tensor product* .

Remark

Let H and K be Hilbert spaces and $\{e_i\}_{i\in I} \subseteq H$ and $\{f_j\}_{j\in J} \subseteq K$ be orthonormal bases. Then, $\{e_i \otimes f_j\}_{(i,j)\in I\times J}$ is an orthonormal basis for the Hilbert space $H\hat{\otimes}_{hs}K$

Examples

1. $H \otimes_{hs} \ell^2([n]) = H^n$, where H^n is the direct sum of n-copies of H, that turns into a Hilbert space w.r.t. the norm $\|(h_i)_{i=1}^n\|_2^2 = \sum_{i=1}^n \|h_i\|_H^2$.
2. $\ell^2(I)\hat{\otimes}_{hs}\ell^2(J) = \ell^2(I \times J)$, where I, J are arbitrary sets.
3. $L^2(\mu)\hat{\otimes}_{hs}L^2(\nu) = L^2(\mu \times \nu)$, where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces.

Hilbert Schmidt operators

Now, the following proposition explains the name of this particular tensor product. We denote by $\mathcal{C}_2(K, H)$, the space of Hilbert Schmidt operators, that is, the closure of finite rank operators w.r.t. the norm $\|T\|_{HS} := (\mathrm{Tr}\, T^*T)^{\frac{1}{2}}$.

Proposition

If H and K are two Hilbert spaces, then

$$H \hat{\otimes}_{hs} K^* = \mathcal{C}_2(K, H)$$

Sketch. The map

$$\begin{aligned} \Phi : & H \otimes_{hs} K^* \to (\mathcal{FB}(K, H), \|\cdot\|_{HS}) \\ & x \otimes y^* \mapsto xy^* \end{aligned}$$

is an algebraic isomorphism and an isometry. Thus, the completion w.r.t. $\|\cdot\|_{hs}$ is isomorphic to the closure of $(\mathcal{FB}(K, H), \|\cdot\|_{HS})$, that is, to $\mathcal{C}_2(K, H)$.

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Tensor products of operators

Let H, K be two Hilbert spaces and recall that

$$\mathcal{B}(H) \otimes \mathcal{B}(K) = \mathrm{span}\{T \otimes S : T \in \mathcal{B}(H), S \in \mathcal{B}(K)\}.$$

Note now that we can turn $\mathcal{B}(H) \otimes \mathcal{B}(K)$ into a *-algebra by defining multiplication and involution by

- $(T_1 \otimes T_2) \cdot (S_1 \otimes S_2) = T_1 S_1 \otimes T_2 S_2$
- $(T \otimes S)^* = T^* \otimes S^*$

and extending linearly to the whole space.

Also, recall that the Hilbert space tensor product $H \hat{\otimes}_{hs} K$ is the completion of the tensor product $H \otimes K$ with respect to the norm induced by the inner product

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle_{hs} = \langle h_1, h_2 \rangle_H \langle k_1, k_2 \rangle_K, \quad h_i \in H, k_i \in K.$$

How is the *-algebra $\mathcal{B}(H) \otimes \mathcal{B}(K)$ related to $\mathcal{B}(H \hat{\otimes}_{hs} K)$?

If $T_1 : H_1 \to K_1$ and $T_2 : H_2 \to K_2$ are linear maps between Hilbert spaces then we saw that there exists a unique linear map

$$\begin{aligned} J : & H_1 \otimes H_2 \to K_1 \otimes K_2 \\ & h_1 \otimes h_2 \mapsto T_1(h_1) \otimes T_2(h_2) \end{aligned}$$

which we denote by $J := T_1 \otimes T_2$.

So the question reformulates as: Can this map be extended to a **bounded** linear map on $\mathcal{B}(H_1 \hat{\otimes}_{hs} H_2, K_1 \hat{\otimes}_{hs} K_2)$?

Operators on Hilbert space tensor products

Proposition

Let $T : H \to H$ and $S : K \to K$ be bounded, linear operators between Hilbert spaces. Then there is a unique bounded linear operator $T \otimes_{sp} S : H\hat{\otimes}_{hs}K \to H\hat{\otimes}_{hs}K$ such that $(T \otimes_{sp} S)(h \otimes k) = T(h) \otimes S(k)$ for every $h \in H$, $k \in K$. Furthermore,

1. $\|T \otimes_{sp} S\| = \|T\| \, \|S\|$
2. $(T_1 + \lambda T_2) \otimes_{sp} S = T_1 \otimes_{sp} S + \lambda(T_2 \otimes_{sp} S)$
3. $T \otimes_{sp} (S_1 + \lambda S_2) = T \otimes_{sp} S_1 + \lambda(T \otimes_{sp} S_2)$
4. $(T_1 \otimes_{sp} S_1)(T_2 \otimes_{sp} S_2) = (T_1 T_2) \otimes_{sp} (S_1 S_2)$
5. $(T \otimes_{sp} S)^* = T^* \otimes_{sp} S^*$

where $\lambda \in \mathbb{C}$, $T_i \in \mathcal{B}(H_i)$ and $S_i \in \mathcal{B}(K_i)$ for Hilbert spaces H_i, K_i, $i = 1, 2$.

Spatial tensor product

Proposition

Let H, K be two Hilbert spaces. The map

$$\mathcal{B}(H) \otimes \mathcal{B}(K) \to \mathcal{B}(H \hat{\otimes}_{hs} K)$$
$$\sum_i T_i \otimes S_i \mapsto \sum_i T_i \otimes_{sp} S_i$$

*is an injective *-homomorphism between *-algebras.*

Now we may define a norm on $\mathcal{B}(H) \otimes \mathcal{B}(K)$ by

$$\left\| \sum_i T_i \otimes S_i \right\|_{sp} := \left\| \sum_i T_i \otimes_{sp} S_i \right\|$$

Spatial tensor product

Define
$$\mathcal{B}(H) \otimes_{sp} \mathcal{B}(K) := \overline{\mathcal{B}(H) \otimes \mathcal{B}(K)}^{\|\cdot\|_{sp}} \subseteq \mathcal{B}(H \hat{\otimes}_{hs} K)$$
and call it the *spatial tensor product*.

Note that $\mathcal{B}(H) \otimes_{sp} \mathcal{B}(K)$ is in fact a C^*-algebra as it is a closed *-subalgebra of $B(H \hat{\otimes}_{hs} K)$.

Also, the norm $\| \cdot \|_{sp}$ is a cross-norm.

• Recall that an *operator space* is a subspace $E \subseteq \mathcal{B}(H)$, where H is a Hilbert space.
If $E \subseteq \mathcal{B}(H)$ and $G \subseteq \mathcal{B}(K)$ are two operator spaces then we have that

$$E \otimes G \subseteq \mathcal{B}(H) \otimes_{sp} \mathcal{B}(K) \subseteq \mathcal{B}(H \hat{\otimes}_{hs} K).$$

So the vector space $E \otimes G$ becomes an operator space with the operator space structure induced by its inclusion in $\mathcal{B}(H \hat{\otimes}_{hs} K)$! That is,

$$\|[u_{i,j}]\|_{M_n(E \otimes G)} := \|[u_{i,j}]\|_{\mathcal{B}((H \hat{\otimes}_{hs} K)^n)}, \quad u_{i,j} \in E \otimes G.$$

We denote this operator space by $E \otimes_{min} G$ and call it the *minimal tensor product* of the operator spaces E and G.

Properties of the minimal tensor product

Remark

*Let H, K be two Hilbert spaces. If $V : H \to K$ is an isometry, then the linear map $u_V : \mathcal{B}(H) \to \mathcal{B}(K)$ defined by $u_V(x) = VxV^*$ is completely isometric. Moreover, if $V : H \to K$ is a unitary, then $u_V : \mathcal{B}(H) \to \mathcal{B}(K)$ is a *-isomorphism between C^*-algebras, hence a completely isometric isomorphism.*

Now recall that if H is a Hilbert space, $\ell^2([n]) \otimes_{hs} H = H^n$ isometrically. By the above remark, we have that

$$\mathcal{B}(\ell^2([n]) \otimes_{hs} H) = \mathcal{B}(H^n) \tag{2}$$

as C^*-algebras.

Now, let $E \subseteq \mathcal{B}(H)$ be an operator space and recall that $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$ via a *-isomorphism.
Of course

$$M_n(E) \subseteq M_n(\mathcal{B}(H)).$$

Also by definition

$$M_n \otimes_{min} E \subseteq \mathcal{B}(\ell^2([n]) \otimes_{hs} H).$$

Remark

The restriction of the isomorphism (2), gives the following completely isometric isomorphism

$$M_n \otimes_{min} E = M_n(E)$$

Example

Now let H, K be two Hilbert spaces with K being finite dimensional. That is, $K = \ell^2([n])$ for some $n \in \mathbb{N}$ and $\mathcal{B}(K) = \mathcal{B}(\ell^2([n])) = M_n$. Consider the minimal tensor product $M_n \otimes_{min} \mathcal{B}(H)$. By the above remark,

$$M_n \otimes_{min} \mathcal{B}(H) = M_n(\mathcal{B}(H))$$

completely isometrically. But, $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n) = \mathcal{B}(\ell^2([n]) \otimes_{hs} H)$ also completely isometrically. Hence,

$$\mathcal{B}(\ell^2([n])) \otimes_{min} \mathcal{B}(H) = \mathcal{B}(\ell^2([n]) \otimes_{hs} H).$$

So, if H, K are Hilbert spaces, with one of them being finite dimensional, then

$$\mathcal{B}(K) \otimes \mathcal{B}(H) = \mathcal{B}(K \otimes_{hs} H)$$

as C^*-algebras, when we endow the left hand side with the operator norm induced by the right hand side.

Remark (Associativity of the minimal tensor product)

Let $X \subseteq \mathcal{B}(H)$, $Y \subseteq \mathcal{B}(K)$ and $Z \subseteq \mathcal{B}(L)$ be operator spaces. Clearly

$$(H \hat{\otimes}_{hs} K) \hat{\otimes}_{hs} L = H \hat{\otimes}_{hs} (K \hat{\otimes}_{hs} L).$$

Hence by the aforementioned remark we have completely isometrically

$$(X \otimes_{min} Y) \otimes_{min} Z = X \otimes_{min} (Y \otimes_{min} Z).$$

Remark (Symmetricity of the minimal tensor space)

Let $X \subseteq B$, $Y \subseteq \mathcal{B}(K)$ *be operator spaces. Since we have* $H \hat{\otimes}_{hs} K = K \hat{\otimes}_{hs} H$, *again by the same remark we have completely isometrically*

$$X \otimes_{min} Y = Y \otimes_{min} X$$

via $x \otimes y \mapsto y \otimes x$.

Remark (Injectivity of the minimal tensor norm)

Let $E_1 \subseteq E_2 \subseteq \mathcal{B}(H)$ and $G_1 \subseteq G_2 \subseteq \mathcal{B}(K)$ be operator spaces (with isometric inclusions) so that $E_1 \otimes G_1 \subseteq E_2 \otimes G_2$. Then for any $x \in E_1 \otimes G_1$ we have

$$\|x\|_{E_1 \otimes_{min} G_1} = \|x\|_{E_2 \otimes_{min} G_2}.$$

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