Definable refinements of classical algebraic invariants

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Table of Contents

Table of Contents

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Invariants in Algebraic Topology

One attaches to topological spaces algebraic invariants such as groups

(All the groups will be abelian.)

From complexes to groups

The final invariant (group) is obtained by passing via complexes.

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- contains automorphism groups of “reasonable” structures
- is closed under countable products and inverse limits
- is closed under closed subgroups and quotients by closed subgroups
- the σ-algebra of Borel sets of a Polish group is standard (isomorphic to the σ-algebra of Borel sets of $\mathbb{R}$)

The homology of a Polish complex

Consider a complex of Polish groups A_*:

$$\cdots \longrightarrow A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \longrightarrow \cdots$$

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In A History of Algebraic and Differential Topology, Dieudonné writes of

> *a trend that was very popular until around 1950 (although later all but abandoned), namely, to consider homology groups as topological groups for suitably chosen topologies.*

The problem with cokernels

Problem: the category $\mathcal{A}$ of abelian Polish groups is not abelian

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In 1976 Calvin C. Moore writes about

> *one final difficulty in considering the cohomology of topological groups which to some extent is incurable, and this is the fact that a continuous group homomorphism need not have closed range.*

Table of Contents

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- $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathrm{LH}(\mathcal{A}))$

More generally the same applies to any quasi-abelian category

An explicit description of the heart of abelian Polish groups

Theorem (L., 2022)

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Objects: *"Formal quotients"* G/N *of abelian Polish groups by Polish subgroups (the topology of* N *need not be induced by* G*)*

Morphisms: *Group homomorphisms* $G/N \to H/M$ *that are Borel-definable, i.e. induced by a Borel function* $G \to H$

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Techniques: advanced tools and recent results from logic

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Techniques: advanced tools and recent results from logic

$\mathrm{LH}(\mathcal{A})$ is the natural framework to develop definable refinements of classical homological algebra and algebraic topology

An explicit description of the heart of other categories

Similar descriptions for the heart of other topological-algebraic structures:

- locally compact abelian Polish groups
- totally disconnected locally compact abelian Polish groups
- non-Archimedean abelian Polish groups

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Similar descriptions for the heart of other topological-algebraic structures:

- locally compact abelian Polish groups
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- R-modules
- real/complex Banach spaces $\longrightarrow$ vector spaces with a Banach cover
- Banach spaces over a non-Archimedean valued field
- Fréchet spaces

Table of Contents

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Virtually all group invariants from algebraic topology can be refined and seen as invariants taking values in the category of groups with Polish cover

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Advantages of the definable versions:

1. finer invariants (distinguish more spaces, more powerful invariants)
2. richer invariants (e.g., one can study their Borel class and Borel rank)
3. rigid invariants (fewer automorphisms, better grasp on the dynamics)

Table of Contents

Finer invariants

Theorem (Bergfalk, L., Panagiotopoulos, 2018–2020)

The following invariants admit definable refinements:

- *Steenrod homology of compact spaces*
- K*-homology of compact spaces and of C*-algebras*
- *Čech cohomology of locally compact spaces*

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Furthermore:

1. *definable Steenrod homology* $H_*(-)$ *is a complete invariant for solenoids (inverse limits of tori)*
2. *definable* K*-homology is a complete invariant for solenoids*
3. *definable Čech cohomology* $H^*(-)$ *is a complete invariant for mapping telescopes of tori or spheres*

Definable homological algebra

The homological invariants

$$\mathrm{Hom}(A, B)$$

$$\mathrm{Ext}(A, B)$$

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The definable homological invariant $\mathrm{Ext}(-, \mathbb{Z})$ is a complete invariant for finite-rank torsion-free abelian groups with no nonzero free summands.

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Definable $\mathrm{Ext}\,(-, \mathbb{Z})$ is a fully faithful functor from finite-rank torsion-free abelian groups with no nonzero free summands to groups w/ a Polish cover

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This does not hold for the purely algebraic Ext.

Spaces with a Banach cover

Theorem

Fix $q < p$ and $q' < p'$

The spaces

$$\ell_p/\ell_q \quad \textit{and} \quad \ell_{p'}/\ell_{q'}$$

are not isomorphic as spaces with a Banach cover when $q \neq q'$.

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However, they are always isomorphic as (seminormed) vector spaces.

Table of Contents

Subobjects

Let $G = \hat{G}/N$ be a group with a Polish cover.

A subgroup with a Polish cover H of G is of the form

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For example, $\{0\}$ inside $\mathbb{R}/\mathbb{Q}$ is $\boldsymbol{\Sigma}_2$ and has rank 2

However, $\{0\}$ inside $\mathbb{R}^{\mathbb{N}}/\mathbb{Q}^{\mathbb{N}}$ is $\boldsymbol{\Pi}_3$ and has rank 3

Solecki subgroups

Theorem (L., 2021, building on Solecki 1999 and Farah–Solecki 2006)

Let G be a group with a Polish cover, and let α be a countable ordinal.

There exists a smallest $\boldsymbol{\Pi}_{1+\alpha+1}$ subgroup with a Polish cover $s_\alpha(G)$ of G.

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Remark

We have that $s_0(G)$ is the closure of $\{0\}$.

Solecki subgroups for Ext of torsion groups

Theorem (L., 2021)

For every countable ordinal α, and torsion groups A and B,

$$s_\alpha\left(\mathrm{Ext}(A, B)\right)$$

is equal to the $(1+\alpha)$-th Ulm subgroup

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For torsion groups A, B, $\{0\}$ can have arbitrarily high rank in $\mathrm{Ext}(A, B)$
The problem of classifying extensions can have arbitrarily high complexity.

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Corollary

For torsion-free A, extensions of A by $\mathbb{Z}$ are classifiable using as invariants countably many binary sequences up to tail equivalence.

Applications to the classification problem for extensions

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We can write $X \approx \lim_n X_n$ where each X_n is a finite polyhedron

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This is a complexity-theoretic consequence of the UCT for K-homology

Table of Contents

Rigidity

Groups with a Polish cover are more rigid than discrete groups: they have fewer automorphisms

The reason is that not all group automorphisms are Borel-definable

p-adic numbers

Let $\mathbb{Q}_p$ be the p-adic numbers (seen as additive locally profinite group)

We have a canonical action $\mathbb{Z}[1/p]^\times \curvearrowright \mathbb{Q}_p$ by multiplication

This induces an action $\mathbb{Z}[1/p]^\times \curvearrowright \mathbb{Q}_p/\mathbb{Z}[1/p]$

Ulam stability of p-adics

Theorem (Bergfalk, L., Panagiotopoulos, 2019)

All Borel-definable automorphisms of $\mathbb{Q}_p/\mathbb{Z}[1/p]$ *are given by the action*

$$\mathbb{Z}[1/p]^\times \curvearrowright \mathbb{Q}_p/\mathbb{Z}[1/p]$$

Thus there exist $\aleph_0$ *Borel-definable automorphisms of* $\mathbb{Q}_p/\mathbb{Z}[1/p]$

In contrast, there exist $2^{2^{\aleph_0}}$ *automorphisms of* $\mathbb{Q}_p/\mathbb{Z}[1/p]$

Solenoids

A solenoid is simply an inverse limit of copies of $\mathbb{T}$

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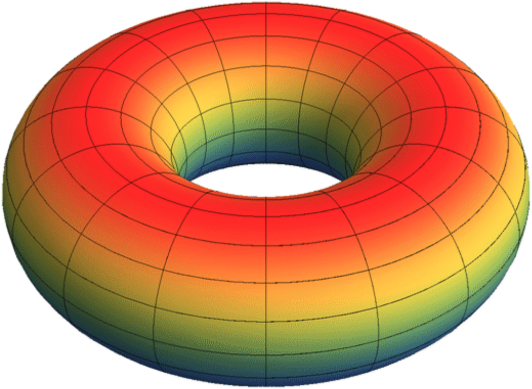
A concrete geometric realization in $\mathbb{R}^3$ of a solenoid can be obtained as intersection of a sequence of nested solid tori



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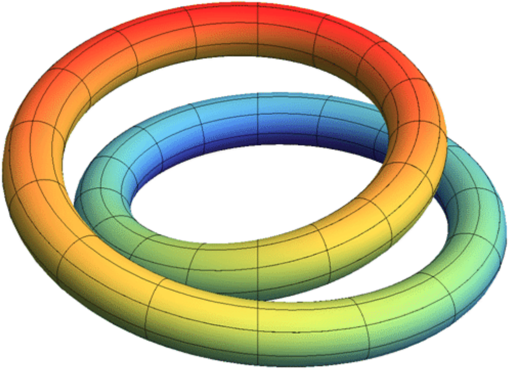

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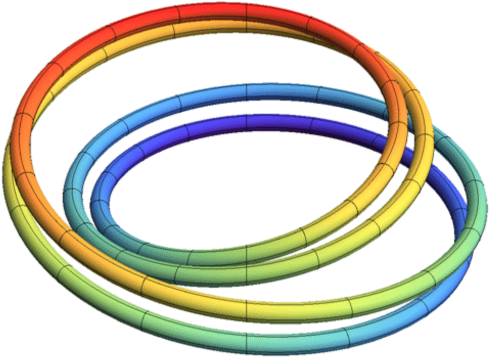

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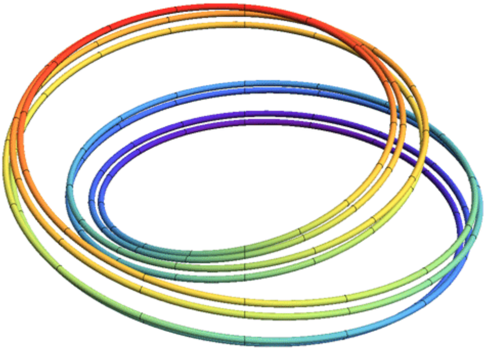

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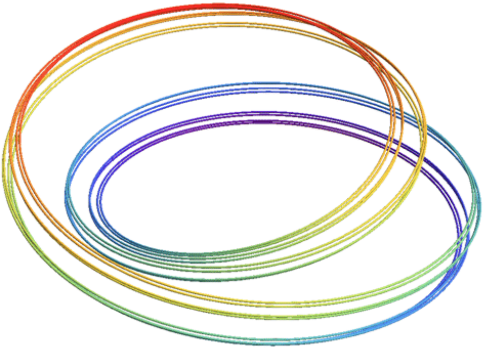

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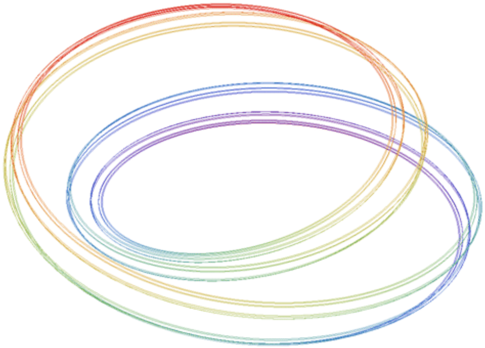

Solenoid complements

We denote by S^d the one-point compactification of $\mathbb{R}^d$

Let $X_p \subseteq S^3$ be a geometric realization of the p-adic solenoid

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Let $[S^3 \setminus X_p, S^2]$ be the space of homotopy classes of maps $S^3 \setminus X_p \to S^2$

Theorem (Bergfalk, L., Panagiotopoulos, 2020)

There is a Borel-definable bijection

$$[S^3 \setminus X_p, S^2] \cong \mathbb{Q}_p/\mathbb{Z}[1/p]$$

Equivariant classification

Let $\mathcal{E}(S^3 \setminus X_p)$ be the space of homotopy automorphisms of $S^3 \setminus X_p$

There is a canonical Borel-definable action

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Using the rigidity of $\mathbb{Q}_p/\mathbb{Z}[1/p]$ we can conclude:

Theorem (Bergfalk, L., Panagiotopoulos, 2020)

The action

$$[S^3 \setminus X_p, S^2] \curvearrowleft \mathcal{E}(S^3 \setminus X_p)$$

corresponds to the canonical action

$$\mathbb{Z}[1/p]^\times \curvearrowright \mathbb{Q}_p/\mathbb{Z}[1/p]$$

Equivariant classification

So the problem of classifying the orbits of

$$[S^3 \setminus X_p, S^2] \curvearrowleft \mathcal{E}(S^3 \setminus X_p)$$

is the same as the problem of classifying the orbits of

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which in turn is the same as the problem of classifying the orbits of

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In particular, there exist $2^{\aleph_0}$ such orbits

Higher dimensions

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In this case we have that the Borel-definable action

$$[S^{d+2} \setminus X_p^d, S^{d+1}] \curvearrowleft \mathcal{E}(S^{d+2} \setminus X_p^d)$$

corresponds to the action

$$\mathrm{GL}_d(\mathbb{Z}[1/p]) \curvearrowright \mathbb{Q}_p^d / \mathbb{Z}[1/p]^d$$

Measuring the complexity

Using tools from

- ergodic theory (superrigidity for profinite actions), and
- algebraic geometry (superrigidity for p-adic Lie groups)

one can compare the Borel complexity of such actions.

Theorem (Bergfalk, L., Panagiotopoulos, 2019)

The Borel complexity of classifying the orbits of

$$[S^{d+2} \setminus X_p^d, S^{d+1}] \curvearrowleft \mathcal{E}(S^{d+2} \setminus X_p^d)$$

or equivalently

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strictly increases with d*.*

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For $d \geq 3$, these problems for different primes are incomparable from the perspective of Borel complexity.

Further directions

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Hierarchies of phantom maps corresponding to Solecki subgroups

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Project

Construct examples of C-algebras and coarse spaces where the UCT and the coarse BC conjecture fail for complexity-theoretic obstructions*