# Separable spaces of continuous functions as Calkin algebras

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Functional Analysis and Operator Algebras in Athens

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# Background

# Banach algebras

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**Example:** For a compact Hausdorff space $K$,

$$C(K) = \{f : K \to \mathbb{C} \text{ continuous}\}$$

with $\|f\| = \sup\{|f(k)| : k \in K\}$ and point-wise multiplication is a unital Banach algebra.

**Example:** the convolution algebra

$$\ell_1(\mathbb{N}_0) = \left\{ a = (a(i))_{i=0}^{\infty} : \ \|a\| = \sum_{i=0}^{\infty} |a(i)| < \infty \right\},$$

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where for $a = (a(i))_{i=0}^{\infty}$ and $b = (b(i))_{i=0}^{\infty}$,

the convolution $a * b = \Big( (a * b)(i) \Big)_{i=0}^{\infty}$ is

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**Comment:** This is related to Taylor series coefficients.

# Banach algebras

**Example:** the coordinte-wise multiplication algebra

$$\ell_p(\mathbb{N}) = \left\{ a = (a(i))_{i=0}^{\infty} : \ \|a\| = \left( \sum_{i=1}^{\infty} |a(i)|^p \right)^{1/p} < \infty \right\},$$

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where for $a = (a(i))_{i=1}^{\infty}$ & $b = (b(i))_{i=1}^{\infty}$, the coordinate-wise product is

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$$a \cdot b = \left( a(i) b(i) \right)_{i=1}^{\infty}.$$

**Comment:** For $1 < p < \infty$ this is a reflexive Banach algebra, i.e.,

$$\ell_p(\mathbb{N}) \equiv \ell_p(\mathbb{N})^{**}.$$

# Banach algebras of operators

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With the operator norm

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$$

and composition $TS$, $\mathcal{L}(X)$ is a Banach algebra.

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**Example:** If $X$ is a Banach space and $S \subset \mathcal{L}(X)$,

$$\mathcal{B}_{X,S} = \overline{\langle \{ T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} : T_1, \dots, T_n \in S \text{ and } k_1, \dots, k_n \in \mathbb{N} \} \rangle}.$$

# Banach algebras of operators

Let $X$ be a Banach space with a basis $(e_i)_{i\in\mathbb{Z}}$ (e.g., $\ell_2(\mathbb{Z})$ or $\ell_1(\mathbb{Z})$).

The **Right Shift** operator $R : X \to X$:

$$R\Big(\sum_{i\in\mathbb{Z}} a_i e_i\Big) = \sum_{i\in\mathbb{Z}} a_i e_{i+1}$$

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**Remark:** on $\ell_2(\mathbb{Z})$ and $\ell_1(\mathbb{Z})$ $L$, $R$ are bounded.

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**Comment:** If $S \subset \mathcal{L}(X)$ then

- the algebraic structure of $S$ and
- the geometric structure of $X$

*together* determine $\mathcal{B}_{X,L,R} \equiv \ell_1(\mathbb{Z})$.

# Quotient algebras of $\mathcal{L}(X)$

An **ideal** of $\mathcal{L}(X)$ is a subspace $\mathcal{A}$ of $\mathcal{L}(X)$

- that is closed with respect to the operator norm topology,
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$$\mathcal{K}(X) = \{T \in \mathcal{L}(X) \text{ compact}\},$$

$$\mathcal{SS}(X) = \{T \in \mathcal{L}(X) \text{ strictly singular}\}.$$

**Remark:** If $\mathcal{A}$ is an ideal of $\mathcal{L}(X)$ then $\mathcal{L}(X)/\mathcal{A}$ is a Banach algebra, called a quotient algebra of $\mathcal{L}(X)$.

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**Definition:** The *Calkin algebra* of $X$ is

$$\mathcal{C}al(X) = \mathcal{L}(X)/\mathcal{K}(X).$$

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**Remark:** $\dim(X) = \infty$ if and only if $\mathcal{C}al(X)$ is a *unital Banach algebra*.

# Origins of the Calkin algebra

**Comment:** $\mathcal{C}al(X)$ is useful for:

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- Classifying ideals of $\mathcal{L}(X)$.

**Theorem:** (Calkin 1941)
$\mathcal{C}al(\ell_2)$ has no non-trivial ideals $\Longrightarrow$
$\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$ are the only ideals in $\mathcal{L}(\ell_2)$.

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- Characterizing Fredholm operators on $X$.

**Theorem:** (Atkinson, 1951)
$T \in \mathcal{L}(X)$ is Fredhold $\iff$ $[T]$ in $\mathcal{Cal}(X)$ is invertible.

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More:

- $K$-theory of $C^*$-algebras (Brown - Douglas - Fillmore, 1977)
- Set theory (Phillips, 2007 and Farah, 2011)

# Explicit Calkin algebras

# Explicit descriptions of $\mathcal{C}al(X)$.

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- $\mathcal{B} = \mathbb{C}$, Argyros-Haydon, 2011.
- $\mathcal{B} = \ell_1(\mathbb{N}_0)$, Tarbard, 2013.
- $\mathcal{B} = C(K)$, for $K$ countable and compact, M-Puglisi-Zisimopoulou, 2016.

and others.

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**Theorem:** (M - 2021) For every *compact metric space $K$* there exists a Banach space $\mathfrak{X}_{C(K)}$ with $\mathcal{C}al(\mathfrak{X}_{C(K)}) \equiv C(K)$.

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E.g., $C[0,1]$, $C(2^{\mathbb{N}})$, and $C(\mathbb{T})$ occur as Calkin algebras.

# Quotients of $\mathcal{L}(X)$ and tight control of operators

# Explicit descriptions of quotient algebras.

**Theorem:** (Gowers - Maurey, 1993) There exists $X_{\mathrm{GM}}$ such that:

- Every $T \in \mathcal{L}(X_{\mathrm{GM}})$ is of the form $T = \lambda I + S$, with $S \in \mathcal{SS}(X_{\mathrm{GM}})$.

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**Theorem:** (Argyros - Haydon, 2011) There exists $\mathfrak{X}_{\mathrm{AH}}$ such that:

- Every $T \in \mathcal{L}(\mathfrak{X}_{\mathrm{AH}})$ is of the form $T = \lambda I + K$, with $K \in \mathcal{K}(\mathfrak{X}_{\mathrm{AH}})$.
- Equivalently, $\mathcal{Cal}(\mathfrak{X}_{\mathrm{AH}})$ is one-dimensional.

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$$T = \lambda I + \sum_{n=1}^{\infty} \lambda_n L^n + \sum_{n=1}^{\infty} \mu_n R^n + S$$

with $S \in \mathcal{SS}(X_{\mathrm{GM}}^{\ell_1})$ and $\sum_n |\lambda_n| + \sum_n |\mu_n| < \infty$.

# Explicit descriptions of quotient algebras.

**Theorem:** (Gowers-Maurey, 1997)
There exists a Banach space $X^{\ell_1}_{\mathrm{GM}}$ with a basis such that:

- The left shift $L$ and the right shift $R$ on $X^{\ell_1}_{\mathrm{GM}}$ are bounded.
- $\forall T \in \mathcal{L}(X^{\ell_1}_{\mathrm{GM}})$,

$$T = \lambda I + \sum_{n=1}^{\infty} \lambda_n L^n + \sum_{n=1}^{\infty} \mu_n R^n + S$$

with $S \in \mathcal{SS}(X^{\ell_1}_{\mathrm{GM}})$ and $\sum_n |\lambda_n| + \sum_n |\mu_n| < \infty$.

Therefore, $\mathcal{L}(X^{\ell_1}_{\mathrm{GM}})/\mathcal{SS}(X^{\ell_1}_{\mathrm{GM}}) \equiv \ell_1(\mathbb{Z})$ (Wiener algebra).

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with $K \in \mathcal{K}(\mathfrak{X}_{\mathrm{T}})$ and $\sum_n |\lambda_n| < \infty$.

Therefore, $\mathcal{C}al(\mathfrak{X}_{\mathrm{T}}) \equiv \ell_1(\mathbb{N}_0)$.

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1. Identify a class of operators $\mathcal{C}$ on a classical Banach space $X_0$ that generates $\mathcal{B}$.

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3. Construct an Argyros-Haydon space $X$ with $\mathcal{C}al(X) \simeq \mathcal{B}$.

# Noteworthy exceptions

- $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ (Laustsen - Kania, 2017), by taking finite sums of Argyros-Haydon spaces.

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And others (e.g., Skillicorn, 2015).

# Idea of proof

# $C(K)$ as a Calkin algebra: step 1.

For given $\mathcal{B} = C(K)$, how to represent $C(K) \simeq \mathcal{C}al(X)$.

1. Identify a class of operators $\mathcal{C}$ on a classical Banach space $X_0$ that generates $C(K)$.
2. Construct a Gowers-Maurey space $X$ with $\mathcal{L}(X)/\mathcal{SS}(X) \simeq C(K)$.
3. Construct an Argyros-Haydon space $X$ with $\mathcal{C}al(X) \simeq C(K)$.

**Proposition:** Let $T_0 \in \mathcal{L}(\ell_2)$ be *normal*. The unital $C^*$-algebra

$$\overline{\langle\{I, T_0^n(T_0^*)^m : n, m \in \mathbb{N}\}\rangle}$$

generated by $T_0$ is $C(\sigma(T_0))$.

# $C(K)$ subalgebras of $\mathcal{L}(\ell_2)$.

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generated by $T_0$ is $C(\sigma(T_0))$.

In fact, any commutative $*$-subalgebra of $\mathcal{L}(\ell_2)$ is some $C(K)$.

# $C(\mathbb{T})$ subalgebras of $\mathcal{L}(\ell_2)$.

**Example:** Fix $\alpha \in (0,1) \setminus \mathbb{Q}$ and put $z_0 = e^{\alpha 2\pi i}$.

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**Notation:** For every continuous $\phi : \mathbb{T} \to \mathbb{C}$ define

$$\hat{\phi} : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}) \text{ with } \hat{\phi}(e_n) = \phi(z_0^n) e_n$$

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**Proposition:** $\hat{\cdot} : C(\mathbb{T}) \to \mathcal{L}(\ell_2(\mathbb{N}))$ is a homomorphic isometry.

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$$\hat{\phi} : X_0 \to X_0 \quad \text{with } \hat{\phi}(e_n) = \phi(z_0^n) e_n$$

**Proposition:** $\hat{\cdot} : C(\mathbb{T}) \to \mathcal{L}(X_0)$ is a homomorphic isometry,

for any $X_0$ with a 1-unconditional basis $(e_n)_n$.

# $C(\mathbb{T})$ as a Calkin algebra: step 2.

For $\mathcal{B} = C(\mathbb{T})$, how to represent $C(\mathbb{T}) \simeq \mathcal{C}al(X)$.

1. Identify a class of operators $\mathcal{C}$ on a classical Banach space $X_0$ that generates $C(\mathbb{T})$.
2. Construct a Gowers-Maurey space $X$ with $\mathcal{L}(X)/\mathcal{SS}(X) \simeq C(\mathbb{T})$.
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In particular, $\mathcal{C}al(\mathfrak{X}_{C(\mathbb{T})}) \equiv C(\mathbb{T})$.

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**Theorem:** (A. Pelczar-Barwacz, 2022) There exists a Banach space $X$ such that $\mathcal{L}(X)/\mathcal{SS}(X)$ is reflexive.

In fact, $\mathcal{L}(X)/\mathcal{SS}(X) \simeq \mathbb{C}e \oplus X^*_{\mathrm{mT}}$.

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**Problem:** Find unital Banach algebras that are **not** Calkin algebras.

Identify non-trivial properties of all Calking algebras.

# Problems.

N. C. Phillips recommended the following.

**Problem:** Find representations of the following non-commutative $C^*$-algebras as Calkin algebras:

- The UHF algebra of type $2^\infty$.
- The Cuntz algebra $\mathcal{O}_n$.
- The reduced $C^*$-algebra of the free group on two generators, $C^*_r(\mathbb{F}_2)$.

Ευχαριστώ!