Operator Spaces: An introduction

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Milestones I

- F. J. Murray and J. Von Neumann.
  On rings of operators.
  Ann. of Math. (2), 37(1):116–229, 1936.
- I. Gelfand and M. Neumark.
  On the imbedding of normed rings into the ring of operators in Hilbert space.
  Rec. Math. [Mat. Sbornik] N.S., 12(54):197–213, 1943.
- W. Forrest Stinespring.
  Positive functions on C^*-algebras.
  Proc. Amer. Math. Soc., 6:211–216, 1955.
- William B. Arveson.
  Subalgebras of C^*-algebras.
  Acta Math., 123:141–224, 1969.

Milestones II

Man Duen Choi and Edward G. Effros.
Injectivity and operator spaces.
J. Functional Analysis, 24(2):156–209, 1977.

Zhong-Jin Ruan.
Subspaces of C^*-algebras.
J. Funct. Anal., 76(1):217–230, 1988.

$\mathscr{B}(H)$

Let H be a Hilbert space. The algebra of all bounded linear operators $T : H \to H$ is denoted $\mathscr{B}(H)$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{ball}(H)\}$$

Moreover, it has an *involution* $T \to T^*$ defined via

$$\langle T^* x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in H.$$

This satisfies

$$\|T^* T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

$\mathscr{B}(H)$ is a C*-algebra

Structure of $\mathscr{B}(H)$:

1. linear space
2. ring for composition of operators [thus, an associative algebra]
3. *-vector space with identity
4. has a complete submultiplicative norm ($\|TS\| \leq \|T\| \|S\|$)
5. norm satisfies the C^* property $\|T^* T\| = \|T\|^2$

Provisional Definitions

- Operator space: A linear subspace of $\mathscr{B}(H)$ (sometimes assumed closed)
- Operator system: A selfadjoint (i.e. *-closed) linear subspace of $\mathscr{B}(H)$ containing the identity
- C*-algebra: A selfadjoint, $\|\cdot\|$-closed subalgebra of $\mathscr{B}(H)$

Function Spaces

A concrete function space is a linear subspace $E \subseteq \ell_\infty(\Gamma)$ for some Γ.
Is E/N a concrete function space on some Γ'?

Remark

Every normed space can be isometrically represented as a concrete function space.

Given $(E, \|\cdot\|)$ and $n \in \mathbb{N}$, consider
$\ell_\infty^n(E) := \{([x_1, \ldots, x_n] : x_i \in E\}$ with sup norm.
Note that if $E \subseteq \ell_\infty(\Gamma)$ then $\ell_\infty^n(E) \subseteq \ell_\infty(\Gamma \times [n])$ isometrically.

Remark

If E is function system, so is $\ell_\infty^n(E)$ for all $n \in \mathbb{N}$.

Operator Spaces

"Quantize": Replace function in $\ell_\infty(\Gamma)$ by operators in $\mathscr{B}(H)$ for some Hilbert space H.

Given a subspace $E \subseteq \mathscr{B}(H)$ and $n \in \mathbb{N}$, then $M_n(E) \subseteq M_n(\mathscr{B}(H))$ as linear spaces.

But note $M_n(\mathscr{B}(H)) \simeq \mathscr{B}(H^n)$ as linear spaces, where $H^n := \{\vec{h} := [h_1, \ldots, h_n] : x_i \in H\}$ with $\left\langle \vec{h}, \vec{h}' \right\rangle := \sum_{k=1}^n \langle h_k, h_k' \rangle_H$ and $M_n(\mathscr{B}(H)) \twoheadrightarrow \mathscr{B}(H^n) : [a_{ij}] \mapsto A$ where $A\vec{h} = [\sum_{j=1}^n a_{ij} h_j]$.

Remark

If E is an operator space on H, then, for all $n \in \mathbb{N}$, $M_n(E)$ is an operator space on H^n.

So the embedding $j : E \rightarrowtail \mathscr{B}(H)$ defines a sequence of norms $\{\|\cdot\|_{M_n(E)} : n \in \mathbb{N}\}$.

Operator Spaces

Definition

An operator space E is a pair (E, j) where E is a linear space and $j : E \rightarrowtail \mathscr{B}(H)$ a linear embedding. If

$$\left\|[x_{ij}]\right\|_{M_n(E)} \stackrel{def}{=} \left\|[j(x_{ij})]\right\|_{\mathscr{B}(H^n)} \quad (x_{ij} \in E),$$

[1] the sequence of norms $\{\|\cdot\|_{M_n(E)} : n \in \mathbb{N}\}$ is called the operator space structure on E induced by j.

Definition

An operator space structure on a normed space $(E, \|\cdot\|)$ is the operator space structure induced by a linear isometric embedding $j : E \rightarrowtail \mathscr{B}(H)$ for some Hilbert space H.

(Thus $\|j(x)\|_{\mathscr{B}(H)} = \|x\|_E$ for all $x \in E$)

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[1] Thanks, Dimos!

Completely bounded maps

Notation Given a linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H')$, for any $n \in \mathbb{N}$ define $\phi_n : \mathscr{B}(H^n) \to \mathscr{B}(H'^n) : [a_{il}] \mapsto [\phi(a_{ij})]$.

Definition

A linear map $\phi : E \to F$ between operator spaces is said to be completely bounded if (each $\phi_n : M_n(E) \to M_n(F)$ is bounded and) $\|\phi\|_{cb} := \sup_n \|\phi_n\| < \infty$.
The map ϕ is said to be completely isometric if each $\phi_n : M_n(E) \to M_n(F)$ is an isometry.

Thus if $j : E \rightarrowtail \mathscr{B}(H)$ is an isometric embedding, the induced operator space structure on E is by definition the one making $j_n : (M_n(E), \|\cdot\|_{M_n(E)}) \to (\mathscr{B}(H^n), \|\cdot\|_{\mathscr{B}(H^n)})$ isometric for all n, i.e. making j a complete isometry.

The minimal operator space structure on $(E, \|\cdot\|_E)$

Remark

Every normed space $(E, \|\cdot\|_E)$ admits an operator space structure.

Proof Embed $E \hookrightarrow \ell_\infty(\Gamma)$ isometrically, then embed $\ell_\infty(\Gamma) \hookrightarrow \mathscr{B}(\ell_2(\Gamma)) = \mathscr{B}(H)$ as diagonal operators. Let j be the composite, so $\|j(x)\|_{\mathscr{B}(H)} = \|x\|_E$ for all $x \in E$. For $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(E)$, define $\|x\|_{M_n(E)} := \|j_n(x)\|_{\mathscr{B}(H^n)}$. □

This op. space structure is called the minimal op. structure $\min E$ on $(E, \|\cdot\|_E)$. It has the universal property:
If F is an op. space and $\phi : F \to E$ a bounded linear map, then $\|\phi\|_{cb} = \|\phi\|$.
For $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(E)$,

$$\left\|[x_{ij}]\right\|_{\min} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H^n)} : \phi : E \to \mathbb{C} \text{ contraction}\}$$

(using $\mathrm{ball}(E^*)$ for Γ).

The maximal operator space structure on $(E, \|\cdot\|_E)$

Let $\mathscr{S}$ be the family of all isometric embeddings $\phi : E \to \mathscr{B}(H_\phi)$ (this is not empty since $\min E$ exists).
The max structure corresponds to the embedding given by the 'direct sum' (suitable defined) of all the embeddings $\phi : E \to \mathscr{B}(H_\phi)$.
Each ϕ induces a norm $\|\cdot\|_n^\phi$ on each $M_n(E)$ given by $\|[x_{ij}]\|_n^\phi = \|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)}$. The max norm is defined to be the supremum of these norms:

$$\|[x_{ij}]\|_{\max} = \sup\{\|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)} : (\phi, H_\phi) \in \mathscr{S}\}, \quad [x_{ij}] \in M_n(E)$$

(this supremum is finite, since [2]
$\|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)} \leq \left(\sum_{i,j} \|\phi(x_{ij})\|_{\mathscr{B}(H_\phi)}^2\right)^{1/2} = \left(\sum_{i,j} \|x_{ij}\|_E^2\right)^{1/2}$).

[2]Thanks, Mihalis!

The maximal and minimal operator space structures

The maximal operator space structure structure $\max E$ on $(E, \|\cdot\|_E)$ has the universal property:
If V is an op. space and $\psi : E \to V$ a bounded linear map, then $\|\psi\|_{cb} = \|\psi\|$.

To compare:

For $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(E)$,

$$\left\|[x_{ij}]\right\|_{\min} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H^n)} : \phi : E \to \mathbb{C} \text{ contraction}\}$$
$$\left\|[x_{ij}]\right\|_{\max} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H_\phi^n)} : \phi : E \to \mathscr{B}(H_\phi) \text{ contraction}\}$$

(by the universal property of max, using contractions instead of isometries does not increase $\|\cdot\|_{\max}$).

Example: $\min(\ell^1[d])$ and $\max(\ell^1[d])$

Let $d \in \mathbb{N}$ and $H = L^2(\mathbb{T}^d)$. For $k \in [d]$ let $V_k \in \mathscr{B}(H)$ be multiplication by the k-th coordinate function: $(V_k f)(z_1, \ldots, z_d) = z_k f(z_1, \ldots, z_d)$ $(f \in H)$. Then $V_1, \ldots V_d \in \mathscr{B}(H)$ are commuting unitaries. Write $\mathscr{C}_d = \operatorname{span}\{V_1, \ldots V_d\} \subseteq \mathscr{B}(H)$.
The map

$$J : \ell^1[d] \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathscr{C}_d \subseteq \mathscr{B}(H) : [a_k] \mapsto \sum_{k=1}^{d} a_k V_k$$

is a linear isometry, and $J(\ell^1[d]) \simeq \min(\ell^1[d])$ completely isometrically.

$\min(\ell^1[d])$ and $\max(\ell^1[d])$ continued

Let $\mathbb{F}_d$ be the free group in d generators $u_1, \ldots, u_d$ and let (π, H_π) be the universal unitary representation of $\mathbb{F}_d$ (the direct sum of all unitary representations on (separable) Hilbert spaces). Let $U_1, \ldots U_d \in \mathscr{B}(H_\pi)$ be the images of the generators: $U_k = \pi(u_k)$. These are free unitaries. Write $\mathscr{Z}_d = \mathrm{span}\{U_1, \ldots U_d\} \subseteq \mathscr{B}(H_\pi)$.
The map

$$J_\pi : \ell^1[d] \rightarrowtail\!\!\!\!\rightarrow \mathscr{Z}_d \subseteq \mathscr{B}(H_\pi) : [a_k] \mapsto \sum_{k=1}^{d} a_k U_k$$

is a linear isometry, and $J_\pi(\ell^1[d]) \simeq \max(\ell^1[d])$ completely isometrically.