Nuclearity of group C*-algebras

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Amenability of groups

In what follows G will always be a countable discrete group. We say that G acts on a space X if

(i)
$$sx \in X$$
, for all $s \in G$
(ii) $ex = x$, if $e \in G$ is the unit.
(iii) $s_1(s_2x) = (s_1s_2)x$, for all $s_1, s_2 \in G$ and $x \in X$.

Whenever G acts on X, it also acts on the (complex valued) functions on X by $sf(x) := f(s^{-1}x)$, $x \in X$.

A group G is said to be amenable iff there exists a state ω on $\ell^{\infty}(G)$, i.e., $\|\omega\| = \omega(\underline{1}) = 1$ which is translation invariant, i.e.,

$$\omega(sf) = \omega(f),$$

for all $f \in \ell^{\infty}(G)$.

By restricting ω on the projections of $\ell^{\infty}(G)$ we obtain a translation invariant finitely additive measure on the powerset of G, denoted again as ω .

Examples of amenable groups: finite groups and direct sums of them, abelian groups.

PROPOSITION. The free group G on two generators a, b is not amenable.

Proof. If $x \in \{a, b, a^{-1}, b^{-1}\}$, then let $G_x \subseteq G$ denote the collection of all words starting with x. Clearly

$$G = G_a \cup G_b \cup G_{a^{-1}} \cup G_{b^{-1}} \cup \{e\}.$$

Now $bG_a \cup b^2G_a \subseteq G_b$ and so if G was amenable

$$2\omega(G_a) = \omega(bG_a) + \omega(b^2G_a) \le \omega(G_b) \le \omega(aG_b) \le \omega(G_a)$$

Hence $\omega(G_a) = \omega(G_b) = 0$ and by symmetry $\omega(G_{a^{-1}}) = \omega(G_{b^{-1}}) = 0$. Also $\omega(\{e\}) = 0$, a contradiction.

Tensor products on $\mathrm{C}^*\text{-}\mathsf{algebras}$

All C*-algebras will be unital.

Let $A \subseteq B(\mathcal{H})$, $B \subseteq B(\mathcal{K})$ be C*-algebras. We define their algebraic tensor product as

 $A \otimes B := \operatorname{span} \{ a \otimes b \mid a \in A, b \in B \} \subseteq B(\mathcal{H} \otimes \mathcal{K})$

and their spatial tensor product as

$$A \otimes_s B := \overline{\operatorname{span}} \{ a \otimes b \mid a \in A, b \in B \} = \overline{A \otimes B}.$$

The spatial tensor product contains a copy of A in the form $A \otimes I$ and similarly for B. If π and ρ are representations of A and B respectively on $\mathfrak{H},$ we say that they form a commuting pair iff

$$\pi(a)
ho(b) =
ho(b)\pi(a), \quad \text{for all } a \in A, b \in B$$

The collection of all commuting pairs of representations of A and B on a space \mathfrak{H} of sufficiently large cardinality is denoted as $\mathcal{C}(A, B)$.

If $(\pi, \rho) \in \mathcal{C}(A, B)$ define $\pi \times \rho : A \otimes B \to B(\mathfrak{H}); a \otimes b \longmapsto \pi(a)\rho(b), \quad a \in A, b \in B$

The maximal tensor product of A and B is defined as

$$A \otimes_m B := \overline{\operatorname{span}} \{ \oplus_{(\pi,\rho) \in \mathcal{C}} (\pi \times \rho) (a \otimes b) \mid a \in A, b \in B \}$$
$$= \overline{\operatorname{span}} \{ \oplus_{(\pi,\rho) \in \mathcal{C}} \pi(a) \rho(b) \mid a \in A, b \in B \}$$

Therefore if $(\pi, \rho) \in C$, then $\pi \times \rho$ extends to a representation of $A \otimes_m B$ by restricting on the appropriate direct summand.

Nuclearity and the extension property

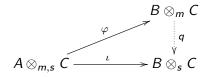
A C*-algebra A is said to be nuclear iff $A \otimes_m B \simeq A \otimes_s B$ for any other C*-algebra B.

A C*-algebra A is said to have the extension property if the maximal tensor product preserves the inclusions of A, i.e., given any C*-algebra B with $A \subseteq B$, then the natural map

$$A \otimes_m C \ni a \otimes c \longmapsto a \otimes c \in B \otimes_m C$$

is injective, for any other C^* -algebra C.

PROPOSITION. If A is nuclear, then A has the extension property. *Proof.* Consider the commuting diagram



where φ is the integrated map of the commuting pair (inclusion, identity) and exists by properties of the maximal tensor product and ι is simply inclusion.

Our goal is to show that the reduced group $\mathrm{C}^*\mbox{-algebra}$ of any non-amenable group fails the extension property and therefore is not nuclear.

Weak expectation property (WEP)

We say that a C*-algebra A enjoys WEP if for any faithful representation $\varphi : A \to B(\mathfrak{H})$, there exists a unital completely positive map E from $B(\mathfrak{H})$ on $\varphi(A)'' = \overline{\varphi(A)}^{sot}$ so that

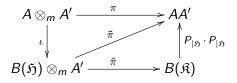
$$E(\varphi(a)x\varphi(b)) = \varphi(a)E(x)\varphi(b), \text{ for all } a, b \in A.$$

THEOREM (Lance 1972) A C^* -algebra has the extension property iff it satisfies WEP.

Proof. We will only show that the extension property implies WEP. Assume that A has been represented by φ on \mathfrak{H} and by the extension property we have the inclusion

$$A \otimes_m A' \subseteq B(\mathfrak{H}) \otimes_m A'.$$

We have a diagram



were:

The map ι is inclusion.

The map π is defined by $\pi(a \otimes a') = aa'$, $a \in A$, $a' \in A'$.

By Arveson's extension theorem, there exists a unital completely positive map $\tilde{\pi}$ extending π

By Stinespring dilation theorem, there exists a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$, and a *-homomorphism

$$\hat{\pi}: B(\mathfrak{H}) \otimes_m A' \to B(\mathfrak{K})$$

so that the above diagram commutes.

Here $P := P_{|\mathfrak{H}}$ is the projection on \mathfrak{H} and commutes with $\hat{\pi}(A \otimes_m A')$.

Define

$$E:B(\mathfrak{H}) o B(\mathfrak{H});s\longmapsto ilde{\pi}(s\otimes I)$$

Notice that if $a' \in A'$ then

$$E(s)a' = \tilde{\pi}(s \otimes I)a'$$

= $P\hat{\pi}(s \otimes I)P\hat{\pi}(I \otimes a')P$
= $P\hat{\pi}(s \otimes I)\hat{\pi}(I \otimes a')P$
= $P\hat{\pi}(s \otimes a')P = P\hat{\pi}(I \otimes a')\hat{\pi}(s \otimes I)P$
= $a'\tilde{\pi}(s \otimes I) = a'E(s),$

Hence $E(s) \in A''$. A similar calculation establishes that $E(\varphi(a)x\varphi(b)) = \varphi(a)E(x)\varphi(b)$ for all $a, b \in A$.

$\textbf{Group}\ C^*\text{-}\textbf{algebras}$

Let G be a group and $\ell^2(G)$ a Hilbert space with an orthonormal basis $\{\delta_s\}_{s\in G}$ parametrized by G.

If $s, t \in G$ then let $L_s, R_t \in B(\ell^2(G))$ be the shifts defined by

$$\begin{split} & L_s \delta_r = \delta_{sr} \\ & R_t \delta_r = \delta_{rt}, \quad \text{ for all } r \in G. \end{split}$$

Notice that R_s, L_t are unitary operators and $R_sL_t = L_tR_s$, for all $s, t \in G$.

The map $G \ni s \mapsto L_s$ is a unitary representation of G which is called the left regular representation.

The reduced group C*-algebra $C_r^*(G)$ is the closed subalgebra of $B(\ell^2(G))$ generated by all L_s , $s \in G$.

The full group C^* -algebra $C^*(G)$ is the universal C^* -algebra for all unitary representations of G.

Non-nuclearity of group $\mathrm{C}^*\text{-}\mathsf{algebras}$

THEOREM (Lance 1972) If $C_r^*(G)$ has satisfies WEP then G is amenable.

Proof. Assume that $C^*_r(G) \subseteq B(\ell^2(G))$ has WEP and let

$$E: B(\ell^2(G)) \longrightarrow \mathrm{C}^*_r(G)'' \subseteq B(\ell^2(G))$$

be the map coming from WEP.

If $f \in \ell^{\infty}(G)$ then let $M_f \in B(\ell^2(G))$ be the "diagonal" operator "multiplication by f", i.e., $M_f \delta_s = f(s) \delta_s$, $s \in G$. Define

$$\omega(f) := \langle E(M_f) \delta_e \mid \delta_e \rangle$$

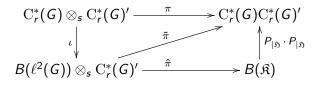
Then

$$\begin{split} \omega(f) &:= \langle E(M_f)\delta_e \mid \delta_e \rangle \\ &= \langle R_s E(M_f)\delta_e \mid R_s \delta_e \rangle \\ &= \langle E(M_f)\delta_s \mid \delta_s \rangle \\ &= \langle E(M_f)L_s \delta_e \mid L_s \delta_e \rangle \\ &= \langle L_s^* E(M_f)L_s \delta_e \mid \delta_e \rangle \\ &= \langle E(L_s^* M_f L_s)\delta_e \mid \delta_e \rangle \\ &= \langle E(M_{s^{-1}f})\delta_e \mid \delta_e \rangle \\ &= \omega(s^{-1}f) \end{split}$$

and so G is amenable.

COROLLARY. If G is the free group with two generators then $C^*_r(G)$ is not nuclear.

REMARK:



From 1972 to today

A C*-algebra A is said to be exact if for any *-homomorphism $\varphi: B \to C$ we have that the *-homomorphism

$$\varphi \otimes \mathsf{id} : B \otimes_{s} A \longrightarrow C \otimes A; b \otimes a \longmapsto \varphi(b) \otimes a$$

satisfies $\ker(\varphi \otimes \operatorname{id}) = \ker \varphi \otimes A$.

THEOREM (Kirchberg) A C^* -algebra is nuclear if and only if it has the extension property (satisfies WEP) and is exact.

A group G is called exact iff $C_r^*(G)$ is exact.

Let \mathbb{F}_∞ be the free group on countably many generators.

THEOREM (Kirchberg) A $\mathrm{C}^*\mbox{-algebra}$ has the extension property (satisfies WEP) iff

$$A \otimes_{s} \mathrm{C}^{*}(\mathbb{F}_{\infty}) \simeq A \otimes_{m} \mathrm{C}^{*}(\mathbb{F}_{\infty}).$$

CONJECTURE (Kirchberg) $\mathrm{C}^*(\mathbb{F}_\infty)$ has the extension property.

The conjecture was shown by Kirchberg to be equivalent to Connes embedding problem which was solved recently in the negative.



- E. Kirchberg, On nonsplit extensions, tensor products and exactness of group c*-algebras, Invent Math **112** (1993), 449–489
- C. Lance, On nuclear C*-algebras, J. Funct. Anal. 12 (1973), 157–176.