Nuclearity of group C^*-algebras

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Amenability of groups

In what follows G will always be a countable discrete group.
We say that G acts on a space X if

(i) $sx \in X$, for all $s \in G$

(ii) $ex = x$, if $e \in G$ is the unit.

(iii) $s_1(s_2x) = (s_1s_2)x$, for all $s_1, s_2 \in G$ and $x \in X$.

Whenever G acts on X, it also acts on the (complex valued) functions on X by $sf(x) := f(s^{-1}x)$, $x \in X$.

A group G is said to be amenable iff there exists a state ω on $\ell^\infty(G)$, i.e., $\|\omega\| = \omega(\underline{1}) = 1$ which is translation invariant, i.e.,

$$\omega(sf) = \omega(f),$$

for all $f \in \ell^\infty(G)$.

By restricting ω on the projections of $\ell^\infty(G)$ we obtain a translation invariant finitely additive measure on the powerset of G, denoted again as ω.

Examples of amenable groups: finite groups and direct sums of them, abelian groups.

PROPOSITION. The free group G on two generators a, b is not amenable.
Proof. If $x \in \{a, b, a^{-1}, b^{-1}\}$, then let $G_x \subseteq G$ denote the collection of all words starting with x. Clearly

$$G = G_a \cup G_b \cup G_{a^{-1}} \cup G_{b^{-1}} \cup \{e\}.$$

Now $bG_a \cup b^2 G_a \subseteq G_b$ and so if G was amenable

$$2\omega(G_a) = \omega(bG_a) + \omega(b^2 G_a) \leq \omega(G_b) \leq \omega(aG_b) \leq \omega(G_a)$$

Hence $\omega(G_a) = \omega(G_b) = 0$ and by symmetry $\omega(G_{a^{-1}}) = \omega(G_{b^{-1}}) = 0$. Also $\omega(\{e\}) = 0$, a contradiction.

Tensor products on C^*-algebras

All C^*-algebras will be unital.

Let $A \subseteq B(\mathcal{H})$, $B \subseteq B(\mathcal{K})$ be C*-algebras. We define their algebraic tensor product as

$$A \otimes B := \mathrm{span}\{a \otimes b \mid a \in A, b \in B\} \subseteq B(\mathcal{H} \otimes \mathcal{K})$$

and their spatial tensor product as

$$A \otimes_s B := \overline{\mathrm{span}}\{a \otimes b \mid a \in A, b \in B\} = \overline{A \otimes B}.$$

The spatial tensor product contains a copy of A in the form $A \otimes I$ and similarly for B.

If π and ρ are representations of A and B respectively on $\mathfrak{H}$, we say that they form a commuting pair iff

$$\pi(a)\rho(b) = \rho(b)\pi(a), \quad \text{for all } a \in A, b \in B$$

The collection of all commuting pairs of representations of A and B on a space $\mathfrak{H}$ of sufficiently large cardinality is denoted as $\mathcal{C}(A, B)$.

If $(\pi, \rho) \in \mathcal{C}(A, B)$ define

$$\pi \times \rho : A \otimes B \to B(\mathfrak{H}); a \otimes b \longmapsto \pi(a)\rho(b), \quad a \in A, b \in B$$

The maximal tensor product of A and B is defined as

$$\begin{aligned} A \otimes_m B &:= \overline{\text{span}}\{\oplus_{(\pi,\rho)\in\mathcal{C}}(\pi \times \rho)(a \otimes b) \mid a \in A, b \in B\} \\ &= \overline{\text{span}}\{\oplus_{(\pi,\rho)\in\mathcal{C}}\pi(a)\rho(b) \mid a \in A, b \in B\} \end{aligned}$$

Therefore if $(\pi, \rho) \in \mathcal{C}$, then $\pi \times \rho$ extends to a representation of $A \otimes_m B$ by restricting on the appropriate direct summand.

Nuclearity and the extension property

A C^*-algebra A is said to be nuclear iff $A \otimes_m B \simeq A \otimes_s B$ for any other C^*-algebra B.

A C^*-algebra A is said to have the extension property if the maximal tensor product preserves the inclusions of A, i.e., given any C^*-algebra B with $A \subseteq B$, then the natural map

$$A \otimes_m C \ni a \otimes c \longmapsto a \otimes c \in B \otimes_m C$$

is injective, for any other C^*-algebra C.

PROPOSITION. If A is nuclear, then A has the extension property.

Proof. Consider the commuting diagram

$$\begin{tikzcd} & B \otimes_m C \arrow[d, dotted, "q"] \\ A \otimes_{m,s} C \arrow[ur, "\varphi"] \arrow[r, "\iota"] & B \otimes_s C \end{tikzcd}$$

where φ is the integrated map of the commuting pair (inclusion, identity) and exists by properties of the maximal tensor product and ι is simply inclusion.

Our goal is to show that the reduced group C^*-algebra of any non-amenable group fails the extension property and therefore is not nuclear.

Weak expectation property (WEP)

We say that a C^*-algebra A enjoys WEP if for any faithful representation $\varphi : A \to B(\mathfrak{H})$, there exists a unital completely positive map E from $B(\mathfrak{H})$ on $\varphi(A)'' = \overline{\varphi(A)}^{\text{sot}}$ so that

$$E(\varphi(a)x\varphi(b)) = \varphi(a)E(x)\varphi(b), \text{ for all } a, b \in A.$$

THEOREM (Lance 1972) A C^*-algebra has the extension property iff it satisfies WEP.

Proof. We will only show that the extension property implies WEP. Assume that A has been represented by φ on $\mathfrak{H}$ and by the extension property we have the inclusion

$$A \otimes_m A' \subseteq B(\mathfrak{H}) \otimes_m A'.$$

We have a diagram

$$\begin{tikzcd}
A \otimes_m A' \arrow[rr, "\pi"] \arrow[d, "\iota"'] & & AA' \\
B(\mathfrak{H}) \otimes_m A' \arrow[urr, "\tilde{\pi}"] \arrow[rr, "\hat{\pi}"] & & B(\mathfrak{K}) \arrow[u, "P_{|\mathfrak{H}} \cdot P_{|\mathfrak{H}}"']
\end{tikzcd}$$

were:
The map ι is inclusion.
The map π is defined by $\pi(a \otimes a') = aa'$, $a \in A$, $a' \in A'$.
By Arveson's extension theorem, there exists a unital completely positive map $\tilde{\pi}$ extending π
By Stinespring dilation theorem, there exists a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$, and a $*$-homomorphism

$$\hat{\pi} : B(\mathfrak{H}) \otimes_m A' \to B(\mathfrak{K})$$

so that the above diagram commutes.
Here $P := P_{|\mathfrak{H}}$ is the projection on $\mathfrak{H}$ and commutes with $\hat{\pi}(A \otimes_m A')$.

Define

$$E : B(\mathfrak{H}) \to B(\mathfrak{H}); s \longmapsto \tilde{\pi}(s \otimes I)$$

Notice that if $a' \in A'$ then

$$\begin{aligned} E(s)a' &= \tilde{\pi}(s \otimes I)a' \\ &= P\hat{\pi}(s \otimes I)P\hat{\pi}(I \otimes a')P \\ &= P\hat{\pi}(s \otimes I)\hat{\pi}(I \otimes a')P \\ &= P\hat{\pi}(s \otimes a')P = P\hat{\pi}(I \otimes a')\hat{\pi}(s \otimes I)P \\ &= a'\tilde{\pi}(s \otimes I) = a'E(s), \end{aligned}$$

Hence $E(s) \in A''$.
A similar calculation establishes that
$E(\varphi(a)x\varphi(b)) = \varphi(a)E(x)\varphi(b)$ for all $a, b \in A$.

Group C^*-algebras

Let G be a group and $\ell^2(G)$ a Hilbert space with an orthonormal basis $\{\delta_s\}_{s\in G}$ parametrized by G.

If $s, t \in G$ then let $L_s, R_t \in B(\ell^2(G))$ be the shifts defined by

$$\begin{aligned} L_s\delta_r &= \delta_{sr} \\ R_t\delta_r &= \delta_{rt}, \quad \text{ for all } r \in G. \end{aligned}$$

Notice that R_s, L_t are unitary operators and $R_s L_t = L_t R_s$, for all $s, t \in G$.

The map $G \ni s \mapsto L_s$ is a unitary representation of G which is called the left regular representation.

The reduced group C^*-algebra $\mathrm{C}^*_r(G)$ is the closed subalgebra of $B(\ell^2(G))$ generated by all L_s, $s \in G$.

The full group C^*-algebra $\mathrm{C}^*(G)$ is the universal C^*-algebra for all unitary representations of G.

Non-nuclearity of group C^*-algebras

THEOREM (Lance 1972) If $\mathrm{C}^*_r(G)$ has satisfies WEP then G is amenable.

Proof. Assume that $\mathrm{C}^*_r(G) \subseteq B(\ell^2(G))$ has WEP and let

$$E : B(\ell^2(G)) \longrightarrow \mathrm{C}^*_r(G)'' \subseteq B(\ell^2(G))$$

be the map coming from WEP.

If $f \in \ell^\infty(G)$ then let $M_f \in B(\ell^2(G))$ be the "diagonal" operator "multiplication by f", i.e., $M_f \delta_s = f(s)\delta_s$, $s \in G$.
Define

$$\omega(f) := \langle E(M_f)\delta_e \mid \delta_e \rangle$$

Then

$$
\begin{aligned}
\omega(f) &:= \langle E(M_f)\delta_e \mid \delta_e\rangle \\
&= \langle R_s E(M_f)\delta_e \mid R_s\delta_e\rangle \\
&= \langle E(M_f)\delta_s \mid \delta_s\rangle \\
&= \langle E(M_f)L_s\delta_e \mid L_s\delta_e\rangle \\
&= \langle L_s^* E(M_f)L_s\delta_e \mid \delta_e\rangle \\
&= \langle E(L_s^* M_f L_s)\delta_e \mid \delta_e\rangle \\
&= \langle E(M_{s^{-1}f})\delta_e \mid \delta_e\rangle \\
&= \omega(s^{-1}f)
\end{aligned}
$$

and so G is amenable.

COROLLARY. If G is the free group with two generators then $\mathrm{C}^*_r(G)$ is not nuclear.

REMARK:

$$\begin{tikzcd}
\mathrm{C}^*_r(G) \otimes_s \mathrm{C}^*_r(G)' \arrow[rr, "\pi"] \arrow[d, "\iota"'] & & \mathrm{C}^*_r(G)\mathrm{C}^*_r(G)' \\
B(\ell^2(G)) \otimes_s \mathrm{C}^*_r(G)' \arrow[rru, "\tilde{\pi}"] \arrow[rr, "\hat{\pi}"] & & B(\mathfrak{K}) \arrow[u, "P_{|\mathfrak{H}} \cdot P_{|\mathfrak{H}}"']
\end{tikzcd}$$

From 1972 to today

A C^*-algebra A is said to be exact if for any $*$-homomorphism $\varphi : B \to C$ we have that the $*$-homomorphism

$$\varphi \otimes \mathrm{id} : B \otimes_s A \longrightarrow C \otimes A; b \otimes a \longmapsto \varphi(b) \otimes a$$

satisfies $\ker(\varphi \otimes \mathrm{id}) = \ker \varphi \otimes A$.

THEOREM (Kirchberg) A C^*-algebra is nuclear if and only if it has the extension property (satisfies WEP) and is exact.

A group G is called exact iff $\mathrm{C}^*_r(G)$ is exact.

Let $\mathbb{F}_\infty$ be the free group on countably many generators.

THEOREM (Kirchberg) A C^*-algebra has the extension property (satisfies WEP) iff

$$A \otimes_s \mathrm{C}^*(\mathbb{F}_\infty) \simeq A \otimes_m \mathrm{C}^*(\mathbb{F}_\infty).$$

CONJECTURE (Kirchberg) $\mathrm{C}^*(\mathbb{F}_\infty)$ has the extension property.

The conjecture was shown by Kirchberg to be equivalent to Connes embedding problem which was solved recently in the negative.

References

E. Kirchberg, *On nonsplit extensions, tensor products and exactness of group c*-algebras*, Invent Math **112** (1993), 449–489

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