New tensor products of C*-algebras and characterization of type I C*-algebras as rigidly symmetric C*-algebras

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A joint work with Hun Hee Lee and Matthew Wiersma

May 5, 2023

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G : \quad &= \quad \text{discrete group (usually finitely generated)},\\
C_r^*(G) \quad &= \quad \text{The reduced } C^*\text{-alg of } G(\subseteq B(l^2(G))),\\
C^*(G) \quad &= \quad \text{The full group } C^*\text{-alg of } G,\\
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Definition

G is **amenable** if it has a finitely additive translation-invariant probability measure.

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Theorem (**Hulanicki**)

G is amenable iff $C_r^*(G) = C^*(G)$ iff $\Lambda : C^*(G) \twoheadrightarrow C_r^*(G)$ is injective.

Question

For G nonamenable, we know that

$$C^*(G) \twoheadrightarrow C^*_r(G)$$

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These algebras, if they exist, are called **exotic $\mathbf{C}^*$-algebra**.

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Their examples are in the from of ℓ_p-C^*-algebras, usually denoted by $C^*_{\ell_p}(G)$. These algebras are C*-completion of $\ell^1(G)$ under the following C*-norm: For every $f \in \ell^1(G)$, we define

$$\|f\|_{C^*_{\ell_p}(G)} := \sup\{\|\pi(f)\| : \pi \text{ is a unitary rep. of } G \\ \text{with enough coefficients in } \ell_p\}.$$

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Here, by "enough coefficients", we mean that there is a dense subset $\mathcal{K} \subseteq \mathcal{H}_\pi$ such that the coefficient functions

$$s \mapsto \langle \pi(s)\xi|\xi\rangle \ , \ \xi \in \mathcal{K}$$

belongs to $\ell_p(G)$.

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$$s \mapsto \langle \pi(s)\xi|\xi\rangle \ , \ \xi \in \mathcal{K}$$

belongs to $\ell_p(G)$. For $2 \leq p \leq p' \leq \infty$, we have

$$C^*_{\ell_\infty}(G) = C^*(G) \twoheadrightarrow C^*_{\ell_{p'}}(G) \twoheadrightarrow C^*_{\ell_p}(G) \twoheadrightarrow C^*_r(G) = C^*_{\ell_2}(G).$$

Constructing exotic C^*-algebras (S-Wiersma)

One ways to construct a C^*-algebra is to use C^*-envelope:

$$\begin{tikzcd}
\ell^1(G) \arrow[rr] \arrow[d] & & C^*_r(G) \arrow[d, equal] \\
C^*(\ell^1(G)) = C^*(G) \arrow[rr] & & C^*_r(G)
\end{tikzcd}$$

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Answer: Yes! And they can be “measured” quite naturally using the *complex interpolation*:

- For every $p \in [1, \infty]$, $\ell^1(G)$ acts on $\ell^p(G)$ by convolution:

$$\begin{aligned} \ell^1(G) \ni f \mapsto T_f &\in B(\ell_p(G)) \\ f \mapsto T_f(g) &= f * g. \end{aligned}$$

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- For $1 \leq p_1 < p_2 < p_3 \leq \infty$, there exists $\theta \in (0,1)$ such that

$$\|T_f\|_{B(\ell_{p_2}(G))} \leq \|T_f\|^{1-\theta}_{B(\ell_{p_1}(G))} \|T_f\|^{\theta}_{B(\ell_{p_3}(G))}.$$

Constructing exotic C^*-algebras (S-Wiersma)

Definition

For any group G, we can define ($1 \leq p \leq \infty$, $1/p + 1/q = 1$)

$$F_p^*(G) := \text{The completion of } \ell^1(G) \text{ in } B(\ell^p(G)) \cap B(\ell^q(G))$$

which has the norm

$$\begin{aligned} \|f\|_{F_p^*(G)} &= \max\{\|T_f\|_{B(\ell^p(G))}, \|T_f\|_{B(\ell^q(G))}\} \\ &= \max\{\|T_f\|_{B(\ell^p(G))}, \|T_{f^*}\|_{B(\ell^p(G))}\}. \end{aligned}$$

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Proposition

- *For every* $p \in [1, \infty]$, $F_p^*(G)$ *is a Banach* $*$*-algebra.*
- $F_p^*(G) = F_q^*(G)$ *if* $\frac{1}{p} + \frac{1}{q} = 1$
- *We have injective* $*$*-inclusions*

$$\ell^1(G) \hookrightarrow F_{p'}^*(G) \hookrightarrow F_p^*(G) \hookrightarrow C_r^*(G)$$

for every $2 \leq p \leq p' \leq \infty$.

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- $F_p^*(G) = F_q^*(G)$ *if* $\dfrac{1}{p} + \dfrac{1}{q} = 1$
- *We have injective $*$-inclusions*

$$\ell^1(G) \hookrightarrow F_{p'}^*(G) \hookrightarrow F_p^*(G) \hookrightarrow C_r^*(G)$$

for every $2 \leq p \leq p' \leq \infty$. Moreover, by taking the C^*-envelopes, we get

$$C^*(G) \twoheadrightarrow C^*(F_{p'}^*(G)) \twoheadrightarrow C^*(F_p^*(G)) \twoheadrightarrow C_r^*(G).$$

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$$\begin{array}{ccccc} \ell^1(G) & \hookrightarrow & F_p^*(G) & \hookrightarrow & C_r^*(G) \\ \downarrow & & \downarrow & & \downarrow \\ C^*(G) & \twoheadrightarrow & C^*(F_p^*(G)) & \twoheadrightarrow & C_r^*(G) \end{array}$$

Constructing exotic C^*-algebras (S-Wiersma)

Question

For G nonamenable, is there a $p \in (2, \infty)$ such that

$$C^*(G) \overset{\text{not } 1-1}{\twoheadrightarrow} C^*(F_p^*(G)) \overset{\text{not } 1-1}{\twoheadrightarrow} C_r^*(G) \quad ?$$

More generally, could $C^*(F_p^*(G))$ be distinct?

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More generally, could $C^*(F_p^*(G))$ be distinct?

Answer: Yes! for nonamenable groups with both *rapid decay and integrable Haagroup property*.

Groups with rapid decay & Haagerup prop.

A **length function** is a function $L : G \to [0, \infty)$ such that

(i) $L(e) = 0$;

(ii) $L(g) = L(g^{-1}), \qquad g \in G$;

(iii) $L(st) \leq L(s) + L(t), \qquad s, t \in G$.

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For any $d > 0$,

$$w_d(s) = (1 + L(s))^d$$

is a (submultiplicative) weight on G. We say that (G, L) has **Rapid decay (RD)** if $\exists\ d > 0$ such that.

$$\ell^2(G, w_d) \subseteq C_r^*(G)$$

iff there is $M > 0$ such that

$$\|f\|_{C_r^*(G)} \leq M \|f\|_{\ell^2(G, w_d)} \ \ (f \in \ell^2(G, w_d)).$$

Groups with rapid decay & Haagerup prop.

A length function L is a **Haagerup length function (IH)** if for every $t \geq 0$,

$$\varphi_t(s) = e^{-tL(s)} \qquad (s \in G)$$

is a positive-definite function on G.

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A length function L is a **Haagerup length function (IH)** if for every $t \geq 0$,

$$\varphi_t(s) = e^{-tL(s)} \qquad (s \in G)$$

is a positive-definite function on G. A group G has **integrable Haagerup (IH) property** if it has a Haagerup length function L such that for every $t > 0$

$$\varphi_t(\cdot) = e^{-tL(\cdot)} \in \bigcup_{1 \leq p < \infty} \ell_p(G).$$

Groups with rapid decay & Haagerup prop.

Example

Groups with RD+IH:

(i) F_n, nonabelian free groups on n-generators.

(ii) Finitely generated Coxeter groups.

(iii) "Some" groups acting properly and cocompactly by isometries on "Some" CAT(0).

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Theorem (**S-Wiersma**)

For G nonamenable with RD+IH,

$$C^*_{\ell_p}(G) = C^*(F^*_p(G)) \quad (p \in [2, \infty]).$$

Moreover, they are all pairwise distinct.

Groups with rapid decay & Haagerup prop.

Key ideals in the proof: We use complex interpolation to obtain the following:

$$\left.\begin{array}{l} \ell^1(G) \to B(\ell^1(G)) \\ \ell^2_w(G) \to B(\ell^2(G)) \end{array}\right\} \Rightarrow \ell^q_{w_q}(G) \subseteq B(\ell^q(G)),$$

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where

$$1 \leq q \leq 2,\ \frac{1}{p} + \frac{1}{q} = 1,\ w_q = w_d^{\frac{2}{p}}.$$

Groups with rapid decay & Haagerup prop.

Hence we have that

$$\ell^q_{w_q}(G) \subseteq F^*_p(G) := \text{The completion of } \ell^1(G) \text{ in } B(\ell^p(G)) \cap B(\ell^q(G)).$$

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We will then have that

$$\begin{tikzcd} \ell^q_{w_q} \arrow[r, hook] \arrow[dr] & F^*_p(G) \arrow[d] \\ & C^*(F^*_p(G)) \end{tikzcd}$$

$$\Rightarrow C^*(F^*_p(G))^* \subseteq \ell^p_{w_q^{-1}}(G).$$

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However we can check exactly which positive definite function of the form $\varphi_t = e^{-tL}$ belongs to $\ell^p_{w_q^{-1}}$ and which does not.

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However we can check exactly which positive definite function of the form $\varphi_t = e^{-tL}$ belongs to $\ell^p_{w_q^{-1}}$ and which does not. Hence using this criterion, we can show that

$$C^*(F_p^*(G)) = C^*_{\ell_p}((G)),$$

and

$$C^*(F_p^*(G)) \neq C^*(F_{p'}^*(G))$$

for all $2 \leq p \neq p' \leq \infty$.

Goal: Apply similar ideas to tensor category of C^*-algebras

Let $A \subseteq B(\mathcal{H})$ and $B \subseteq B(\mathcal{K})$ be C^*-algebras. Then

$$\begin{tikzcd} A \otimes B \arrow[r, hook] \arrow[dr] & A \otimes_{\max} B \arrow[d, two heads] \\ & A \otimes_{\min} B \subseteq B(\mathcal{H} \otimes \mathcal{K}) \end{tikzcd}$$

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We say that (A, B) is a **nuclear pair** if

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Question

Can we construct distinct C^*-tensor norms on $A \otimes B$, when (A, B) is a not a nuclear pair, strictly between $A \otimes_{min} B$ and $A \otimes_{max} B$?

Goal: Apply similar ideas to tensor category of C^*-algebras

- $\mathcal{H}$, a Hilbert space.
- $\mathcal{H}_C := B(\mathbb{C}, \mathcal{H})$ column Hilbert space.
- $\mathcal{H}_R := B(\overline{\mathcal{H}}, \mathbb{C})$ row Hilbert space.

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- Complex interpolation space $(1 \leq p \leq \infty)$

$$\mathcal{H}_{C_p} := [\mathcal{H}_C, \mathcal{H}_R]_{\frac{1}{p}} \ , \ \mathcal{H}_{R_p} := [\mathcal{H}_R, \mathcal{H}_C]_{\frac{1}{p}} .$$

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- $\mathcal{H}_{C_2} = \mathcal{H}_{R_2} (= \mathcal{H}_{oh})$ operator Hilbert space uniquely determined by

$$\mathcal{H}_{oh}^* \cong \overline{\mathcal{H}}_{oh}.$$

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- $\otimes_h$ is the **Haagerup tensor product**.
- **Compatibility with the complex interpolation:**

$$[X_0, X_1]_\theta \otimes_h [Y_0, Y_1]_\theta \cong [X_0 \otimes_h Y_0, X_1 \otimes_h Y_1]_\theta,$$

where (X_0, X_1) & (Y_0, Y_1) are compatible operator spaces & $0 \le \theta \le 1$.

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Definition

Let $1 \leq p \leq \infty$. For $A \subseteq B(\mathcal{H})$, $B \subseteq B(\mathcal{K})$, C^*-algebras. We consider the following mapping:

$$\pi_p : A \otimes B \to CB(\mathcal{H}_{C_p} \otimes_h \mathcal{K}_{R_p})$$
$$\pi_p(a \otimes b)(\xi \otimes \eta) = a\xi \otimes b\eta.$$

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We can use it to define a tensor product norm on $A \otimes B$:

$$A \otimes_p B := \text{The completion of } \pi_p(A \otimes B)$$
$$\text{inside } CB(\mathcal{H}_{C_p} \otimes \mathcal{K}_{R_p}).$$

Another reformulation

$$\mathcal{H}_{C_1} \otimes_h \mathcal{K}_{R_1} = \mathcal{H}_R \otimes_h \mathcal{K}_C \cong T(\overline{\mathcal{K}}, \mathcal{H})$$

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$$\mathcal{H}_{C_p} \otimes_h \mathcal{K}_{R_p} \cong S_p(\overline{\mathcal{K}}, \mathcal{H}),$$

where $S_p(\overline{\mathcal{K}}, \mathcal{H})$ is the space of p-Schatten class operators.

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where $S_p(\overline{\mathcal{K}}, \mathcal{H})$ is the space of p-Schatten class operators. In this case, we can write

$$\begin{array}{l}\pi_p : A \otimes B \to CB(S_p(\overline{\mathcal{K}}, \mathcal{H})) \\ \pi_p(a \otimes b)T = aT\tilde{b} \\ [b \in B(\mathcal{H}) \to \tilde{b} \in B(\overline{\mathcal{H}})].\end{array}$$

Another reformulation

Remark

$$F_p(G) \subseteq B(\ell^p(G)) \qquad (\ell^p(G) \textit{ is a commutative } L_p\textit{-space}).$$
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- $\otimes_1$, $\otimes_2$, $\otimes_\infty$ does not depend on the representations of C^*-algebras.

Another reformulation

Definition

Let $A \subseteq B(\mathcal{H})$, $B \subseteq B(\mathcal{K})$ be C^*-algebras, and let $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We let

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- *There is* $\theta \in [0,1]$ *such that*

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For discrete groups G_1 & G_2, we define

$$S_p(G_1 \times G_2) := \{f : G_1 \times G_2 \to \mathbb{C} : \ [f(s,t)] \in S_p(\overline{\ell^2(G_2)}, \ell^2(G_1))\}.$$

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- *The identity map on $\ell^1(G_1 \times G_2)$ extends to a surjective $*$-homomorphism*

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- If G_1 & G_2 have RD+IH, then

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In particular, they are all distinct for $1 \leq p \neq p' \leq 2$ if $G_1 = G_2$ is a nonamenable group having RD+IH.

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Theorem

*(i) (**Ludwig-1979**) Finite extensions of nilpotent groups (e.g. finitely generated groups with polynomial growth) are Hermitian.*
*(ii) (**Jenkins 1970**) Any group containing free subsemigroup on two generators is not Hermitian.*

Hermitian groups are amenable

Theorem (Hulanicki-Leinert)

Let A be $*$-semisimple Banach $*$-algebra. Then TFAE:
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The proof heavily uses the intermediate Banach $*$-algebras $F_p^*(G)$ (between $\ell^1(G)$ and $C_r^*(G)$) and the complex interpolation relation

$$\|f\|_{F_p^*(G)} \leq \|f\|_1^{1-\theta} \|f\|_{C_r^*(G)}^{\theta} \quad (f \in \ell^1(G)).$$

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$$C^*(A \otimes_{1,\infty} B) = A \otimes_{\max} B.$$