Operator systems, coproducts and quantum information theory

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Nonlocality scenarios

We assume that A, B, X, Y are finite sets. Suppose that we have two parties, named **Alice** and **Bob**, located in spatially separated labs. From a common source, they both receive a quantum system on which they conduct measurements.

• For Alice, the set X indexes the observables and A the outcomes. Alice's system is described by a Hilbert space H_A and each observable $x \in X$ is measured using a POVM $\{E_{x,a}\}_{a\in A} \subseteq \mathcal{B}(H_A)$.

• For Bob, the set Y indexes the observables and B the outcomes. Bob's system is described by a Hilbert space H_B and each observable $y \in Y$ is measured using a POVM $\{F_{y,b}\}_{b\in B} \subseteq \mathcal{B}(H_B)$.

• Positive operator valued measure **(POVM)**: $\{E_i\}_{i=1}^n \subseteq (\mathcal{B}(\mathcal{H}))^+$ s.t. $\sum_{i=1}^n E_i = 1$.

The tensor product assumption

• Suppose that the Hilbert space describing the composite system is modelled by the tensor product $H_A \otimes_{hs} H_B$ and that H_A, H_B are finite dimensional.
The probability that we observe $a \in A$ and $b \in B$ when measuring $x \in X$ and $y \in Y$ is given by

$$p(a, b|x, y) = \langle (E_{x,a} \otimes_{sp} F_{y,b})\psi, \psi\rangle$$

where $\psi \in H_A \otimes_{hs} H_B$ is the state of the system, that is, a unit vector.
So, $p = \{(p(a, b|x, y))_{(a,b)\in A\times B} : (x, y) \in X \times Y\}$ is a family of probability distributions.

The set of all families p of this type for fixed A, B, X, Y, is called the set of **quantum correlations** and is denoted by $Q_\otimes$.

The commutativity assumption

• Suppose that Alice and Bob share a common quantum system $H_A = H_B = H$, where H is infinite dimensional. So that the composite system is modelled by H.
The probability that we observe $a \in A$ and $b \in B$ when measuring $x \in X$ and $y \in Y$ is given by

$$p(a, b|x, y) = \langle (E_{x,a} F_{y,b})\psi, \psi \rangle$$

where $\psi \in H$ is the state of the system, and that $E_{x,a} F_{y,b} = F_{y,b} E_{x,a}$ for all x, y, a, b.
So, $p = \{(p(a, b|x, y))_{(a,b)\in A\times B} : (x, y) \in X \times Y\}$ is again a family of probability distributions.

The set of all families p of this type for fixed A, B, X, Y, is called the set of **quantum commuting correlations** and is denoted by Q_{qc}.

Tsirelson's problem

Note that both $Q_\otimes, Q_{qc}$ are subsets of $\mathbb{R}^{ABXY}$.

Tsirelson's problem was if

$$\overline{Q}_\otimes = Q_{qc}$$

for all X, Y, A, B

In fact, the equality $\overline{Q_\otimes} = Q_{qc}$ was shown to be equivalent to an affirmative answer to **Kirchberg's conjecture** (1993) in operator algebra theory which in turn is equivalent a positive answer to **Connes' embedding conjecture** in von Neumann algebras, from the 1970's!

All three answers were anounced negative in 2020, by Ji, Natarajan, Vidick, Wright, Yuen [JNV$^+$21] (computational complexity).

A universal operator system

A, X will denote finite sets throughout the talk.
Consider the C*-algebra ℓ_A^∞, and let $\{e_a\}_{a\in A}$ denote its canonical basis.

Proposition

For any POVM $\{E_a\}_{a\in A}$ in $\mathcal{B}(H)$, there is a unital completely positive map

$$\phi : \ell_A^\infty \to \mathcal{B}(H)$$

such that $\phi(e_a) = E_a$, $a \in A$. Conversely, any such ucp map defines a POVM by setting $E_a := \phi(e_a)$.

Moreover, there is an operator system, let's call it $S_{A,X}$ that contains X-copies of ℓ_A^∞ and let $\{e_{a,x}\}_{a\in A}$ denote the canonical basis of the x-th copy of ℓ_A^∞.

Proposition

For any family of POVM's $\{E_{a,x}\}_{a\in A} \subseteq \mathcal{B}(H)$, $x \in X$ there is a ucp map

$$\phi : S_{X,A} \to \mathcal{B}(H)$$

such that $\phi(e_a^x) = E_{a,x}$, for every a, x. Conversely, any such map defines a family of POVM's $\{E_{a,x}\}_{a\in A}$ with this equation.

The operator system $S_{A,X}$ is what is called the coproduct of X-copies of the operator system ℓ_A^∞ and is denoted by $\underbrace{\ell_A^\infty \oplus_1 \cdots \oplus_1 \ell_A^\infty}_{X}$.

• Projection valued measure **(PVM)**: $\{P_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$ s.t. $\sum_{i=1}^n P_i = 1$ and $P_i^2 = P_i = P_i^*$.

Theorem (Naimark's dilation)

For every POVM $\{E_a\}_{a\in A} \subseteq \mathcal{B}(H)$, there exist a Hilbert space K, an isometry $V \in B(H,K)$ and a PVM $\{P_a\}_{a\in A} \subseteq \mathcal{B}(K)$ such that $E_a = V^* P_a V$, $a \in A$.

This can be done by applying Stinespring's dilation theorem to the ucp map ϕ induced by the POVM. By a similar argument we can show that

Theorem (A simultaneous Naimark's dilation)

For any family $\{E_{a,x}\}_{a\in A}$, $x \in X$ of POVM's there exist a Hilbert space K, an isometry $V \in B(H,K)$ and PVM's $\{P_{a,x}\}_{a\in A} \subseteq \mathcal{B}(K)$, $x \in X$ such that $E_{a,x} = V^* P_{a,x} V$, $a \in A$, $x \in X$.

With this in hand, one can replace the POVM's in the definition of the correlation sets $Q_\otimes$, Q_{qc} with PVM's without changing the probabilities!

Kirchberg's conjecture

Let $\mathbb{F}_n$ be the free group on n generators, and $C^*(\mathbb{F}_n)$ be the full group C*-algebra of the group $\mathbb{F}_n$. The full group C*-algebra $C^*(\mathbb{F}_n)$ is the completion of the group ring $\mathbb{C}[\mathbb{F}_n]$ with respect to the norm $\|x\| = \sup_\pi \|\pi(x)\|$, where the supremum is taken over all unitary representations π of the group $\mathbb{F}_n$ into some $\mathcal{B}(H)$.

(A reformulation of) **Kirchberg's problem** was whether

$$C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$$

Kavruk's reformulation

Now, let $\mathcal{S}_n = \mathrm{span}\{1, g_i, g_i^* : i = 1, \dots n\} \subseteq C^*(\mathbb{F}_n)$, where the g_i are the unitary generators of $C^*(\mathbb{F}_n)$.
In particular, $\mathcal{S}_2 = \{1, g_1, g_2, g_1^*, g_2^*\}$ is a five-dimensional operator subsystem of $C^*(\mathbb{F}_2)$. Kavruk's remarkable result, states the following,

Theorem ([Kav14])

The following are equivalent,

1. *Kirchberg's conjecture has an afirmative answer.*
2. $\mathcal{S}_2 \otimes_{min} \mathcal{S}_2 = \mathcal{S}_2 \otimes_c \mathcal{S}_2$.

Now let $\mathcal{S}_1 := \mathrm{span}\{1, z, z^*\} \subseteq C(\mathbb{T})$, where z is the identity function, i.e. $z(e^{i\theta}) = e^{i\theta}$.

It can be shown that the op. system $\mathcal{S}_2$ from before, equals

$$\mathcal{S}_2 = \mathcal{S}_1 \oplus_1 \mathcal{S}_1$$

where the right-hand side is what is called, the **coproduct** (or unital direct sum) of $\mathcal{S}_1$ with itself.

The coproduct of two operator systems, can be defined via a universal property.

Universal property

The coproduct of two operator systems S_1 and S_2 [Kav14], is an operator system $S_1 \oplus_1 S_2$, together with two ucp maps $\phi_i : S_i \to S_1 \oplus_1 S_2$, $i = 1, 2$ s.t. whenever R is another o.s. with ucp maps $\psi_i : S_i \to R$, there exists a ucp map $\Psi : S_1 \oplus_1 S_2 \to R$ with $\Psi \circ \phi_i = \psi_i$, $i = 1, 2$, i.e. the following diagram commutes

$$\begin{tikzcd}
S_1 \arrow[r, "\phi_1"] \arrow[dr, "\psi_1"'] & S_1 \oplus_1 S_2 \arrow[d, "\Psi"] & S_2 \arrow[l, "\phi_2"'] \arrow[dl, "\psi_2"] \\
& R &
\end{tikzcd}$$

$S_1 \oplus_1 S_2$ is unique up to a unital complete order isomorphism.

The coproduct is a categorical notion that exists in several categories.
• The coproduct or free product of two unital C*-algebras [VDN92] A, B is again a unital C*-algebra $A *_1 B$ satisfying a similar universal property. $A *_1 B$ is the universal C*-algebra generated by copies of A and B with identification of their units.

$A *_1 B$ can also be thought of as completion of the *-algebra free product, which is spanned by linear combinations of "words" in A and B with identification of their units.

• More generally, if D is a C*-algebra contained in A and B, we may form the amalgamated free product [Bla78] $A *_D B$, which is the universal C*-algebra generated by copies of A and B so that the copies of D are identified.

Example

Let G_1, G_2 be discrete groups. Then, the coproduct in this category is the free product of groups $G_1 * G_2$.

Example

$C^*(G_1) *_1 C^*(G_2) = C^*(G_1 * G_2)$

The free products of C*-algebras can get quite large. Consider the cyclic group $\mathbb{Z}_m$ of order m, we have the isomorphism $C^*(\mathbb{Z}_m) = \mathbb{C}^m$. In particular, $C^*(\mathbb{Z}_2) = \mathbb{C}^2$ and their free product $\mathbb{C}^2 *_1 \mathbb{C}^2 = C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ is the universal C*-algebra generated by two projections. More concretely,

Example (see [RS89])

$\mathbb{C}^2 *_1 \mathbb{C}^2 = \{f \in C([0,1], M_2) : f(0), f(1) \text{ are diagonal}\}$

If $S \subseteq A$ and $T \subseteq B$ are two operator systems, where A, B are unital C*-algebras. We have two (complete order isomorphic) ways of realising their coproduct $S \oplus_1 T$.

1. $S \oplus_1 T = \mathrm{span}\{s + t : s \in S, t \in T\} \subseteq A *_1 B$
2. $S \oplus_1 T = (S \oplus T)/\{\lambda(1_S \oplus -1_T) : \lambda \in \mathbb{C}\}$

Where,

1. The 1st form, makes use of Boca's theorem [Boc91], which implies that, whenever we have ucp maps $\phi : A \to \mathcal{B}(H)$ and $\psi : B \to \mathcal{B}(H)$, there exists a ucp map $\Phi : A *_1 B \to \mathcal{B}(H)$ with $\Phi|_A = \phi$ and $\Phi|_B = \psi$.
2. For the 2nd one, one needs to prove first that $J = \{\lambda(1_S \oplus -1_T) : \lambda \in \mathbb{C}\}$ is an "appropriate" subspace of $S \oplus T$ so that the resulting quotient is an operator system.

Now, we consider a generalisation of the notion of the operator systems, namely, the operator A-systems.
For a unital C^*-algebra A, an operator system S with unit $e \in S$, is called *operator A-system* if

1. S is a bimodule over A
2. $(a \cdot s)^* = s^* \cdot a^*$
3. $a \cdot e = e \cdot a, \quad e \in \mathcal{S}$
4. $[a_{i,j}] \cdot [s_{i,j}] \cdot [a_{i,j}]^* \in M_n(S)^+$, for all $[a_{i,j}] \in M_{n,m}(A)$, $[s_{i,j}] \in M_m(S)^+$, $n, m \in \mathbb{N}$

For example, every operator system, is an operator A-system, for $A = \mathbb{C}$. Also, every unital C*-algebra B is an operator A-system over any unital C*-subalgebra A.

Note that, the "appropriate" morphisms for operator A-systems, are the ucp A-bimodule maps, i.e. ϕ is ucp and $\phi(a \cdot s) = a \cdot \phi(s)\ \forall a \in A,\ s \in S$.

There is a Choi-Effros-type representation theorem for operator A-systems,

Theorem ([Pau03])

Let A be a unital C^*- algebra and S be an operator A-system. There exists a Hilbert space H, a unital complete order embedding $\phi : S \to \mathcal{B}(H)$ and a unital $*$-homomorphism $\pi : A \to \mathcal{B}(H)$, such that

$$\phi(a \cdot s) = \pi(a)\,\phi(s),$$

for all $a \in A$ and $s \in S$.

So, we can identify an (abstract) operator A-system S with its image under ϕ, and consider it as $S \subseteq \mathcal{B}(H)$ while the module action becomes $a \cdot s = \pi(a)s \in \mathcal{B}(H)$.

Denote the module action $A \times S \to S$, by $a \cdot s$.

- S is a *faithful operator A-system* if:

$$a \cdot e \neq 0, \quad \text{for all} \quad a \in \mathcal{A} \setminus \{0\}.$$

By the previous theorem, if S is a faithful operator A-system, there exists a unital complete order embedding $\phi : S \to \mathcal{B}(H)$ and a unital *-homomorphism $\pi : A \to \mathcal{B}(H)$ s.t. $\phi(a \cdot s) = \pi(a)\phi(s)$. From this, we obtain that if $a \neq 0$, then

$$\pi(a) = \phi(a \cdot e) \neq 0$$

since S is faithful and ϕ is injective. So, π is injective as well.

- More concretely this means that: $A \subseteq S \subseteq B(H)$, for some H.

Operator A-system coproducts

A notion of a coproduct also exists in this context.

Theorem ([Cha22])

Let S_1 and S_2 be two faithful operator A-systems. There exists a unique faithful operator A-system $S_1 \oplus_A S_2$, along with unital complete order embeddings $\phi_i : S_i \hookrightarrow S_1 \oplus_A S_2$, $i = 1, 2$ that are also A-bimodule maps, such that the following universal property holds: For every operator A-system R and u.c.p. A-bimodule maps $\psi_i : S_i \to R$, $i = 1, 2$, there exists a unique u.c.p. A-bimodule map $\Psi : S_1 \oplus_A S_2 \to R$ such that $\Psi \circ \phi_i = \psi_i$ for $i = 1, 2$.

Let A, B_1 and B_2 be unital C*-algebras. Suppose that we have two faithful operator A-systems S and T s.t. $A \subseteq S \subseteq B_1$ and $A \subseteq T \subseteq B_2$. In analogy to the operator system case, we can realise the coproduct $S \oplus_A T$ of two faithful operator A-systems S and T as

1. $S \oplus_A T = \mathrm{span}\{s + t : s \in S, t \in T\} \subseteq B_1 *_A B_2$,
2. $S \oplus_A T = (S \oplus T)/\{a \oplus (-a) : a \in A\}$.

Where,

1. The 1st form makes use of an extension of Boca's theorem by Davidson and Kakariadis [DK19], which roughly says that we can lift ucp maps from the C* algebras B_1 and B_2 that agree on A, to a ucp map on their amalgamated free product.
2. For the 2nd form, on must first prove that $(S \oplus T)/\{a \oplus (-a) : a \in A\}$ admits an operator A-system structure.

Graph operator systems

Graph operator systems, are examples of operator A-systems. If $G = (V, E)$ is a simple, undirected graph, with $|V| = n$, the graph operator system S_G is defined as

$$S_G := \mathrm{span}\{e_{i,j} \in M_n : i = j \text{ or } (i, j) \in E\}$$

where $\{e_{i,j}\}$ are the matrix units of M_n.

It can be shown that an operator system $S \subseteq M_n$ is a graph operator system, if and only if S is a bimodule over D_n.

So, graph operator systems are operator A-systems for $A = D_n$, $n \in \mathbb{N}$. In particular they are faithful since, $D_n \subseteq S_G \subseteq M_n$.

Now, we let G be the complete graph on two vertices, so $S_G = M_2$. Then, we can form the "amalgamated" coproduct $M_2 \oplus_{D_2} M_2$, which is a faithful operator D_2-system, i.e., an operator system, that is a bimodule over D_2 and such that $D_2 \subseteq M_2 \oplus_{D_2} M_2$. Also, $M_2 \oplus_{D_2} M_2$ is 6-dimensional.

More concretely, we can imagine $M_2 \oplus_{D_2} M_2$ as

$$M_2 \oplus_{D_2} M_2 = \left\{ \left(\begin{array}{cc|cc} a & b & & \\ c & d & & \\ \hline & & e & f \\ & & g & h \end{array} \right) : a, \ldots, h \in \mathbb{C} \right\}.$$

where we identify

$$\left(\begin{array}{cc|cc} a & 0 & & \\ 0 & d & & \\ \hline & & 0 & 0 \\ & & 0 & 0 \end{array} \right) = \left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & a & 0 \\ & & 0 & d \end{array} \right).$$

However, $M_2 \oplus_{D_2} M_2$ is not a graph operator system for any graph G. In fact, it cannot be represented faithfully in a finite dimensional Hilbert space!

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