C^*-algebras I

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November 2022

1. C^*-algebras
2. the spectrum
3. Gelfand theory for commutative C*-algebras

C^*-algebras

The space of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathcal{H})\}$$

($\mathrm{b}_1(\mathcal{X})$ the closed unit ball of a normed space $\mathcal{X}$) and is an algebra under composition. Moreover, because it acts on a Hilbert space, it has additional structure: an *involution* $T \to T^*$ defined via

$$\langle T^*x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

This satisfies

$$\|T^*T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

C^*-algebras

These fundamental properties of $\mathcal{B}(\mathcal{H})$ (norm-completeness, involution, C^* property) motivate the definition of an abstract C*-algebra.

C^*-algebras

Definition

(a) A Banach algebra $\mathcal{A}$ is a complex algebra equipped with a complete norm which is sub-multiplicative:

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } \; a, b \in \mathcal{A}.$$

(b) An involution is a map on $\mathcal{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

(c) A C^*-algebra $\mathcal{A}$ is a Banach algebra equipped with an involution $a \to a^*$ satisfying the C^*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \; a \in \mathcal{A}.$$

C^*-algebras

If $\mathcal{A}$ has a unit **1** then necessarily $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$.

Definition

If $\mathcal{A}$ is a C*-algebra let

$$\mathcal{A}^\sim =: \mathcal{A} \oplus \mathbb{C}$$

$$(a, z)(b, w) =: (ab + wa + zb, zw)$$

$$(a, z)^* =: (a^*, \bar{z})$$

$$\|(a, z)\| =: \sup\{\|ab + zb\| : b \in \mathrm{b}_1 \mathcal{A}\}$$

Thus the norm of $\mathcal{A}^\sim$ is defined by identifying each $(a, z) \in \mathcal{A}^\sim$ with the operator $L_{(a,z)} : \mathcal{A} \to \mathcal{A} : b \to ab + zb$ acting on the Banach space $\mathcal{A}$.

C^*-algebras

$\mathbb{C}^2$ with norm

$$\|(x,y)\| = |x| + |y|$$

is not a C^*-algebra.

$$\|a^*a\| = \|(1,1)(1,1)\| = \|(1,1)\| = 2$$

$$\|a\|^2 = \|(1,1)\|^2 = 4$$

C^*-algebras

A morphism $\phi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is a linear map that preserves products and the involution.

C^*-algebras

- $\mathbb{C}$, the set of complex numbers.
- $C(K)$, the set of all continuous functions $f : K \to \mathbb{C}$, where K is a compact Hausdorff space. With pointwise operations, $f^*(t) = \overline{f(t)}$ and the sup norm, $C(K)$ is an abelian, unital algebra.
- $C_0(X)$, where X is a locally compact Hausdorff space. This consists of all functions $f : X \to \mathbb{C}$ which are continuous and 'vanish at infinity': given $\varepsilon > 0$ there is a compact $K_{f,\varepsilon} \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K_{f,\varepsilon}$. With the same operations and norm as above, this is an abelian C*-algebra.

C^*-algebras

- $M_n(\mathbb{C})$, the set of all $n \times n$ matrices with complex entries. With matrix operations, $A^* =$ conjugate transpose, and $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$, this is a non-abelian, unital algebra.
- $\mathcal{B}(\mathcal{H})$ is a non-abelian, unital C*-algebra.
- $\mathcal{K}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \overline{A(\mathrm{b}_1(\mathcal{H}))}$ compact in $\mathcal{H}\}$: the compact operators. This is a closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence a C*-algebra.

C^*-algebras

If X is an index set and $\mathcal{A}$ is a C*-algebra, the Banach space $\ell^\infty(X, \mathcal{A})$ of all bounded functions $a : X \to \mathcal{A}$ (with norm $\|a\|_\infty = \sup\{\|a(x)\|_\mathcal{A} : x \in X\}$) becomes a C*-algebra with pointwise product and involution.
Its subspace $c_0(X, \mathcal{A})$ consisting of all $a : X \to \mathcal{A}$ such that $\lim\limits_{x\to\infty} \|a(x)\|_\mathcal{A} = 0$ is a C*-algebra. (for each $\varepsilon > 0$ there is a finite subset $X_{a,\varepsilon} \subseteq X$ s.t. $x \notin X_{a,\varepsilon} \Rightarrow \|a(x)\|_\mathcal{A} < \varepsilon$).

C^*-algebras

If X is a locally compact Hausdorff space then $C_b(X, \mathcal{A})$ is the *-subalgebra of $\ell^\infty(X, \mathcal{A})$ consisting of continuous bounded functions. It is closed, hence a C*-algebra.
The C*-algebra $C_0(X, \mathcal{A})$ consists of those $f \in C_b(X, \mathcal{A})$ which `vanish at infinity', i.e. such that the function $t \to \|f(t)\|_{\mathcal{A}}$ is in $C_0(X)$.

C^*-algebras

Consider subsets of the Cartesian product $\prod \mathcal{A}_i$ of a family of C*-algebras:
(i) The direct sum $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ of C*-algebras is a C*-algebra under pointwise operations and involution and the norm

$$\|(a_1, \ldots, a_n)\| = \max\{\|a_1\|, \ldots, \|a_n\|\}.$$

(ii) Let $\{\mathcal{A}_i\}$ be a family of C*-algebras. Their direct product or ℓ^∞-direct sum $\bigoplus_{\ell^\infty} \mathcal{A}_i$ is the subset of the Cartesian product $\prod \mathcal{A}_i$ consisting of all $(a_i) \in \prod \mathcal{A}_i$ such that $i \to \|a_i\|_{\mathcal{A}_i}$ is bounded. It is a C*-algebra under pointwise operations and involution and the norm

$$\|(a_i)\| = \sup\{\|a_i\|_{\mathcal{A}_i} : i \in I\}$$

.

C^*-algebras

(iii) The direct sum or c_0-direct sum $\bigoplus_{c_0} \mathcal{A}_i$ of a family $\{\mathcal{A}_i\}$ of C*-algebras is the closed selfadjoint subalgebra of their direct product consisting of all $(a_i) \in \prod \mathcal{A}_i$ such that $i \to \|a_i\|_{\mathcal{A}_i}$ vanishes at infinity. In case $\mathcal{A}_i = \mathcal{A}$ for all i, the direct product is just $\ell^\infty(I, \mathcal{A})$ and the direct sum is $c_0(X, \mathcal{A})$.

C^*-algebras

If $\mathcal{A}$ is a C*-algebra and $n \in \mathbb{N}$, the space $M_n(\mathcal{A})$ of all matrices $[a_{ij}]$ with entries $a_{ij} \in \mathcal{A}$ becomes a *-algebra with product $[a_{ij}][b_{ij}] = [c_{ij}]$ where $c_{ij} = \sum_k a_{ik} b_{kj}$ and involution $[a_{ij}]^* = [d_{ij}]$ where $d_{ij} = a_{ji}^*$.

Define a norm on $M_n(\mathcal{A})$ satisfying the C*-condition.

C^*-algebras

Suppose $\mathcal{A}$ is $C_0(X)$. Identify $M_n(C_0(X))$ with $C_0(X, M_n)$, i.e. M_n-valued continuous functions on X vanishing at infinity: each matrix $[f_{ij}] \in M_n(C_0(X))$ defines a function $F : X \to M_n : x \to [f_{ij}(x)]$ which is continuous with respect to the norm on M_n. Conversely, if $F : X \to M_n$ is continuous, then its entries f_{ij} given by $f_{ij}(x) = \langle F(x)e_j, e_i \rangle$ form an $n \times n$ matrix of continuous functions.

Define

$$\|[f_{ij}]\| = \|F\|_\infty = \sup\{\|F(x)\|_{M_n} : x \in X\}.$$

This satisfies the C*-condition, because the norm on M_n satisfies the C*-condition.

C^*-algebras

Suppose $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Identify $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$: Given a matrix $[a_{ij}]$ of bounded operators a_{ij} on $\mathcal{H}$, we define an operator A on $\mathcal{H}^n$ by

$$A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}\xi_j \\ \vdots \\ \sum_j a_{nj}\xi_j \end{bmatrix}$$

Conversely any $A \in \mathcal{B}(\mathcal{H}^n)$ defines an $n \times n$ matrix of operators a_{ij} on $\mathcal{H}$ by $\langle a_{ij}\xi, \eta\rangle_{\mathcal{H}} = \langle A\xi_j, \eta_i\rangle_{\mathcal{H}^n}$, where $\xi_j \in \mathcal{H}^n$ is the vector having ξ at the j-th entry and zeroes elsewhere (and η_i is defined analogously).

C^*-algebras

Hence one defines the norm $\|[a_{ij}]\|$ of $[a_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ to be the norm $\|A\|$ of the corresponding operator on $\mathcal{H}^n$.

For $n = 2$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} A\xi + B\eta \\ C\xi + D\eta \end{bmatrix}$$

This applies also if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Group algebras

Definition

A topological group is a group G which is a topological space such that the maps

$$(x, y) \mapsto xy$$

$$x \mapsto x^{-1}$$

are continuous.

Group algebras

Examples

- G any group with the discrete topology
- $(\mathbb{R}, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{R}^*_+, \cdot)$
- $(\mathbb{T}, \cdot)$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$
- $GL(n, \mathbb{R}) = \{A = (a_{ij}) : n \times n \,\mathrm{matrix}, a_{ij} \in \mathbb{R}, \ \det A \neq 0\}$
- F_n the free group with n-generators

Group algebras

Proposition

G locally compact topological group. Then G has a left invariant measure. This measure is unique up to a scalar, is called the Haar measure and is denoted by $d\mu$.

It satisfies

$$\int_G f(ax)d\mu(x) = \int_G f(x)d\mu(x)$$

for $f \in L^1(G)$, $a \in G$.

Group algebras

Definition

$\mathcal{H}$ Hilbert space and G topological group. A unitary representation π of G is a map $G \to B(\mathcal{H})$ such that:

1. $\pi(x)^*\pi(x) = \pi(x)\pi(x)^* = I, \quad \forall x \in G.$
2. $x \to \pi(x)$ is a homomorphism of groups from G into the group of unitary operators on $\mathcal{H}$.
3. For each $v \in \mathcal{H}$ the map $x \mapsto \pi(x)v$ is continuous.

Group algebras

Examples

- The trivial representation
- $L^2(G)$ the Hilbert space with inner product

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)}d\mu(x).$$

The representation λ defined by:

$$\lambda(y)f(x) = f(y^{-1}x)$$

is called the left regular representation of G.

Group algebras

For $f, g \in L^1(G)$, define

$$f * g(x) = \int_{y \in G} f(xy^{-1})g(y)d\mu(y)$$

$$f^*(x) = \overline{f(x^{-1})}$$

group algebras

(π, H) representation of G. For $f \in L^1(G)$, $\xi, \eta \in H$ define an operator on $\mathcal{H}$

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x) \langle \pi(x)\xi, \eta \rangle \, d\mu(x).$$

Then, $\|\pi(f)\| \leq \|f\|$.

Proposition

$(\pi, \mathcal{H})$ representation of G. Then $f \mapsto \pi(f) = \int_G f(x)\pi(x)d\mu(x)$ satisfies

1. $\pi : L^1(G) \to B(\mathcal{H})$ *is linear.*
2. $\pi(f * g) = \pi(f)\pi(g)$
3. $\pi(f^*) = \pi(f)^*$
4. $\overline{\pi(L^1(G))\mathcal{H}} = \mathcal{H}$

$C^*(G)$

Definition

Define a norm on $L^1(G)$

$$\|f\| = sup_{\pi \in \hat{G}} \|\pi(f)\|.$$

The C^* algebra of G, $C^*(G)$ is the completion of $L^1(G)$ wrt this norm.

$\hat{G}$: the set of equivalence classes of irreducible representations.
π irreducible: there are no invariant subspaces for $\{\pi(g) : g \in G\}$.

$C^*(G)$

Examples

- $C^*(\mathbb{R}) \simeq C_0(\mathbb{R})$.
- $C^*(\mathbb{T}) \simeq C_0(\mathbb{Z})$.
- $C^*(\mathbb{Z}) \simeq C(\mathbb{T})$.

$vN(G)$

G, λ left regular representation.

Definition

von Neumann algebra of G is the wot closure of the span of $\{\lambda(x) : x \in G\}$.

Examples

- $vN(\mathbb{R}) = L^\infty(\mathbb{R})$.
- $vN(\mathbb{T}) = \ell^\infty(\mathbb{Z})$.
- $vN(\mathbb{Z}) = L^\infty(\mathbb{T})$.

Example

$G = \mathbb{Z}$

$$\lambda(1) = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

- $C^*(F_m)$ not isomorphic to $C^*(F_n)$, for $n \neq m$
- Is $vN(F_2)$ isomorphic to $vN(F_3)$?

Nonexamples:

- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$ (upper triangular matrices).
- $M_{oo}(\mathbb{C})$: infinite matrices with finite support. *To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete.*

The spectrum

Definition

$\mathcal{A}$ unital C*-algebra and $GL(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$.
The spectrum of an element $a \in \mathcal{A}$ is

$$\sigma(a) = \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin GL(\mathcal{A})\}.$$

If $\mathcal{A}$ is non-unital, the spectrum of $a \in \mathcal{A}$ is defined by

$$\sigma(a) = \sigma_{\mathcal{A}^\sim}(a).$$

In this case, necessarily $0 \in \sigma(a)$.

The spectrum

Examples

- $\mathcal{A} = M_n(\mathbb{C})$ and $a \in \mathcal{A}$, then $\sigma(A)$ is the set of eigenvalues of A.
- $\mathcal{A} = C([0,1])$ and $f \in \mathcal{A}$, then:

$$f - \lambda \mathbf{1} \text{ invertible} \Leftrightarrow f(x) - \lambda \mathbf{1}(x) \neq 0, \forall x$$

$$\Leftrightarrow f(x) - \lambda 1 \neq 0, \forall x \Leftrightarrow \lambda \neq f(x), \forall x.$$

$$\Rightarrow \sigma(f) = \{f(x) : x \in [0,1]\}$$

The spectrum

Proposition

The spectrum $\sigma(a)$ is a compact nonempty subset of $\mathbb{C}$.

The spectrum

The spectral radius of $a \in \mathcal{A}$ is defined to be

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

It satisfies $\rho(a) \leq \|a\|$, but equality may fail. In fact, it can be shown that

$$\rho(a) = \lim_n \|a^n\|^{1/n}$$

This is the Gelfand-Beurling formula.

The spectrum

Lemma

If $a = a^*$ *then* $\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|a\|$.

proof

$\|a\|^2 = \|a^2\|$ and inductively $\|a\|^{2^n} = \|a^{2^n}\|$ for all n. Thus, by the Gelfand - Beurling formula, $\rho(a) = \lim \left\|a^{2^n}\right\|^{2^{-n}} = \|a\|$. $\square$

Proposition

*There is at most one norm on a *-algebra making it a C*-algebra.*

proof

$$\|a\|^2 = \|a^* a\| = \rho(a^* a)$$

The spectrum

Theorem

A morphism $\pi : \mathcal{A} \to \mathcal{B}$ is contractive (i.e. $\|\pi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$).

proof If $x, y \in \mathcal{A}$ and $xy = \mathbf{1} \Rightarrow \pi(x)\pi(y) = \mathbf{1}$.

$a - \lambda\mathbf{1}$ invertible implies $\pi(a) - \lambda\mathbf{1}$ invertible and hence, $\sigma(\pi(a) \subseteq \sigma(a)$ and hence $\rho(\pi(a)) \leq \rho(a)$.

$$\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\|$$

$$= \|\pi(a^*a)\| = \rho(\pi(a^*a)) \leq \rho(a^*a) = \|a^*a\| = \|a\|^2$$

$\square$

The spectrum

An element $a \in \mathcal{A}$ is said to be normal if $a^*a = aa^*$, selfadjoint if $a = a^*$ and unitary if ($\mathcal{A}$ is unital and) $u^*u = \mathbf{1} = uu^*$.

Proposition

(i) $a = a^* \implies \sigma(a) \subseteq \mathbb{R}$
(ii) $a = b^*b \implies \sigma(a) \subseteq \mathbb{R}^+$
(iii) $u^*u = \mathbf{1} = uu^* \implies \sigma(u) \subseteq \mathbb{T}$.

Gelfand theory for commutative C*-algebras

Theorem (Gelfand-Naimark 1)

Every commutative C-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is the set of nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. For each $a \in \mathcal{A}$ the function $\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} : \ \phi \to \phi(a)$ is in $C_0(\hat{\mathcal{A}})$. The Gelfand transform:*

$$\mathcal{A} \to C_0(\hat{\mathcal{A}}) : \ a \to \hat{a}$$

*is an isometric *-isomorphism. The space $\hat{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ is unital.*

Commutative C*-algebras

$\mathcal{A}$ unital.

- $\hat{\mathcal{A}}$ is the set of all nonzero multiplicative linear forms (characters) $\phi : \mathcal{A} \to \mathbb{C}$.
  $\phi(\mathbf{1})^2 = \phi(\mathbf{1}) \Rightarrow \phi(\mathbf{1}) = 1$ (for if $\phi(\mathbf{1}) = 0$ then $\phi(a) = \phi(a\mathbf{1}) = 0$ for all a, a contradiction).
  Each $\phi \in \hat{\mathcal{A}}$ satisfies $\|\phi\| \leq 1$ and $\|\phi\| = \phi(\mathbf{1}) = 1$. The topology on $\hat{\mathcal{A}}$ is pointwise convergence: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathcal{A}$.

Commutative C*-algebras

- The inequality $|\phi(a)| \leq \|a\|$ shows that $\hat{\mathcal{A}}$ is contained in the space $\Pi_{a\in\mathcal{A}}\mathbb{D}_a$, the Cartesian product of the compact spaces $\mathbb{D}_a = \{z \in \mathbb{C} : |z| \leq \|a\|\}$; and the product topology is the topology of pointwise convergence.

  $\hat{\mathcal{A}}$ is closed in this product: if $\phi_i \to \psi$ pointwise, then ψ is linear and multiplicative, because each ϕ_i is linear and multiplicative, and $\psi \neq 0$ because $\psi(\mathbf{1}) = \lim_i \phi_i(\mathbf{1}) = 1$; thus $\psi \in \widehat{\mathcal{A}}$.

Commutative C*-algebras

- The Gelfand map $\mathcal{G} : a \to \hat{a}$. For each $a \in \mathcal{A}$ the function

$$\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} \quad \text{where} \quad \hat{a}(\phi) = \phi(a), \ (\phi \in \hat{\mathcal{A}})$$

is continuous by the definition of the topology on $\hat{\mathcal{A}}$. This gives a well defined map

$$\mathcal{G} : \mathcal{A} \to C(\hat{\mathcal{A}}) : a \to \hat{a}\,.$$

- If $a, b \in \mathcal{A}$, since each $\phi \in \hat{\mathcal{A}}$ is linear, multiplicative and *-preserving, we have

$$\widehat{(a+b)}(\phi) = \phi(a+b) = \phi(a) + \phi(b) = \hat{a}(\phi) + \hat{b}(\phi)$$

$$\widehat{(ab)}(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi)$$

$$\widehat{(a^*)}(\phi) = \phi(a^*) = \overline{\phi(a)} = \overline{\hat{a}(\phi)}$$

- therefore

$$\mathcal{G}(a+b) = \mathcal{G}(a)+\mathcal{G}(b), \quad \mathcal{G}(ab) = \mathcal{G}(a)\mathcal{G}(b) \quad \text{and} \quad \mathcal{G}(a^*) = \mathcal{G}(a)^*$$

- The map $\mathcal{G}$ is isometric.

$$\|\mathcal{G}(a)\|^2 = \|\mathcal{G}(a)^*\mathcal{G}(a)\| = \|\mathcal{G}(a^*a)\| = \rho(a^*a) = \|a^*a\| = \|a\|^2$$

Commutative C*-algebras

- The Gelfand map is onto $C(\hat{\mathcal{A}})$. Consider the range $\mathcal{G}(\mathcal{A})$: it is a *-subalgebra of $C(\hat{\mathcal{A}})$, because $\mathcal{G}$ is a *-homomorphism. It contains the constants, because $\mathcal{G}(\mathbf{1}) = \mathbf{1}$. It separates the points of $\hat{\mathcal{A}}$, because if $\phi, \psi \in \hat{\mathcal{A}}$ are different, they must differ at some $a \in \mathcal{A}$, so

$$\mathcal{G}(a)(\phi) = \phi(a) \neq \psi(a) = \mathcal{G}(a)(\psi).$$

By the Stone -- Weierstrass Theorem, $\mathcal{G}(\mathcal{A})$ must be dense in $C(\hat{\mathcal{A}})$. But it is closed, since $\mathcal{A}$ is complete and $\mathcal{G}$ is isometric. Hence $\mathcal{G}(\mathcal{A}) = C(\hat{\mathcal{A}})$. $\square$

Commutative C*-algebras

When $\mathcal{A}$ is abelian but non-unital every $\phi \in \hat{\mathcal{A}}$ extends uniquely to a character $\phi^\sim \in \widehat{\mathcal{A}^\sim}$ by $\phi^\sim(\mathbf{1}) = 1$, and there is exactly one $\phi_\infty \in \widehat{\mathcal{A}^\sim}$ that vanishes on $\mathcal{A}$. Thus $\mathcal{A}$ is *-isomorphic the algebra of those continuous functions on the 'one-point compactification' $\hat{\mathcal{A}} \cup \{\phi_\infty\}$ of $\hat{\mathcal{A}}$ which vanish at ϕ_∞; this algebra is in fact isomorphic to $C_0(\hat{\mathcal{A}})$.

Commutative C*-algebras

Example

c_0 the space of sequences converging to 0.

$\phi_n : c_0 \to \mathbb{C}$, $\phi_n((a_k)) = a_n$. Then $\hat{c_0} \simeq \mathbb{N}$.

(ϕ_n) converges pointwise to the zero character, since

$$\lim_n \phi_n((a_k)) = \lim_n a_n = 0.$$

Thus, $\hat{c_0}$ is not compact.

Commutative C*-algebras

Example

Consider the unitization c of c_0 which is the space of convergent sequences.

Extend ϕ_n to c by the same formula $\phi_n^\sim((a_k)) = a_n$.

A new nonzero character appears: $\phi_\infty((a_k)) = \lim(a_k)$.
This is the pointwise limit of the $\phi_n^\sim$, since

$$\lim_n \phi_n^\sim((a_k)) = \lim_n(a_n) = \phi_\infty((a_n)).$$

$\hat{c}$ is the one point compactification of $\mathbb{N}$.

Commutative C*-algebras

Remark

When $\mathcal{A}$ is non-abelian there may be no characters. $M_2(\mathbb{C})$ has no ideals, hence the only character is the trivial one.