Operator Spaces and Operator Systems: An introduction

Aristides Katavolos

November 4, 2022

Operator Spaces

Definition

An operator space E is a pair (E, j) where E is a linear space and $j : E \rightarrowtail \mathscr{B}(H)$ a linear embedding. If

$$\left\|[x_{ij}]\right\|_{M_n(E)} \stackrel{def}{=} \left\|[j(x_{ij})]\right\|_{\mathscr{B}(H^n)} \ (x_{ij} \in E),$$

the sequence of norms $\{\|\cdot\|_{M_n(E)} : n \in \mathbb{N}\}$ is called the operator space structure on E induced by j.

Definition

An operator space structure on a normed space $(E, \|\cdot\|)$ is the operator space structure induced by a linear isometric embedding $j : E \rightarrowtail \mathscr{B}(H)$ for some Hilbert space H.

(Thus $\|j(x)\|_{\mathscr{B}(H)} = \|x\|_E$ for all $x \in E$)

Completely bounded maps

Notation Given a linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H')$, for any $n \in \mathbb{N}$ define $\phi_n : \mathscr{B}(H^n) \to \mathscr{B}(H'^n) : [a_{il}] \mapsto [\phi(a_{ij})]$.

Definition

A linear map $\phi : E \to F$ between operator spaces is said to be completely bounded if (each $\phi_n : M_n(E) \to M_n(F)$ is bounded and) $\|\phi\|_{cb} := \sup_n \|\phi_n\| < \infty$.
The map ϕ is said to be completely isometric if each $\phi_n : M_n(E) \to M_n(F)$ is an isometry.

Write $CB(E,F)$ for the space of all cb maps $E \to F$. These are the natural morphisms for the category of operator spaces. When F is complete, $(CB(E,F), \|\cdot\|_{cb})$ is a Banach space.

The minimal operator space structure on $(E, \|\cdot\|_E)$

Remark

Every normed space $(E, \|\cdot\|_E)$ admits an operator space structure.

Idea: Embed $E \hookrightarrow \ell_\infty(\Gamma)$ isometrically, then embed $\ell_\infty(\Gamma) \hookrightarrow \mathscr{B}(\ell_2(\Gamma)) = \mathscr{B}(H)$ as diagonal operators.

This op. space structure is called the minimal op. structure $\min E$ on $(E, \|\cdot\|_E)$. It has the universal property:
If F is an op. space and $\phi : F \to E$ a bounded linear map, then ϕ is completely bounded and in fact $\|\phi\|_{cb} = \|\phi\|$.

The maximal operator space structure on $(E, \|\cdot\|_E)$

Let $\mathscr{S}$ be the family of all isometric embeddings $\phi : E \to \mathscr{B}(H_\phi)$ (this is not empty since $\min E$ exists).
The max structure corresponds to the embedding given by the 'direct sum' (suitable defined) of all the embeddings $\phi : E \to \mathscr{B}(H_\phi)$.
Each ϕ induces a norm $\|\cdot\|_n^\phi$ on each $M_n(E)$ given by $\|[x_{ij}]\|_n^\phi = \|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)}$. The max norm is defined to be the supremum of these norms:

$$\|[x_{ij}]\|_{\max} = \sup\{\|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)} : (\phi, H_\phi) \in \mathscr{S}\}, \quad [x_{ij}] \in M_n(E)$$

(this supremum is finite, since [1]
$\|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)} \leq \left(\sum_{i,j} \|\phi(x_{ij})\|^2_{\mathscr{B}(H_\phi)}\right)^{1/2} = \left(\sum_{i,j} \|x_{ij}\|_E^2\right)^{1/2}$).

[1] Thanks, Mihalis!

The maximal and minimal operator space structures

The maximal operator space structure structure $\max E$ on $(E, \|\cdot\|_E)$ has the universal property:
If V is an op. space and $\psi : E \to V$ a bounded linear map, then ψ is completely bounded and in fact $\|\psi\|_{cb} = \|\psi\|$.

To compare:

For $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(E)$,

$$\|[x_{ij}]\|_{\min} = \sup\{\|[\phi(x_{ij})]\|_{\mathscr{B}(H^n)} : \phi : E \to \mathbb{C} \text{ contraction}\}$$
$$\|[x_{ij}]\|_{\max} = \sup\{\|[\phi(x_{ij})]\|_{\mathscr{B}(H_\phi^n)} : \phi : E \to \mathscr{B}(H_\phi) \text{ contraction}\}$$

The column and row spaces C_n and R_n

$R_n \subseteq \mathscr{B}(\ell^2[n])$ consists of all $n \times n$ matrices having zeroes except for the first row $\simeq \mathscr{B}(\ell^2[n], \mathbb{C}) \simeq M_{1,n}$.

$C_n \subseteq \mathscr{B}(\ell^2[n])$ consists of all $n \times n$ matrices having zeroes except for the first column $\simeq \mathscr{B}(\mathbb{C}, \ell^2[n]) \simeq M_{n,1}$.

Both are isometric to $\ell^2[n]$, hence to one another.

The column and row spaces C_n and R_n

Now $M_{p,q}(C_n) \simeq M_{np,q}$ while $M_{p,q}(R_n) \simeq M_{p,nq}$.

Thus $M_{1,n}(C_n) \simeq M_{n,n}$ while $M_{1,n}(R_n) \simeq M_{1,n^2}$.

Take $[e_1, \ldots, e_n] \in M_{1,n}(C_n)$ to get $I_n \in M_n$ which has norm 1, while $[e_1, \ldots, e_n] \in M_{1,n}(R_n)$ gives a row vector of length n^2 with n 1's so its norm is $\sqrt{n}$.

Thus the identity mapping $\imath : C_n \to R_n$ has $\|\imath\| = 1$ but $\|\imath\|_{cb} \geq \sqrt{n}$.
It follows that the identity mapping $\imath : C \to R$ between the infinite-dimensional analogues, which is an isometry, is not completely bounded.

The dual of an operator space E

- A bounded linear map $\phi : E \to \mathbb{C}$ is automatically completely bounded with $\|\phi\|_{cb} = \|\phi\|$.

Proposition

The (Banach space) dual E^* of an operator space E has a natural operator space structure.

Idea: Let $D = \bigcup_n \mathrm{ball}(M_n(E))$ (disjoint union). For each $x \in D$ there is $n(x) \in \mathbb{N}$ with $x = [x_{ij}] \in \mathrm{ball}(M_{n(x)}(E))$. Define

$$v_x : E^* \to M_{n(x)}(\mathbb{C}) : \phi \in E^* \mapsto [\phi(x_{ij})].$$

The map

$$J : E^* \to \bigoplus_{x \in D} M_{n(x)}(\mathbb{C}) : \phi \mapsto \bigoplus_{x \in D} v_x(\phi)$$

is an isometric embedding into a direct sum of matrix algebras, which consists of operators on the Hilbert space direct sum $\bigoplus_{x \in D} \ell^2[n(x)]$.

Positivity

An $A \in \mathscr{B}(H)$ is positive if ($A = A^*$ and) $\langle Ax, x\rangle \geq 0$ for all $x \in H$.
Equivalently if $\exists B \in \mathscr{B}(H)$ with $A = B^* B$ (!)

(For a *complex Hilbert space* H, $\langle Ax, x\rangle \geq 0$ for all $x \in H$ implies $A = A^*$ automatically.)

Συστήματα τελεστών (Operator Systems)

Σύστημα τελεστών (operator system) είναι ένας γραμμικός υπόχωρος $E \subseteq \mathscr{B}(H)$ (ή $E \subseteq \mathscr{B}$ όπου $\mathscr{B}$ μια C^* άλγεβρα με μονάδα) που είναι αυτοσυζυγής και περιέχει την μονάδα.

Παρατήρηση Κάθε $x \in E$ γράφεται μοναδικά $x = x_1 + ix_2$ όπου $x_k \in E^h$ (Ερμιτιανά ή αυτοσυζυγή). Κάθε $y \in E^h$ γράφεται (όχι μοναδικά) ως διαφορά δυο θετικών στοιχείων του E^h, π.χ. $y = (\|y\|\mathbf{1} + y) - \|y\|\mathbf{1}$. Άρα $E = E^h + iE^h$ και $E^h = E^+ - E^+$.

Η μονάδα είναι μονάδα διάταξης (order unit) δηλαδή για κάθε $y \in E^h$ υπάρχει $r > 0$ με $-r\mathbf{1} \le y \le r\mathbf{1}$. Η μονάδα είναι επιπλέον Αρχιμήδεια, δηλαδή αν $-\varepsilon\mathbf{1} \le y \le \varepsilon\mathbf{1}$ για κάθε $\varepsilon > 0$ τότε $y = 0$.

Συστήματα τελεστών (Operator Systems)

Αν $E \subseteq \mathscr{B}$ είναι σύστημα τελεστών, τότε για κάθε $n \in \mathbb{N}$ ο χώρος $E_n := M_n(E) \subseteq M_n(\mathscr{B}(H))$ είναι σύστημα τελεστών, αν εφοδιασθεί με την δομή που κληρονομεί από την C^* άλγεβρα $M_n(\mathscr{B}(H)) \simeq \mathscr{B}(H^n)$ (δηλ. ενέλιξη, μονάδα και θετικό κώνο $M_n(E)^+ := M_n(\mathscr{B}(H))^+ \cap M_n(E)$).

Αν E, F είναι συστήματα τελεστών, μια γραμμική απεικόνιση $\phi : E \to F$ λέγεται θετική αν $\phi(E^+) \subseteq F^+$. Λέγεται *n-θετική* αν η απεικόνιση $\phi_n : M_n(E) \to M_n(F) : [x_{ij}] \mapsto [\phi(x_{ij})]$ είναι θετική και πλήρως θετική αν είναι n-θετική για κάθε n.

Operator Systems

Thus, from each operator system there may be derived a sequence of order unit spaces $(E_n, E_n^+, \mathbf{1}_n)$. Here $\mathbf{1}_n = \mathrm{diag}(\mathbf{1}, \ldots, \mathbf{1})$ where $\mathbf{1}$ is the unit in E. Moreover, there is a natural family of connecting maps defined as follows. For $a \in M_{n,k}$ define

$$\mathrm{ad}(a) : E_n \to E_k : x \mapsto a^* x a.$$

Notice that $\mathrm{ad}(a)$ preserves the order structure.

Higher order cone determines norm

Exercise For $x \in \mathscr{B}(H)$ and $\lambda \geq 0$,

$$\|x\| \leq \lambda \iff \begin{bmatrix} \lambda \mathbf{1} & x \\ x^* & \lambda \mathbf{1} \end{bmatrix} \in \mathscr{B}(H^2)^+.$$

Proposition

Let $E \subset \mathscr{B}(H)$ be an operator system. For $x \in E_n$ we have that

$$\|x\|_n = \inf \left\{ \lambda > 0 : \begin{bmatrix} \lambda 1 & x \\ x^* & \lambda 1 \end{bmatrix} \in E_{2n}^+ \right\}.$$

Complete positivity and complete boundedness

Lemma

Κάθε μοναδιαία και 2-θετική απεικόνιση $\phi : E \to F$ είναι συστολή.

Αλλά δεν αρκεί η θετικότητα:

Example

Έστω $E \subseteq C(\mathbb{T})$ η γραμμική θήκη των $\{\mathbf{1}, \zeta, \bar{\zeta}\}$ όπου $\zeta(z) = z$. Δηλαδή κάθε $f \in E$ είναι της μορφής $f(e^{it}) = a + be^{it} + ce^{-it}$ με $a, b, c \in \mathbb{C}$. Ορίζουμε

$$\phi : E \to M_2(\mathbb{C}) : f \mapsto \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

Τότε η ϕ είναι θετική και μοναδιαία αλλά $\|\phi\| > 1$.

Corollary

Κάθε μοναδιαία και πλήρως θετική απεικόνιση $\phi : E \to F$ είναι πλήρης συστολή.

Complete positivity and complete boundedness

Proposition

Any unital, contractive linear map between operator systems is positive.

Compare Any complex measure μ on a space X with total variation $\|\mu\| = 1$ and $\mu(X) = 1$ is a positive (probability) measure.

Corollary

Every complete order embedding of operator systems $\phi : E \to F$ is a completely isometric embedding, that is, $\phi_n : E_n \to F_n$ and $(\phi^{-1})_n : \phi(E)_n \to E_n$ are isometries for all $n = 1, 2, \ldots$.
Every unital completely isometric embedding is a complete order embedding.

Choi-Kraus decomposition

Recall that if E is an operator system and $a \in M_{n,k}$ the map $\phi : E_n \to E_k : x \mapsto a^* x a$ is CP.

Proposition (Choi's Theorem)

Every completely positive map $\phi : M_n \to M_k$ is of the form $\phi(x) = \sum_{i=1}^{nk} a_i^* x a_i$ for some $a_1, \ldots, a_{nk} \in M_{n,k}$.

The minimal number of a_i's required is the Choi rank of ϕ.

Abstract Operator Systems

Definition

A matrix ordered vector space is a *-vector space E together with a family of proper cones $C_n \subseteq M_n(E)^h$ (i.e. $C_n \cap (-C_n) = \{0\}$) for each $n \in \mathbb{N}$, which are compatible in the sense that

$$a^* C_n a \subseteq C_k \text{ for every } \ a \in M_{n,k}(\mathbb{C}).$$

A matrix ordered vector space E is called an abstract operator system if E^h has an order unit e such that, for all $n \in \mathbb{N}$, $e_n := \mathrm{diag}(e, \ldots, e)$ is an *Archimedean order unit* for $M_n(E)^h$.

Η e είναι μονάδα διάταξης (order unit) δηλαδή για κάθε $y \in E^h$ υπάρχει $r > 0$ με $-re \leq y \leq re$. Η e είναι επιπλέον Αρχιμήδεια, αν ικανοποιεί την: αν $y \in E^h$ και $-\varepsilon e \leq y \leq \varepsilon e$ για κάθε $\varepsilon > 0$ τότε $y = 0$.

The Choi - Effros Theorem (1975)

Theorem

Every abstract operator system 'is' a concrete operator system: For every abstract operator system E there exists a Hilbert space H and a complete order embedding

$$J : E \to \mathscr{B}(H).$$

Idea: Let $D = \bigcup_n UCP(E, M_n)$ (unital CP maps). For each $\phi \in D$ there is $n(\phi) \in \mathbb{N}$ with $\phi \in UCP(E, M_{n(\phi)})$. Define

$$J : E \to \bigoplus_{\phi \in D} M_{n(\phi)} : x \mapsto \bigoplus_{\phi \in D} \phi(x).$$

This is a complete order embedding of E into a direct sum of matrix algebras, which consists of operators on the Hilbert space direct sum $\bigoplus_{\phi \in D} \ell^2[n(\phi)]$.

The Arveson Extension Theorem (1969)

Theorem (Arveson)

Αν $E \subseteq F$ είναι συστήματα τελεστών και $\Phi : E \to \mathscr{B}(H)$ μια πλήρως θετική γραμμική απεικόνιση, τότε η Φ έχει επέκταση σε μια γραμμική απεικόνιση $\tilde{\Phi} : F \to \mathscr{B}(H)$ που είναι πλήρως θετική.

$$\begin{array}{ccc} F & & \\ \uparrow & \overset{\exists \tilde{\Phi}}{\searrow} & \\ E & \xrightarrow{\ \Phi\ } & \mathscr{B}(H) \end{array}$$

The Dual of an operator system

Given an operator system E, consider the dual operator *space* $F = E^*$. We let $F^+ := \{\phi \in F : \phi(x) \geq 0 \, \forall x \in E^+\}$.
For $n \in \mathbb{N}$, each $\phi = [\phi_{ij}] \in M_n(F)$ defines a $\tilde{\phi} : E \to M_n : x \mapsto [\phi_{ij}(x)]$.
We put

$$M_n(E^*)^+ = F_n^+ := \{\phi = [\phi_{ij}] \in M_n(F) : \tilde{\phi} \in CP(E, M_n)\}.$$

This defines a matrix ordered vector space structure $(M_n(E^*), M_n(E^*)^+)$ on E^*.
However E^* is *not* in general an operator system, because it may fail to have an order unit.
In the special case $\dim E < \infty$ it is possible to choose an Archimedean order unit and thus to give an operator space structure to E^*.