A comparison property for Cartan subalgebras

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Classification theory

Theorem (Elliott, 1976)

Two unital AF algebras A, B are isomorphic if and only if $K_0 A \cong K_0 B$ as ordered abelian groups.

Motivation: Can we classify all unital, simple, separable nuclear C*-algebras by their K-theory and traces (i.e. the Elliott invariant $\mathrm{Ell}(\cdot)$)?

Theorem (Jiang – Su, 1999)

There exists a unique infinite dimensional, unital, separable, simple, nuclear C*-algebra $\mathcal{Z}$ s.t. $\mathrm{Ell}(\mathcal{Z}) \cong \mathrm{Ell}(\mathbb{C})$. Moreover, $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$.

Regularity conditions needed to be imposed in the "to-be-classified" class: $\mathrm{Ell}(\cdot)$ cannot tell A and $A \otimes \mathcal{Z}$ apart.

Theorem (...)

The class of unital, separable, simple, nuclear, $\mathcal{Z}$-stable C*-algebras that satisfy the UCT are classified by K-theory and tracial data.

Regularity properties of C*-algebras

Conjecture (Toms – Winter, 2008)

Let A be a unital, simple, separable, nuclear, infinite dimensional C*-algebra. TFAE:

1. $\mathcal{Z}$-stability: $A \cong A \otimes \mathcal{Z}$
2. Finite nuclear dimension: $\dim_{\mathrm{nuc}}(A) < \infty$
3. strict comparison

Status of the TW conjecture:

- $A \cong A \otimes \mathcal{Z} \implies$ strict comparison (Rørdam, 2004)
- $\dim_{\mathrm{nuc}}(A) < \infty \implies A \cong A \otimes \mathcal{Z}$ (Winter, 2012)
- $A \cong A \otimes \mathcal{Z} \implies \dim_{\mathrm{nuc}}(A) < \infty$ (CETWW, 2019)
- strict comparison $+$ uniform property $\Gamma \implies A \cong A \otimes \mathcal{Z}$ (CETWW, 2019)

Strict comparison

Definition

For $a, b \in A_+$, we say a is *Cuntz subequivalent* to b (symb. $a \precsim b$) if for any $\varepsilon > 0$ there exists $r \in A$ s.t. $\|a - rbr^*\| < \varepsilon$.

Examples

- In $C_0(X)$, $f \precsim g$ if-f $\mathrm{supp}(f) \subset \mathrm{supp}(g)$.
- In $\mathcal{K}$ or M_n, $a \precsim b$ if-f $\mathrm{rank}(a) \leq \mathrm{rank}(b)$.

For a tracial state $\tau \in \mathrm{T}(A)$, let $d_\tau \colon (A \otimes \mathcal{K})_+ \to [0, \infty]$ denote the *dimension function* $d_\tau(a) := \lim_n \tau(a^{1/n})$.

Strict comparison (cont.)

Examples

- For $\tau(\cdot) = \int_X \cdot d\mu \in \mathrm{T}(C(X))$, $d_\tau(f) = \mu(\mathrm{supp}(f))$.
- For the unique tracial state τ on M_n, d_τ is the rank function.

Notice that $a \precsim b \implies d_\tau(a) \le d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$.

Definition

We say that A has *strict comparison* when, for all $a, b \in (A \otimes \mathcal{K})_+$, if $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim b$ (in $A \otimes \mathcal{K}$).

Classifiability of crossed products

Given an action $G \curvearrowright X$, when is the crossed product $C(X) \rtimes_{\mathtt{r}} G$ classifiable?

- G is countable, X is compact & metrizable: $C(X) \rtimes G$ is unital, separable
- G is amenable: $C(X) \rtimes G$ is nuclear, satisfies the UCT
- $G \curvearrowright X$ is topologically free & minimal: $C(X) \rtimes G$ is simple.

As for $\mathcal{Z}$-stability, some first results:

Theorems

- $C(X) \rtimes \mathbb{Z}$ is $\mathcal{Z}$-stable for free minimal actions $\mathbb{Z} \curvearrowright X$ where $\dim(X) < \infty$ (Toms – Winter, 2009)
- $C(X) \rtimes \mathbb{Z}^d$ is $\mathcal{Z}$-stable for free minimal actions $\mathbb{Z}^d \curvearrowright X$ where $\dim(X) < \infty$ and $E_{\mathbb{Z}^d}(X)$ weak-* compact (Winter, 2015).

Dynamical comparison, SBP, almost finiteness

Definition

Let $G \curvearrowright X$ be an action on a zero dimensional space. A *castle* is a finite collection $\{(S_i, V_i)\}_{i \in I}$ where $S_i \subset_{\text{fin.}} G$ (*shapes*), $V_i \subset X$ are clopen (*bases*) such that $\{sV_i : s \in S_i, i \in I\}$ (*levels*) are pairwise disjoint.

Definition (Kerr 2017; $\dim(X) = 0$ version)

An action $G \curvearrowright X$ on a compact metric space with $\dim(X) = 0$ is called *almost finite* when for any $\varepsilon > 0$, $K \subset_{\text{fin.}} G$ there exists a castle $\{(S_i, V_i)\}_{i \in I}$ such that

- $\mathrm{diam}(sV_i) < \varepsilon$ for all $s \in S_i, i \in I$
- S_i is (K, ε) invariant, i.e. $\max_{g \in K} \frac{|gS_i \triangle S_i|}{|S_i|} < \varepsilon$ for all $i \in I$
- $\bigsqcup_{i \in I} S_i V_i = X$.

Dynamical comparison, SBP, almost finiteness (cont.)

Definition

Let $G \curvearrowright X$ be an action and $A, B \subset X$ open. We say A is *dynamically below* B (symb. $A \prec B$) when for each compact $K \subset A$, there exists a finite open cover $\mathcal{U}$ of K and group elements $\{g_U\}_{U \in \mathcal{U}} \subset G$ s.t.

$$\bigsqcup_{U \in \mathcal{U}} g_U U \subset B.$$

We say $G \curvearrowright X$ has *dynamical comparison* when, for all open $A, B \subset X$, if $\mu(A) < \mu(B)$ for all $\mu \in M_G(X)$, then $A \prec B$.

Definition

We say $G \curvearrowright X$ has the *small boundary property* (SBP) when there exists a basis for the topology on X consisting of open sets U such that $\mu(\partial U) = 0$ for all $\mu \in M_G(X)$.

Dynamical comparison, SBP, almost finiteness (cont..)

Remark

Almost finiteness is defined for general metric spaces X as well: a remainder is allowed to exist, but it is required to be dynamically below an arbitrary prescribed proportion of the castle.

Theorem (Kerr 2017)

Let $G \curvearrowright X$ be a free minimal action of an amenable group on a compact metric space. If $G \curvearrowright X$ is almost finite, then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

Theorem (Kerr – Szabó 2018)

A free action $G \curvearrowright X$ is almost finite if and only if $G \curvearrowright X$ has dynamical comparison and the SBP.

$\rightsquigarrow$ For $\dim(X) = 0$, almost finiteness is equivalent to dynamical comparison.

Dynamical comparison, SBP, almost finiteness (cont...)

Remarks

- Free $G \curvearrowright X$ with $\dim(X) < \infty$ have the SBP (Szabó 2015)
- $G \curvearrowright X$ has comparison when
  - $G \curvearrowright X$ is free, G is subexponential growth, $\dim(X) < \infty$ (Downarowicz – Zhang 2017; Kerr – Szabó 2018)
  - $G \curvearrowright X$ is minimal, G is polynomial growth (Naryshkin 2022)
  - $G \curvearrowright X$ is free and minimal, G is elementary amenable, $\dim(X) < \infty$ (Kerr – Naryshkin 2021)
- It is conjectured that dynamical comparison is automatic.
- Dynamical comparison alone does not suffice for classification: there exists a free minimal action $\mathbb{Z} \curvearrowright X$ with $\dim(X) = \infty$ such that $C(X) \rtimes \mathbb{Z}$ does not have strict comparison $\rightsquigarrow$ $C(X) \rtimes \mathbb{Z}$ is not $\mathcal{Z}$-stable (Giol – Kerr 2010).
- If $G \curvearrowright X$ is free and has the SBP, then $C(X) \rtimes G$ has uniform property Γ (Kerr – Szabó 2018).

Cartan subalgebras

Definition

Let $D \subset A$ be an inclusion of C*-algebras. We say that D is a *Cartan subalgebra* of A when

- D is maximal abelian in A
- D contains an approximate unit of A
- $\mathcal{N}_A(D) := \{n \in A : nDn^* \cup n^*Dn \subset D\}$ generates A as a C*-algebra, i.e. $\overline{\mathrm{span}}\mathcal{N}_A(D) = A$
- there exists a unique faithful conditional expectation $\Phi\colon A \to D$.

If moreover every pure state of D extends *uniquely* to a pure state on A, D is called a *C*-diagonal* of A.

Example: For a topologically free action $G \curvearrowright X$,the canonical copy $C(X) \subset C(X) \rtimes_{\mathrm{r}} G$ is a Cartan subalgebra. It is a C*-diagonal when $G \curvearrowright X$ is free.

Recast: dynamical comparison

Remarks

$\rightsquigarrow$ Denote by $\mathcal{LN}_A(D) := \{w \in A : wDw^* \subset D\}$ the left-sided normalizers.

Let $G \curvearrowright X$ be a free action.

- For $f, g \in C(X)_+$, we have $\mathrm{supp}(f) \prec \mathrm{supp}(g)$ if-f there exists a sequence $(w_n) \subset \mathcal{LN}_{C(X)\rtimes_{\mathrm{r}} G}(C(X))$ s.t. $w_n g w_n^* \to f$.
- We have an affine w*-w* homeomorphism $M_G(X) \cong \mathrm{T}(C(X) \rtimes_{\mathrm{r}} G)$ given by $\mu \mapsto \tau_\mu(\cdot) := \int_X \Phi(\cdot) d\mu$

$\rightsquigarrow$ Dynamical comparison can be rephrased as follows:
for all $f, g \in C(X)_+$, if $d_\tau(f) < d_\tau(g)$ for all $\tau \in \mathrm{T}(C(X) \rtimes G)$, then $f \precsim g$ via left-sided normalizers.

Cartan comparison

Definition (K., Winter)

Let $D \subset A$ be a Cartan inclusion and let $a, b \in A_+$. Say that a is *Cartan below* b (symb. $a \precsim_{\text{Cartan}} b$) when, for any $\varepsilon > 0$ there exist $u, w, v \in A$ such that

- $\|a - uwvbv^*w^*u^*\| < \varepsilon$
- $\mathrm{supp}(\Phi(u^*u)) \subset \mathrm{supp}(\Phi(a))$
- $\mathrm{supp}(\Phi(vv^*)) \subset \mathrm{supp}(\Phi(b))$
- $w \in \mathcal{LN}_A(D)$.

Definition

Let $D \subset A$ be a Cartan inclusion with $\dim(\widehat{D}) = 0$ and $\mathrm{T}(A) \neq \emptyset$. We say that $D \subset A$ has *Cartan comparison* when, for all $a, b \in (A \otimes \mathcal{K})_+$, if $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_{\text{Cartan}} b$ in $(D \otimes D_{\mathcal{K}} \subset A \otimes \mathcal{K})$.

Cartan comparison (cont.)

Theorem (K., Winter)

Let $G \curvearrowright X$ be a free minimal action of a countable amenable group on a compact zero dimensional metric space. TFAE:

1. $G \curvearrowright X$ has dynamical comparison (and $C(X) \rtimes G$ has strict comparison)
2. $C(X) \subset C(X) \rtimes G$ has Cartan comparison.

Sketch of proof

(2) $\implies$ (1): let $p, q \in C(X)$ projections s.t. $p \precsim_{\text{Cartan}} q$. obtain u, w, v s.t. $\|p - uwvq(uwv)^*\| < 1$.
Use stable finiteness to conclude that u is invertible in the corner $\mathrm{Her}(p)$ and elementary inequalities to conclude that $(pw)q(pw)^*$ is invertible in $C(X) \cap \mathrm{Her}(p)$. Obtain a normalizer w' s.t. $w'qw'^* = p$ (hence $p \prec q$).

(1) $\implies$ (2): start with $a, b \geq 0$ s.t. $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$.
Heuristic for the unique trace case:
-Show first that $x \in \mathrm{Her}(\Phi(x))$ for all $x \geq 0$ (thus $d_\tau(a) \leq d_\tau(\Phi(a)) =$ $= \mu(\mathrm{supp}(\Phi(a)))$.
-Distinguish cases based on whether 0 is isolated in the spectrum of a: if not, wlog assume $d_\tau(a) < d_\tau(\Phi(a))$.
-Use intermediate value theorem of diffuse measures to "interpolate" a projection $p \in C(X)$ with $\mathrm{supp}(p) \subset \mathrm{supp}(\Phi(a))$ and s.t. $d_\tau(a) < d_\tau(p) < d_\tau(b)$.
-In the same way obtain a projection $q \in C(X)$ s.t. $\mathrm{supp}(q) \subset \mathrm{supp}(\Phi(b))$ and $d_\tau(p) < d_\tau(q) < d_\tau(b)$.

Sketch of proof (cont.)

- Since $d_\tau(a) < d_\tau(p) < d_\tau(q) < d_\tau(b)$, use strict comparison to compare a with p and q with b (obtaining u, v resp.) and dynamical comparison to compare p with q (obtaining the normalizer w).

For an arbitrary trace space: first show the following "tracial divisibility" lemma

Lemma

Let $G \curvearrowright X$ be a free action on a zero dimensional space and let $\lambda \in [0,1]$, $\varepsilon > 0$. For a clopen set $A \subset X$, there exists clopen $C \subset A$ s.t. $\sup_{\mu \in M_G(X)} |\mu(C) - \lambda \cdot \mu(A)| < \varepsilon$.

which allows for a similar approach to be carried out simultaneously for all traces $\tau \in \mathrm{T}(A)$.

Thank you!