C^*-rigidity for certain exponential Lie groups

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The problem (short version)

Exponential Lie groups

■ A Lie group G is *exponential* $\iff$ $\exp_G\colon \mathfrak{g} = \mathrm{Lie}(G) \to G$ is a diffeomorphism

• A connected and simply connected Lie group G is exponential $\iff$ for every $X \in \mathfrak{g}$ the eigenvalues of $\mathrm{ad}_X = [X, \cdot]\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ are not of the form $\mathrm{i}\alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$.

▲ G exponential $\Rightarrow$

- G is amenable
- $C^*(G)$ is type I
- $\widehat{G} \simeq \mathfrak{g}^*/G$ (Kirillov-Bernat-Leptin-Ludwig)

Problem

If G is an exponential Lie group, does $C^*(G)$ uniquely determine G?

▲ *Short answer*

Not always.

Group C^*-algebras

- $\mathcal{A}$ C^*-algebra
  - $\mathrm{Id}(\mathcal{A}) := \{\mathcal{J}$ closed linear subspace $\subseteq \mathcal{A} \mid \mathcal{A}\mathcal{J} + \mathcal{J}\mathcal{A} \subseteq \mathcal{A}\}$
  - $\widehat{\mathcal{A}}$: set of equiv. classes $[\pi]$ of irreducible $*$-repres. $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$
  - *Topology of* $\widehat{\mathcal{A}}$: open sets are $\widehat{\mathcal{J}} := \{[\pi] \in \widehat{\mathcal{A}} \mid \pi|_{\mathcal{J}} \not\equiv 0\}$

for $\mathcal{J} \in \mathrm{Id}(\mathcal{A})$

▲ Separable C^*-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are *stably isomorphic*
(or *Morita equivalent*) if $\mathcal{A}_1 \widehat{\otimes} \mathcal{K} \simeq \mathcal{A}_2 \widehat{\otimes} \mathcal{K}$

• $\mathcal{A}_1 \widehat{\otimes} \mathcal{K} \simeq \mathcal{A}_2 \widehat{\otimes} \mathcal{K} \implies \widehat{\mathcal{A}_1} \simeq \widehat{\mathcal{A}_2}$. (noncanonical homeom.)

G locally compact group

• $\pi\colon G \to \mathrm{U}(\mathcal{H}_\pi)$ unitary representation
$\rightsquigarrow \pi\colon L^1(G) \to \mathcal{B}(\mathcal{H}_\pi)$, $\pi(f) = \int\limits_G f(x)\pi(x)\mathrm{d}x$

$\rightsquigarrow \pi\colon C^*(G) \to \mathcal{B}(\mathcal{H}_\pi)$

- $\widehat{G} \simeq \widehat{C^*(G)}$

First examples

• G locally compact *abelian*, the Fourier transform extends to an isom. of C^*-algebras

$$C^*(G) \simeq \mathcal{C}_0(\widehat{G}).$$

$\rightsquigarrow$ $C^*(G)$ is uniquely determined by the *topological space* $\widehat{G}$

• $\mathbb{T}^n := (\mathbb{T}^n, \cdot)$ the n-dimensional torus

Then $\mathbb{Z}^n \xrightarrow{\sim} \widehat{\mathbb{T}^n}$, via $\alpha \mapsto \chi_\alpha$, $\chi_\alpha(z) := z^\alpha$.

$\rightsquigarrow$ $\mathbb{T}^n \not\simeq \mathbb{T}^m$ as Lie groups if $n \neq m$.

However, $\widehat{\mathbb{T}^n} \simeq \widehat{\mathbb{T}^m}$ as topological spaces, hence $C^*(\mathbb{T}^n) \simeq C^*(\mathbb{T}^m)$.

Conclusion: Stick to *simply connected* Lie groups.

Eigenspaces

- $\mathcal{V}$ real finite-dimensional vector space of dim. n.
- $D \in \mathrm{End}(\mathcal{V})$.
  - $D\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$ $\mathbb{C}$-linear extension, $\sigma(D)$ its spectrum.
  - $\mu \in \sigma(D) \setminus \mathbb{R}$, $E^D(\mu) = \{v_1, v_2 \mid v_1 + \mathrm{i}v_2 \in \mathrm{Ker}\,(D - \mu I)^n\}$.
  - $\mu \in \sigma(D) \cap \mathbb{R}$, $E^D(\mu) = \mathrm{Ker}\,(D - \mu I)^n$.
  - $\mathcal{V}^D_\pm := \sum_{\pm \mathrm{Re}\,\nu > 0} E^D(\nu)$, $n^D_\pm = \dim \mathcal{V}^D_\pm$
  - $\mathcal{V}^D_0 := \sum_{\mathrm{Re}\,\nu = 0} E^D(\nu)$, $n^D_0 = \dim \mathcal{V}^D_0$
  - Then $\mathcal{V} = \mathcal{V}^D_+ \oplus \mathcal{V}^D_- \oplus \mathcal{V}^D_0$.

Non-rigid exponential Lie groups

- $\mathcal{V}$ real finite-dimensional vector space of dim. n.
- $D \in \mathrm{End}(\mathcal{V})$
- $n_{\pm}^D$, n_0^D as above

▲ $G_D := \mathcal{V} \rtimes \mathbb{R}$ with $(b_1, t_1) \cdot (b_2, t_2) = (b_1 + \mathrm{e}^{t_1 D} b_2, t_1 + t_2)$.

Theorem

Let $D_j \in \mathrm{End}\,(\mathcal{V})$, $j = 1, 2$. Assume $\mathcal{V}_0^{D_j} = \mathrm{Ker}\, D_j$ for $j = 1, 2$. Then

- $G_{D_1} \simeq G_{D_2}$ as Lie groups iff $\sigma(D_1) = r \cdot \sigma(D_2)$ for some $r \in \mathbb{R} \setminus \{0\}$
- $C^*(G_{D_1}) \simeq C^*(G_{D_2})$ iff $\{n_+^{D_1}, n_-^{D_1}\} = \{n_+^{D_2}, n_-^{D_2}\}$ and $n_0^{D_1} = n_0^{D_2}$.

Y.-F. Lin, J. Ludwig (2013): special case for diagonalizable matrices D_j

Non-rigid exponential Lie groups-2

For the proof:

• $C^*(G_D) = \mathcal{C}_0(\mathcal{V}^*) \rtimes_{\alpha_{D^*}} \mathbb{R}$ where
$\alpha_{D^*} \colon \mathbb{R} \times \mathcal{V}^* \to \mathcal{V}^*$, $\alpha_{D^*}(t, \xi) = \mathrm{e}^{tD^*}\xi$ for $\xi \in \mathcal{V}^*$

▲ $D_1, D_2 \in \mathrm{End}\,(\mathcal{V})$ as above, *without necessarily* $\mathcal{V}_0^{D_j} = \mathrm{Ker}\, D_j$. The following assert. are equiv.:

(i) There is a homeomorphism $\Phi \colon \mathcal{V} \to \mathcal{V}$ with $\Phi \circ \mathrm{e}^{tD_1} = \mathrm{e}^{tD_2} \circ \Phi$, for all $t \in \mathbb{R}$.

(ii) $\{n_+^{D_1}, n_-^{D_1}\} = \{n_+^{D_2}, n_-^{D_2}\}$, and there is an isomorphism $T \colon \mathcal{V}_0^{D_1} \to \mathcal{V}_0^{D_2}$ such that $TD_1|_{\mathcal{V}_0^{D_1}} = D_2 T$.

(Ladis (1973)).

Non-rigid exponential Lie groups-3

▲ $D \in \mathrm{End}\,(\mathcal{V})$ with $\mathcal{V}_0^D = \mathrm{Ker}\, D$.
$\mathcal{V}^* \simeq \mathbb{R}^{n_+^D} \times \mathbb{R}^{n_1^D} \times \mathbb{R}^{n_0^D}$

▲ $\widehat{G_D} = \mathcal{Q}_1^D \cup \mathcal{Q}_2^D \cup \mathcal{Q}_3^D$, where

- $\mathcal{Q}_1^D \simeq \{(0, \xi_+, \xi_-, 0) \in \mathbb{R}^{n_+^D} \times \mathbb{R}^{n_-^D} \times \mathbb{R}^{n_0^D} \times \mathbb{R} \mid |\xi_+| = |\xi_-| \neq 0\}$,
- $\mathcal{Q}_2^D \simeq (\{0\} \times S_1^{n_+^D} \times \{0\} \times \{0\}) \cup (\{0\} \times \{0\} \times S_1^{n_-^D} \times \{0\})$,
- $\mathcal{Q}_3^D \simeq \{0\} \times \mathbb{R}^{n_0^D} \times \mathbb{R}$.

Moreover,

- $\mathcal{Q}_1^D$ is open, dense and Hausdorff, the corresponding repres. are CCR ($= \emptyset$ if $n_+^D = 0$ or $n_- = 0$.)
- $\mathcal{Q}_3^D$ are the characters of G_D.

$\downarrow$

$C^*(G_{D_1}) \simeq C^*(G_{D_2}) \Rightarrow \mathcal{Q}_j^{D_1} \simeq \mathcal{Q}_j^{D_2}$, $j = 1, 2, 3$,
$\Rightarrow \{n_+^{D_1}, n_-^{D_1}\} = \{n_+^{D_2}, n_-^{D_2}\}$ and $n_0^{D_1} = n_0^{D_2}$.

Non-rigid exponential Lie groups-4

Corollary

$D_j \in \mathrm{End}\,(\mathcal{V})$ with $\mathcal{V}_0^{D_j} = \mathrm{Ker}\,D_j,\ j = 1,2.$ Then
$C^*(G_{D_1}) \simeq C^*(G_{D_2})$
$\iff \{n_+^{D_1}, n_-^{D_1}\} = \{n_+^{D_2}, n_-^{D_2}\}$ and $n_=^{D_1} = n_0^{D_2}$
$\iff$ there is an equivariant $*$-isomorphisms between
$(\mathcal{C}_0(\mathcal{V}^*), \mathbb{R}, \alpha_{D_1^*})$ and $(\mathcal{C}_0(\mathcal{V}^*), \mathbb{R}, \alpha_{D_2^*})$.

Nilpotent Lie groups

- $G = (\mathbb{R}^n, \cdot)$ with $\begin{cases} \cdot\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \text{ \textit{polynomial} mapping} \\ (\forall t, s \in \mathbb{R})(\forall x \in \mathbb{R}^n) \quad (tx)\cdot(sx) = (t+s)x \end{cases}$

- *Lie algebra* $\mathfrak{g} = (\mathbb{R}^n, [\cdot,\cdot])$, $\quad [x,y] := \frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} (tx)\cdot(sy)\cdot(-tx)$

  - G and $\mathfrak{g}$ are called *nilpotent* since $[x, \dots, [x,y]] = 0$ for all $x, y \in \mathfrak{g}$.
  - $G \simeq (\mathfrak{g}, \cdot)$ with $x \cdot y = x + y + \frac{1}{2}[x,y] + \frac{1}{6}[x,[x,y]] + \cdots$

- *Adjoint action* of G on $\mathfrak{g}$: $\mathrm{Ad}_G\colon G \times \mathfrak{g} \to \mathfrak{g}$

$$(\forall x \in G,\ y \in \mathfrak{g}) \quad (\mathrm{Ad}_G x)y := \frac{\partial}{\partial s}\Big|_{s=0} x\cdot(sy)\cdot(-x)$$

- *Coadjoint action* of G on $\mathfrak{g}^*$: $\mathrm{Ad}_G^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$

$$(\forall x \in G, \xi \in \mathfrak{g}^*) \quad (\mathrm{Ad}_G x)\xi := \xi \circ (\mathrm{Ad}_G x)^{-1}$$

Representations of nilpotent Lie groups

▲ $G = (\mathfrak{g}, \cdot)$ nilpotent Lie group with Lie algebra $\mathfrak{g}$

Kirillov correspondence:

$$\widehat{G} \longleftrightarrow \mathfrak{g}^*/G,\ [\pi] \mapsto \mathcal{O}_\pi$$

- continuous bijection (A. Kirillov)
- homeomorphism (I. Brown)

A sketch:

$\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$ irreducible representation

$\rightsquigarrow \pi(f) := \int\limits_{\mathfrak{g}} f(x)\pi(x)\mathrm{d}x \in \mathcal{B}(\mathcal{H}_\pi)$ *trace-class* operator for all $f \in \mathcal{C}_c^\infty(\mathfrak{g})$

$\rightsquigarrow$ the character $\chi_\pi := \mathrm{Tr} \circ \pi\colon \mathcal{C}_c^\infty(\mathfrak{g}) \to \mathbb{C}$ is a *tempered* distribution

$\rightsquigarrow$ the Fourier transform $\widehat{\chi_\pi}\colon \mathcal{C}_c^\infty(\mathfrak{g}^*) \to \mathbb{C}$ is a *positive measure*

$\rightsquigarrow \mathcal{O}_\pi := \mathrm{supp}\,\widehat{\chi_\pi} \subseteq \mathfrak{g}^*$ is an *orbit* of the coadjoint action of G in $\mathfrak{g}^*$

- $C^*(G)$ is CCR (liminary).

C^*-rigidity of nilpotent Lie groups

- G is *C^*-rigid* if

$$G_1 \text{ exponential Lie group, } C^*(G_1) \simeq C^*(G) \implies G_1 \simeq G$$

- G is *stably C^*-rigid* if:

$$G_1 \text{ exponential Lie group, } C^*(G_1)\widehat{\otimes}\mathcal{K} \simeq C^*(G)\widehat{\otimes}\mathcal{K} \implies G_1 \simeq G$$

• If G is an exponential Lie group such that $C^*(G)$ is CCR $\Rightarrow$ G is nilpotent (Auslander-Moore)

Hence it is enough to compare nilpotent Lie groups.

Special $\mathbb{R}$-spaces

• *Special $\mathbb{R}$-space* is a top. space X with a cont. map $\mathbb{R} \times X \to X$, $(t, x) \mapsto t \cdot x$ and a distinguished point $x_0 \in X$ such that

- $x \in X$, $t \in \mathbb{R} \Rightarrow 0 \cdot x = t \cdot x_0 = x_0$ and $1 \cdot x = x$.
- $t, s \in \mathbb{R}$ and $x \in X \Rightarrow t \cdot (s \cdot x) = ts \cdot x$.
- For $x \in X \setminus \{x_0\}$ the map $\psi_x \colon \mathbb{R} \to X$, $t \mapsto t \cdot x$ is a homeomorphism onto its image.

$\mathbb{R}$ *spaces of finite length*

• X *$\mathbb{R}$-space of finite length* if it is 2nd count., loc. quasi-compact, has the property T_1, and there is finite family of open subsets,

$$\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = X,$$

such that $V_j \setminus V_{j-1}$ is Hausdorff in its relative topology and dense in $X \setminus V_{j-1}$, for $j = 1, \ldots, n$, and

- X special $\mathbb{R}$-space, $\Gamma_j := V_j \setminus V_{j-1} \subseteq X$ is an $\mathbb{R}$-subspace for $j = 1, \ldots, n$.
- $\Gamma_n := X \setminus V_{n-1}$ is isom. (as a special $\mathbb{R}$-space) to some $\mathbb{R}^m$ with whose origin corresp. to dist. point x_0 of X.
- For $j = 1, \ldots, n-1$ the points of Γ_{j+1} are closed and separated in $X \setminus V_j$.
- For $j = 1, \ldots, n$, Γ_j is isom. (as a special $\mathbb{R}$-space) to a cone C_j in a finite-dim. vector space. In addition, C_1 is an semi-algebraic Zariski open set, and the dimension of the correp. ambient vector space is called the *index of X* ($:= \operatorname{ind} X$.)

Useful properties of $C^*(G)$

Theorem (D. Beltita, I.B, J. Ludwig)

G is a nilpotent Lie group $\Rightarrow \widehat{G}$ is a $\mathbb{R}$-space of finite length.

$\rightsquigarrow \operatorname{ind}\widehat{G}$ is an invariant of $C^*(G)$.

Proposition

• $[\mathfrak{g},\mathfrak{g}]^\perp \simeq \operatorname{Hom}(G,\mathbb{T}) \subseteq \widehat{G}$ is a maximal element of the set
$Q(\widehat{G}) := \{S \subseteq \widehat{G} \mid S$ is closed, connected, its relative topology is Hausdorff$\}$.

• $\operatorname{RR}(C^*(G)) = \dim[\mathfrak{g},\mathfrak{g}]^\perp$ (but this is not always preserved under Morita equivalence).

The Heisenberg group

▲ *Heisenberg algebra/group*:
$\mathfrak{h}_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $[(q,p,t),(q',p',t')] = [(0,0,\langle p,q'\rangle - \langle p',q\rangle)]$
$\mathbb{H}_{2n+1} = (\mathfrak{h}_{2n+1},\cdot)$, $x\cdot y := x + y + \frac{1}{2}[x,y]$

• Schrödinger representation $\pi_\lambda\colon \mathbb{H}_{2n+1} \to \mathcal{B}(L^2(\mathbb{R}^n))$,

$$\pi_\lambda(q,p,t)f(x) = \mathrm{e}^{\mathrm{i}\lambda(\langle p,x\rangle + \frac{1}{2}\langle p,q\rangle + t)} f(q+x)$$

for $\lambda \in \mathbb{R}^* \hookrightarrow \widehat{\mathbb{H}}_{2n+1}$

• Generic coadjoint orbits in $\mathfrak{h}^*_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, for $\lambda \in \mathbb{R}^*$:

$$\mathcal{O}_\lambda = \mathbb{R}^n \times \mathbb{R}^n \times \{\lambda\} \qquad .$$

• $0 \to \mathcal{C}_0(\mathbb{R}^*, \mathcal{K}) \to C^*(\mathbb{H}_{2n+1}) \to \mathcal{C}_0(\mathbb{R}^{2n}) \to 0.$

Picture of $\widehat{\mathbb{H}}_{2n+1}$:

• $\widehat{\mathbb{H}}_{2n+1} = (\mathbb{R}\setminus\{0\}) \sqcup \mathbb{R}^{2n}$

• If $\lim\limits_{n\to\infty} t_n = 0$ in $\mathbb{R}\setminus\{0\}$, then $\mathrm{Lim}_{n\to\infty} t_n = \mathbb{R}^{2n}$ in $\widehat{\mathbb{H}}_{2n+1}$

• $\mathrm{ind}\, \widehat{\mathbb{H}}_{2n+1} = 1$

Rigidity of Heisenberg groups

Theorem

Let $\mathbb{H}_{2n+1}$ be the Heisenberg group and G another nilpotent Lie group. Then

$$\mathbb{H}_{2n+1} \simeq G \iff \widehat{G} \simeq \widehat{\mathbb{H}_{2n+1}}$$

Corollary

Heisenberg groups are stably C^*-rigid.

A direct proof of the corollary:

Lemma

G_1 nilpotent Lie groups with centre Z_1 and $\operatorname{ind} G_1 = 1$. Then

- $\dim Z_1 = 1$.

Let G_2 another nilpotent Lie group, with centre Z_2 and assume that $C^*(G_1)$ is Morita equivalent to $C^*(G_2)$, Then

- $\operatorname{ind} G_2 = \dim Z_2 = \dim Z_1 = 1$
- $C^*(G_1/Z_1)$ is Morita equivalent to $C^*(G_2/Z_2)$.

Filiform groups

▲ *Filiform (threadlike) Lie algebra*: $\mathfrak{f}_n = \mathrm{span}\{X_1, \ldots, X_n\}$

$$[X_n, X_j] = X_{j-1}, \quad j = 2, \ldots, n-1.$$

$F_n = (\mathfrak{f}_n, \cdot)$.

Theorem

Filiform groups are stably C^*-rigid.

• $F_n = \mathbb{R}^{n-1} \rtimes \mathbb{R}$, where the action $(t, x) \mapsto \mathrm{e}^{tD} x$ is given by the nilpotent matrix

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Remark: The topology of unitary duals of filiform groups was studied by R.J. Archbold, E. Kaniuth, J. Ludwig, G. Schlichting, D. W. B. Somerset (1990-2007)

Low dimensional groups

Theorem

All nilpotent Lie groups with $\dim \leq 5$ are stably C^*-rigid.

$X_1, \ldots, X_n$ be a basis of a Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} = n \leq 5$.

- *Case* $n = 3$:
  - $\mathfrak{g}_3$: $[X_3, X_2] = X_1$
- *Case* $n = 4$:
  - $\mathfrak{g}_4$: $[X_4, X_3] = X_2$, $[X_4, X_2] = X_1$
- *Case* $n = 5$:
  - $\mathfrak{g}_{5,1}$: $[X_5, X_4] = X_1$, $[X_3, X_2] = X_1$
  - $\mathfrak{g}_{5,2}$: $[X_5, X_4] = X_2$, $[X_5, X_3] = X_1$
  - $\mathfrak{g}_{5,3}$: $[X_5, X_4] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$
  - $\mathfrak{g}_{5,4}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_4, X_3] = X_1$
  - $\mathfrak{g}_{5,5}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_5, X_2] = X_1$
  - $\mathfrak{g}_{5,6}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$

A 6-dimensional group

Theorem

The free 6-dimensional nilpotent Lie group

- $G_{6,15} = (\mathfrak{g}_{6,15}, \cdot)$ for $\mathfrak{g}_{6,15} = \mathrm{span}\{X_1, X_2, X_3, X_4, X_5, X_6\}$ with

$$[X_6, X_5] = X_3,\ [X_6, X_4] = X_1,\ [X_5, X_4] = X_2$$

is stably C^*-rigid.