Discrete logarithmic Sobolev inequalities in Banach spaces

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The hypercube

We shall be studying functional inequalities for functions of the form $f : \{-1,1\}^n \to X$, where $(X, \|\cdot\|_X)$ is a Banach space. As a toy model, we will also consider functions $f : (\mathbb{R}^n, \gamma_n) \to X$, where γ_n is the standard Gaussian measure on $\mathbb{R}^n$.

For a function $f : \{-1,1\}^n \to X$, we define $\partial_i f : \{-1,1\}^n \to X$ as

$$\partial_i f(x) \stackrel{\mathrm{def}}{=} \frac{f(x) - f(x_1, \ldots, -x_i, \ldots, x_n)}{2},$$

where $x \in \{-1,1\}^n$ and $i \in \{1, \ldots, n\}$.

The Efron–Stein inequality

Efron–Stein inequality. Every $f : \{-1,1\}^n \to \mathbb{R}$ satisfies

$$\mathrm{Var}(f) = \|f - \mathbb{E}f\|_2^2 \leq \sum_{i=1}^{n} \|\partial_i f\|_2^2.$$

- Observe that this bound is *dimension-free*.

- The proof is an immediate consequence of Parseval's identity for the Walsh basis (which is an orthonormal basis of characters for the discrete hypercube).

Talagrand's L_p Poincaré inequality

For $p \in [1, \infty)$ and $f : \{-1, 1\}^n \to \mathbb{R}$, we use the notation

$$\|\nabla f\|_p \stackrel{\text{def}}{=} \left[\mathbb{E}\left(\sum_{i=1}^{n} \left(\partial_i f\right)^2\right)^{p/2}\right]^{1/p}.$$

Talagrand's L_p Poincaré inequality. (1993) For any $p \in [1, \infty)$, there exists $\mathsf{C}_p > 0$ such that for any $n \in \mathbb{N}$, every function $f : \{-1, 1\}^n \to \mathbb{R}$ satisfies

$$\|f - \mathbb{E}f\|_p \leq \mathsf{C}_p \|\nabla f\|_p.$$

Many known **proofs** relying on isoperimetry, martingales, Bellman functions, non-commutative tools and semigroup methods.

Vector-valued Poincaré inequalities

Question. For which normed spaces $(X, \|\cdot\|_X)$ do these estimates extend to X-valued functions?

For $f : \{-1,1\}^n \to X$, we denote $\|f\|_{L_p(X)} \stackrel{\mathrm{def}}{=} \big(\mathbb{E}\|f\|_X^p\big)^{1/p}$.

In the scalar case $f : \{-1,1\}^n \to \mathbb{R}$, Khintchine's inequality gives

$$\|\nabla f\|_p \asymp_p \left(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i \partial_i f\Big\|_p^p\right)^{1/p}.$$

Therefore, for a function $f : \{-1,1\}^n \to X$ we denote

$$\|\nabla f\|_{L_p(X)} \stackrel{\mathrm{def}}{=} \left(\mathbb{E}\Big\|\sum_{i=1}^n \delta_i \partial_i f\Big\|_{L_p(X)}^p\right)^{1/p}.$$

Vector-valued Poincaré inequalities (continued)

This vector-valued gradient first appeared in work of Pisier.

Pisier's inequality. (1986) For any $p \in [1, \infty)$ and any normed space $(X, \|\cdot\|_X)$, every function $f : \{-1,1\}^n \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_p(X)} \leq 2e \log n \|\nabla f\|_{L_p(X)}.$$

This is the discrete counterpart of a famous result in Gauss space.

Maurey–Pisier inequality. (1986) For any $p \in [1, \infty)$ and any normed space $(X, \|\cdot\|_X)$, every function $f : (\mathbb{R}^n, \gamma_n) \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_p(X)} \leq \frac{\pi}{2} \left(\mathbb{E}_{g,g'} \Big\| \sum_{i=1}^n g_i' \partial_i f(g) \Big\|_X^p \right)^{1/p}.$$

Vector-valued Poincaré inequalities (continued)

Definition. A normed space $(X, \|\cdot\|_X)$ has *finite cotype* if there exists $m \in \mathbb{N}$ and $\varepsilon > 0$ such that $\ell_\infty^m = (\mathbb{R}^m, \|\cdot\|_\infty)$ does not embed into X with distortion smaller than $1+\varepsilon$.

Talagrand's counterexample. (1993) The $\log n$ factor in Pisier's inequality is needed if the space X does not have finite cotype.

Ivanisvili–van Handel–Volberg theorem. (2020) If X has finite cotype, then for any $p \in [1, \infty)$ there exists $\mathsf{C}_p(X) > 0$ such that for any $n \in \mathbb{N}$, every $f : \{-1,1\}^n \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_p(X)} \leq \mathsf{C}_p(X) \|\nabla f\|_{L_p(X)}.$$

The Bonami–Gross inequality

For a function $h : (\Omega, \mu) \to \mathbb{R}_+$ denote

$$\mathrm{Ent}(h) \stackrel{\mathrm{def}}{=} \int_\Omega h \log\left(\frac{h}{\int_\Omega h \, \mathrm{d}\mu}\right) \mathrm{d}\mu.$$

Bonami–Gross inequality. (1970-1975) For any $n \in \mathbb{N}$, every function $f : \{-1,1\}^n \to \mathbb{R}$ satisfies

$$\mathrm{Ent}(f^2) \leq 2 \sum_{i=1}^{n} \|\partial_i f\|_2^2.$$

The Bonami–Gross inequality (continued)

It is elementary to prove the two-sided bound

$$\frac{1}{2}\max\left\{\|h\|_2^2, \mathrm{Ent}(h^2)\right\} \le \|h\|_{L_2(\log L)}^2 \le 14\max\left\{\|h\|_2^2, \mathrm{Ent}(h^2)\right\},$$

where if $\psi_{p,a}(x) = x^p \log^a(e + x^p)$, we denote

$$\|h\|_{L_p(\log L)^a} \stackrel{\mathrm{def}}{=} \inf\left\{\gamma > 0 : \int_\Omega \psi_{p,a}\big(|f|/\gamma\big)\,\mathrm{d}\mu \le 1\right\}.$$

As logarithmic Sobolev inequalities are stronger than Poincaré inequalities, the Bonami–Gross inequality is equivalent to

$$\|f - \mathbb{E}f\|_{L_2(\log L)} \le \mathrm{C}\|\nabla f\|_2,$$

for a universal constant $\mathrm{C} > 0$.

Talagrand's L_p logarithmic Sobolev inequality

In order to obtain a quantitative version of Margulis' graph connectivity theorem, Talagrand proved the following deep extension of the Bonami–Gross inequality.

Talagrand's L_p logarithmic Sobolev inequality. (1993) For any $p \in [1, \infty)$, there exists $\mathsf{K}_p \in (0, \infty)$ such that for any $n \in \mathbb{N}$, every $f : \{-1, 1\}^n \to \mathbb{R}$ satisfies

$$(*) \qquad \|f - \mathbb{E}f\|_{L_p(\log L)^{p/2}} \leq \mathsf{K}_p \|\nabla f\|_p.$$

The Gaussian version of Talagrand's inequality was previously proven by **Ledoux** (1988).

Proofs of Talagrand's inequality $(*)$

1. (Talagrand) *Step 1.* Prove $(*)$ for characteristic functions of sets via an intricate induction on the dimension n. This is currently known as Talagrand's isoperimetric inequality.

Step 2. Use a layer cake decomposition for the function f and combine the isoperimetric inequality with a delicate approximate version of the co-area formula.

This argument is modeled after **Ledoux's** proof (1988) in Gauss space which combines the co-area formula with the Gaussian isoperimetric inequality.

Proofs of Talagrand's inequality $(*)$

2. (forlklore in Strasbourg of late 1980s?) Concatenate the lower Riesz transform inequality of **Lust-Piquard** (1998) asserting that for every $p \in (1, \infty)$,

$$\|\nabla f\|_p \gtrsim_p \|\Delta^{1/2} f\|_p$$

with a delicate result of **Bakry** and **Meyer** (1984) according to which if $\mathscr{L}$ is the negative generator of any hypercontractive semigroup, then for $p \in (1, \infty)$ and $\alpha > 0$,

$$\left\|(-\mathscr{L})^a f\right\|_p \gtrsim_{p,a} \|f - \mathbb{E} f\|_{L_p(\log L)^{pa}}.$$

This argument *fails* to capture the endpoint case $p = 1$.

Vector-valued log-Sobolev inequalities

Question. Are there vector-valued versions of Talagrand's L_p logarithmic Sobolev inequality?

• The scalar proofs do not extend to interesting normed spaces.

• The semigroup argument of **Ivanisvili**, **van Handel** and **Volberg** (2020) shows that if X is a normed space of cotype $q < \infty$, then for any $p \in [1, \infty)$ every $f : \{-1, 1\}^n \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_p(\log L)^a(X)} \lesssim_{X,p,a} \|\nabla f\|_{L_p(X)}$$

for any $a < \frac{p \min\{p,2\}}{2 \max\{p,q\}}$ which is very far from the scalar case.

Gaussian interlude

In 1988, **Ledoux** proved that for any normed space $(X, \|\cdot\|_X)$, every smooth function $f : (\mathbb{R}^n, \gamma_n) \to X$ satisfies the estimate

$$\mathrm{Ent}(\|f\|_X^2) \leq 2\ \mathbb{E}_{g,g'} \Big\| \sum_{i=1}^{n} g_i' \partial_i f(g) \Big\|_X^2.$$

Combined with the **Maurey–Pisier** inequality (1986) and the elementary fact about Orlicz norms, we conclude that every smooth function $f : (\mathbb{R}^n, \gamma_n) \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_2(\log L)(X)} \leq \mathsf{C}\ \mathbb{E}_{g,g'} \Big\| \sum_{i=1}^{n} g_i' \partial_i f(g) \Big\|_X^2.$$

for some universal $\mathsf{C} > 0$.

Proof of Ledoux's inequality

WLOG assume that $\gamma_n\{f = 0\} = 0$ and that $\|\cdot\|_X$ is smooth on $X \setminus \{0\}$, i.e. for any $v \neq 0$ there exists a linear functional $\mathsf{D}_v^* \in X^*$ with $\|\mathsf{D}_v^*\|_{X^*} \leq 1$ such that for any smooth curve $\beta : (-\varepsilon, \varepsilon) \to X$ with $\beta(0) = v$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \|\beta(t)\|_X = \left\langle \mathsf{D}_v^*, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \beta(t) \right\rangle.$$

Then, the Bonami–Gross inequality gives

$$\mathrm{Ent}(\|f\|_X^2) \leq 2 \sum_{i=1}^n \big\|\partial_i \|f\|_X\big\|_2^2 = 2\ \mathbb{E}_{g,g'} \Big| \sum_{i=1}^n g_i' \partial_i \|f\|_X(g) \Big|^2$$

$$= 2\ \mathbb{E}_{g,g'} \Big| \sum_{i=1}^n g_i' \langle \mathsf{D}_{f(g)}^*, \partial_i f(g) \rangle \Big|^2 = 2\mathbb{E}_{g,g'} \Big| \Big\langle \mathsf{D}_{f(g)}^*, \sum_{i=1}^n g_i' \partial_i f(g) \Big\rangle \Big|^2$$

and the conclusion follows as $\|\mathsf{D}_{f(g)}^*\|_{X^*} \leq 1$ almost surely. $\square$

The main result

⚠ This use of chain rule seems difficult to adapt when dealing with discrete derivatives on a graph (e.g. the hypercube).

Theorem (Cordero-Erausquin, E., 2023)

Let $(X, \|\cdot\|_X)$ *be a normed space of finite cotype. For every* $p \in [1, \infty)$*, there exists* $\mathsf{K}_p(X) > 0$ *such that for any* $n \in \mathbb{N}$*, every function* $f : \{-1, 1\}^n \to X$ *satisfies*

$$\|f - \mathbb{E}f\|_{L_p(\log L)^{p/2}(X)} \leq \mathsf{K}_p(X) \|\nabla f\|_{L_p(X)}.$$

The proof

We shall use a technical inequality from Talagrand's proof. For a scalar function $h : \{-1,1\}^n \to \mathbb{R}$, consider the *asymmetric* gradient

$$\mathrm{M}h(x) = \Big(\sum_{i=1}^{n} \partial_i h(x)_+^2 \Big)^{1/2},$$

where $a_+ = \max\{a, 0\}$ and $x \in \{-1,1\}^n$.

Proposition (Talagrand, 1993)

Let $h : \{-1,1\}^n \to \mathbb{R}_+$ be a nonnegative function for which $\mathbb{P}\{h = 0\} \geq \frac{1}{2}$. Then,

$$\|h\|_{L_p(\log L)^{p/2}} \leq \kappa_p \|\mathrm{M}h\|_p. \tag{1}$$

The proof (continued)

Fix a function $f : \{-1,1\}^n \to X$ with $\mathbb{E}f = 0$, let $h = \|f\|_X$ and consider $m \geq 0$ a median of h so that

$$\mathbb{P}\{h \leq m\} \geq \tfrac{1}{2} \qquad \text{and} \qquad \mathbb{P}\{h \geq m\} \geq \tfrac{1}{2}.$$

As $0 \leq h \leq (h-m)_+ + m$, we have

$$\|f\|_{L_p(\log L)^{p/2}(X)} = \|h\|_{L_p(\log L)^{p/2}} \leq \|(h-m)_+\|_{L_p(\log L)^{p/2}} + m.$$

For the second term observe that

$$m \leq \frac{1}{\mathbb{P}\{h \geq m\}} \int_{\{h \geq m\}} h \leq 2\|f\|_{L_1(X)} \leq 2\mathsf{C}_1(X)\|\nabla f\|_{L_1(X)},$$

where the last inequality follows from the the vector-valued L_1 Poincaré inequality under finite cotype.

The proof (continued)

To control the first term notice that $\mathbb{P}\{(h-m)_+ = 0\} \geq \frac{1}{2}$, so

$$\|(h-m)_+\|_{L_p(\log L)^{p/2}} \leq \kappa_p \|\mathrm{M}(h-m)_+\|_p,$$

by Talagrand's inequality. Moreover, the elementary inequality

$$\left(a_+ - b_+\right)_+ \leq (a-b)_+$$

which holds for $a, b \in \mathbb{R}$ shows that we can further upper bound this Orlicz norm by

$$\|(h-m)_+\|_{L_p(\log L)^{p/2}} \leq \kappa_p \|\mathrm{M}(h-m)\|_p = \kappa_p \|\mathrm{M}h\|_p.$$

The key lemma

Lemma. For any $f : \{-1,1\}^n \to X$, we have the pointwise bound

$$\mathrm{M}\|f\|_X(x)^2 \leq \mathbb{E}_\delta \Big\| \sum_{i=1}^n \delta_i \partial_i f(x) \Big\|_X^2.$$

Proof. Let v_x^* be a normalizing functional of $f(x)$. Then, for every $i \in \{1,\dots,n\}$, we have

$$\begin{aligned}\big(\|f(x)\|_X - &\|f(x_1,\dots,-x_i,\dots,x_n)\|_X\big)_+ \\ &\leq \big(\langle v_x^*, f(x)\rangle - \langle v_x^*, f(x_1,\dots,-x_i,\dots,x_n)\rangle\big)_+\end{aligned}$$

which implies that

$$\mathrm{M}\|f\|_X(x) \leq \sum_{i=1}^n \big\langle v_x^*, \partial_i f(x)\big\rangle^2 = \mathbb{E}_\delta \left\langle v_x^*, \sum_{i=1}^n \delta_i \partial_i f(x) \right\rangle^2$$

and the conclusion follows as $\|v_x^*\|_{X^*} \leq 1$.

Finishing the proof

By Kahane's inequality,

$$\left(\mathbb{E}_\delta \Big\| \sum_{i=1}^n \delta_i \partial_i f(x) \Big\|_X^2\right)^{1/2} \leq \sqrt{2} \left(\mathbb{E}_\delta \Big\| \sum_{i=1}^n \delta_i \partial_i f(x) \Big\|_X^p\right)^{1/p}$$

and thus the lemma implies that

$$\|\mathsf{M}h\|_p \leq \sqrt{2}\|\nabla f\|_{L_p(X)}.$$

Putting everything together, if $\mathbb{E}f = 0$, then we have

$$\|f\|_{L_p(\log L)^{p/2}(X)} \leq \sqrt{2}\kappa_p \|\nabla f\|_{L_p(X)} + 2\mathsf{C}_1(X)\|\nabla f\|_{L_1(X)}$$

and this completes the proof. $\square$

A refined Pisier inequality

Corollary. For any normed space $(X, \|\cdot\|_X)$, $p \in [1, \infty)$ and $n \in \mathbb{N}$, every function $f : \{-1, 1\}^n \to X$ satisfies

$$\|f - \mathbb{E}f\|_{L_p(\log L)^{p/2}(X)} \leq \sqrt{2}\kappa_p \|\nabla f\|_{L_p(X)} + 4e \log n \|\nabla f\|_{L_1(X)}.$$

Other results

- An application to the bi-Lipschitz distortion of quotients of the discrete hypercube with the Hamming metric.

- A general mechanism to boost metric Poincaré inequalities to metric log-Sobolev inequalities. In particular, we deduce new vector-valued log-Sobolev inequalities on the symmetric group for target spaces of martingale type 2.

- Vector-valued Beckner inequalities.

- Some vector-valued inequalities of isoperimetric type improving recent results of **Beltran**, **Ivanisvili** and **Madrid** (2023).

Thank you!