Dynamical Alternating Groups

an operator-algebraic overview

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Conventions and Notation

- Γ, Λ, G, H will always denote discrete groups. Γ, Λ will always be infinite.
  $\rightarrow$ usually countable, amenable
- X, Y will always denote compact metrizable spaces.
  $\rightarrow$ almost always the Cantor space
- All actions are by homeomorphisms.

Dynamical Preliminaries

Let $\Gamma \overset{\alpha}{\curvearrowright} X$ be an action.

Definition

α is called:

- *minimal* if every orbit is dense in X.
- *topologically free* if the set of points with trivial stabilizer is dense in X.
- *expansive* if, fixing a compatible metric d on X, there exists $\varepsilon > 0$ such that for any distinct $x, y \in X$ there exists $s \in \Gamma$ satisfying $d(sx, sy) > \varepsilon$.

Dynamical Preliminaries (cont.)

Definition

Let $\Lambda \overset{\beta}{\curvearrowright} Y$ be another action. α and β are said to be *continuously orbit equivalent* if there exists a homeomorphism $\Phi : X \to Y$ and continuous maps $\kappa : \Gamma \times X \to \Lambda$ and $\lambda : \Lambda \times Y \to \Gamma$ such that

$$\Phi(\alpha_s x) = \beta_{\kappa(s,x)}\Phi(x),$$
$$\Phi^{-1}(\beta_t y) = \alpha_{\lambda(t,y)}\Phi^{-1}(y)$$

for all $x \in X, y \in Y, s \in \Gamma, t \in \Lambda$.

Topological Full Groups

Definition

The *topological full group* $[[\Gamma \curvearrowright X]]$ (also denoted by $[[\alpha]]$) of the action is the group of homeomorphisms T of X with the property that there exists a clopen partition $\{P_1, \ldots, P_n\}$ of X and elements $s_1, \ldots, s_n$ in Γ such that $T|_{P_i} = \alpha_{s_i}|_{P_i}$ for all $i \in \{1, \ldots, n\}$.

Note that $[[\alpha]]$ is countable whenever Γ is, since there are at most countably many clopen partitions of X.
TFGs where first defined in 1999 by Giordano, Putnam, and Skau for minimal Cantor systems, proving they serve as a complete invariant for flip conjugacy.
General definition given for étale groupoids by Matui in 2012.

Amenability

Grigorchuk and Medynets conjectured that the TFG of any minimal Cantor system is amenable. The conjecture was proven in the celebrated work of Juschenko and Monod in 2012. More generally, a Tits alternative type of theorem holds.

Theorem (Szőke '18)

Let Γ be finitely generated.

- If Γ is virtually cyclic, then $[[\Gamma \curvearrowright C]]$ is amenable for any minimal action on any compact Hausdorff space C.
- If Γ is not virtually cyclic, then there exists a minimal free action on the Cantor space such that the corresponding TFG contains $\mathbb{F}_2$.

Symmetric and Alternating Groups

Let $n \in \mathbb{N}$ and consider the collection of all group homomorphisms $\phi : \mathsf{S}_n \to [[\Gamma \curvearrowright X]]$ for which there exist disjoint clopen subsets $\{A_1, \ldots, A_n\}$ of X such that:

- $\phi_\sigma|_{X \setminus \cup_{i=1}^{n} A_i}$ is the identity for all $\sigma \in \mathsf{S}_n$.
- $\phi_\sigma(A_i) = A_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$ and all $\sigma \in \mathsf{S}_n$.
- For all $\sigma \in \mathsf{S}_n$ and all $i \in \{1, \ldots, n\}$, there exists $s_{\sigma,i} \in \Gamma$ such that $\phi_\sigma|_{A_i} = \alpha_{s_{\sigma,i}}|_{A_i}$.

We define $\mathsf{S}_n(\Gamma, X)$ (also denoted by $\mathsf{S}_n(\alpha)$) to be the subgroup of $[[\Gamma \curvearrowright X]]$ generated by the images of all such homomorphisms and $\mathsf{A}_n(\Gamma, X)$ (also denoted by $\mathsf{A}_n(\alpha)$) to be the subgroup of $[[\Gamma \curvearrowright X]]$ generated by the images of $\mathsf{A}_n \leq \mathsf{S}_n$ under all such homomorphisms.

Symmetric and Alternating Groups (cont.)

Definition

We define the *symmetric group* $\mathsf{S}(\Gamma, X)$ (also denoted by $\mathsf{S}(\alpha)$) and the *alternating group* $\mathsf{A}(\Gamma, X)$ (also denoted by $\mathsf{A}(\alpha)$) of the action to be $\mathsf{S}_2(\Gamma, X)$ and $\mathsf{A}_3(\Gamma, X)$, respectively.

These groups were first defined in the groupoid setting. They complete the following diagram of groups, all of which completely determine the topological groupoid $\mathfrak{G}$. In our case, they determine the action up to continuous orbit equivalence.

Symmetric and Alternating Groups (cont.)

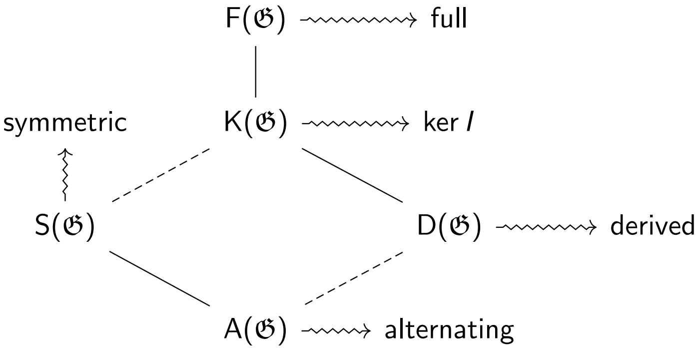

Figure: Important subgroups of the TFG.

Algebraic Properties

Theorem (Nekrashevych '15)

If α is minimal, then $\mathsf{A}(\alpha)$ is simple and is contained in every non-trivial normal subgroup of $[[\alpha]]$.

Theorem (Nekrashevych '15)

If α is expansive and every orbit has at least five points, then $\mathsf{A}(\alpha)$ is finitely generated.

Both these theorems generalise prior work of Matui from 2006, which - coupled with the work of Juschenko and Monod in 2012 - gave the first examples of finitely generated infinite simple amenable groups (in fact, a whole continuum of them).

C^*-Simplicity

Definition

G is called C^**-simple* if the reduced group C^*-algebra $C_r^*(G)$ is simple.

Theorem (Kerr, Tucker-Drob '19)

Let Γ be finitely generated and amenable. Suppose furthermore that it is either:

- torsion-free, ICC, and residually finite, or
- of the form $\Gamma_0 \times \mathbb{Z}$ where Γ_0 is torsion-free.

Then there is an uncountable family of topologically free expansive minimal actions α_λ of Γ on the Cantor space such that $\mathsf{A}(\alpha_\lambda)$ is C^*-simple for all λ, and such that the groups $\mathsf{A}(\alpha_\lambda)$ are pairwise non-isomorphic.

Amenability/C^*-Simplicity Dichotomy

We have now seen that the alternating group, even if we restrict ourselves to nice actions of nice groups, is in some cases amenable, while in others it is C^*-simple (so, in a sense, *very* non-amenable). The following theorem tells us that there is no in-between.

Theorem (Scarparo '21)

Let α be minimal, let Γ be countable and let X be the Cantor set. TFAE:

1. $\mathsf{A}(\alpha)$ is non-amenable.
2. Any group H such that $\mathsf{A}(\alpha) \leq H \leq [[\alpha]]$ is C^*-simple.
3. There exists a C^*-simple group H such that $\mathsf{A}(\alpha) \leq H \leq [[\alpha]]$.

Property (Γ)

Definition

A II_1-factor (M, τ) is said to have *property* (Γ) if for any $\varepsilon > 0$ and any finite subset $\Omega \subseteq M$ there exists a unitary $u \in M$ with $\tau(u) = 0$ and $\|[u, a]\|_2 < \varepsilon$ for all $a \in \Omega$. An ICC countable discrete group G is said to have property (Γ) if $\mathcal{L}G$ has it.

Proposition

Let X be the Cantor set and assume that the set of points with orbits of size at least 4 is dense in X. Then every subgroup of $[[\alpha]]$ containing $\mathrm{A}(\alpha)$ is ICC.

Theorem (Kerr, Tucker-Drob '19)

Let α be topologically free, let X be the Cantor space, and let Γ be countable and amenable. Then every subgroup of $[[\alpha]]$ containing $\mathrm{A}(\alpha)$ has property (Γ).

Almost Finiteness

Definition

Let $F, K \subset\subset G$ and let $\varepsilon > 0$. K is called (F, ε)-*invariant* if

$$\left| K \cap \bigcap_{g \in F} g^{-1} K \right| \geq (1 - \varepsilon)|K|.$$

Definition

Suppose that $G \overset{\alpha}{\curvearrowright} X$ is a free action of a discrete amenable group on a zero-dimensional space. We say that α is *almost finite* if for every $F \subset\subset G$ and every $\varepsilon > 0$ there exists a finite collection $\{(S_i, V_i)_{i=1}^n\}$ such that V_i are clopen subsets of X, $S_i \subset\subset G$ are (F, ε)-invariant, and

$$X = \bigsqcup_{i=1}^{n} \bigsqcup_{g \in S_i} g V_i.$$

Almost Finiteness (cont.)

The definition can be extended to spaces of arbitrary dimension, but since we will only care about finite-dimensional spaces, the following theorem guarantees that we can restrict ourselves to zero-dimensional ones.

Theorem (Kerr, Szabo '18)

Let G be a countably infinite amenable group. Suppose that all free actions of G on zero-dimensional compact metrizable spaces are almost finite. Then all free actions of G on finite-dimensional compact metrizable spaces are also almost finite.

Classifiability of Crossed Products

The completion of Elliott's classification program boils down to the following theorem.

Theorem

Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by the Elliott invariant.

If we restrict ourselves to free minimal actions $G \curvearrowright X$ of amenable groups, the only property that needs to be checked for the associated crossed product is $\mathcal{Z}$-stability. Almost finiteness was defined precisely for this reason.

Theorem (Kerr '17)

Suppose that G is infinite. Let $G \curvearrowright X$ be a free minimal action which is almost finite. Then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

Comparison

Let $G \curvearrowright X$ be an action.

Definition

Let A, B be two open subsets of X. We say that A is *(dynamically) subequivalent* to B if for any closed set $C \subseteq A$ there exists a finite open cover $\mathcal{U}$ of C and elements $(g_U)_{u \in \mathcal{U}}$ in G such that the sets $(g_U U)_{U \in \mathcal{U}}$ are pairwise disjoint and contained in B. We denote $A \precsim_G B$.

Definition

We define the *type semigroup* to be the abelian semigroup generated by symbols $\{[U] : U \subseteq X \text{ open}\}$ subject to the relations $[U \sqcup V] = [U] + [V]$ and $[U] = [gU]$ for all $g \in G$.

Comparison (cont.)

One can easily check that $\precsim_G$ extends to a partial order on the type semigroup. We will keep the same notation for this extension.

Definition

We say that the action has *comparison* if for any open sets $U, V \subseteq X$ we have

$$\mu(U) < \mu(V) \ \forall \mu \in M^G(X) \implies U \precsim_G V.$$

Similarly, the action has *comparison on multisets* if for any elements $\sum_{i=1}^n [U_i], \sum_{j=1}^m [V_j]$ in the type semigroup we have

$$\sum_{i=1}^n \mu(U_i) < \sum_{j=1}^m \mu(V_j) \ \forall \mu \in M^G(X) \implies \sum_{i=1}^n [U_i] \precsim_G \sum_{j=1}^m [V_j].$$

Almost Finiteness vs Comparison

In general, almost finiteness implies comparison, a fact already shown by Kerr in the same paper he defined the former. In fact, the same proof can be used to show that almost finiteness implies comparison on multisets. However, there is a missing piece for the converse to hold.

Theorem (Kerr, Szabo '18)

Suppose that $G \curvearrowright X$ is free and G is countably infinite and amenable. TFAE:

1. The action is almost finite.
2. The action has comparison and the small boundary property.

The TFG Case

Theorem (Naryshkin, P. '23)

Let $\Gamma \curvearrowright Z$ be a topologically free action of a countable group on an infinite zero-dimensional space Z and let $\mathsf{A}(\Gamma, Z) \leq G \leq [[\Gamma \curvearrowright Z]]$. Then, whenever G is amenable, any free action of G on a finite-dimensional space X is almost finite.

The proof relies on a modification of a recent result of Naryshkin, and on the following lemma.

Lemma (Naryshkin, P. '23)

Let $\Gamma \overset{\alpha}{\curvearrowright} Z$ be a topologically free action of a countable group on an infinite zero-dimensional space Z. Then for any $g_1, g_2, \dots, g_k \in [[\Gamma \curvearrowright Z]]$ and any $N \in \mathbb{N}$ there exist $M \geq N$ and embeddings $A_N \hookrightarrow \mathsf{A}(\alpha)$, $A_M \hookrightarrow \mathsf{A}(\alpha)$ such that $g_i A_N g_i^{-1} \subseteq A_M$ for any $i \in \{1, 2, \dots, k\}$, where we identified the groups A_N, A_M with their images in $\mathsf{A}(\Gamma, Z)$.

Sketch of Proof

1. It suffices to prove comparison in dimension zero. To that end, let $A, B \subseteq X$ be clopen sets such that $\mu(A) < \mu(B)$ for all $\mu \in M^G(X)$.
2. Choose a Følner set F in G such that the F-translates of A have measure strictly less than the F-translates of B on average, for any measure in $M(X)$.
3. Using the lemma and the fact that free actions of finite groups on zero-dimensional spaces are almost finite, we partition X into towers with sufficiently large shapes in some A_N which are F-conjugated to shapes in some $A_M, M \geq N$.
4. Using the towers, construct two particular elements of the type semigroup. One, call it $\tilde{A}$, that sits dynamically above A and one, call it $\tilde{B}$, that sits dynamically below B.
5. By the choices that have been made thus far, one shows that $\nu(\tilde{A}) < \nu(\tilde{B})$ for all $\nu \in M^{A_M}(X)$. Using almost finiteness again, this implies that $\tilde{A} \precsim_{A_M} \tilde{B}$, which finishes the proof.

The Automatic AF Conjecture

Conjecture

Every free action of an infinite amenable group on a finite-dimensional space is almost finite.

The largest class of amenable groups for which the conjecture has been proven is the class SG of subexponentially amenable groups. Dynamical alternating groups, however, are in most cases outside of this class and are not covered by past results. It is worth noting that the only known examples of amenable groups outside SG that do not relate to TFGs are the Basilica group and its generalizations (p-Basilica groups).

Thank you!