Meaningful decay behavior of higher dimensional continuous wavelet transforms.

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Classical wavelet transform on $\mathbb{R}$

Fix a "good" vector $\phi \in L^2(\mathbb{R})$. Define $\phi_{b,a}(x) = \frac{1}{\sqrt{|a|}}\phi(\frac{x-b}{a})$. Then,

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}^*} \langle f, \phi_{b,a} \rangle \phi_{b,a} \frac{dbda}{a^2}.$$

- Representation
  $\pi : \mathbb{R} \rtimes \mathbb{R}^* \to \mathcal{U}(L^2(\mathbb{R})), \quad \pi(b,a)\psi\ (x) = \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}).$
- Wavelet transform
  $\mathcal{W}_\phi : L^2(\mathbb{R}) \to L^2(\mathbb{R} \rtimes \mathbb{R}^*), \quad \mathcal{W}_\phi(f)(b,a) = \langle f, \pi(b,a)\phi \rangle.$

Thm: $\mathcal{W}_\phi(f)$ decays rapidly near x_0 iff f is smooth at x_0.

$\mathcal{W}_\phi(f)$ decays rapidly near x_0 if for some nbhd $x_0 \in U$, for all $N > 0$, $|\mathcal{W}_\phi(f)(b,a)| = O(a^N)$ as $a \to 0, \forall b \in U$.

Continuous wavelet transform

- Dilation group H closed subgroup of $\mathrm{GL}_d(\mathbb{R})$.
- $G = \mathbb{R}^d \rtimes H$, with
  - $(x,h)(y,k) = (x + hy, hk), \quad (x,h)^{-1} = (-h^{-1}x, h^{-1})$.
  - $d(x,h) = \frac{dx\, dh}{|det(h)|}$.
- Quasi-regular representation of G acts on $L^2(\mathbb{R}^d)$ as

$$\pi(a,h)g(b) = \frac{1}{|\det(h)|^{1/2}} g(h^{-1} \cdot (b-a)).$$

- To construct continuous wavelet transform (CWT), fix $\psi \in L^2(\mathbb{R}^d)$, and define

$$\mathcal{W}_\psi : L^2(\mathbb{R}^d) \to C_b(G), \quad \mathcal{W}_\psi f(x,h) = \langle f, \pi(x,h)\psi \rangle.$$

A dilation group H is irreducibly admissible if the quasi-regular rep π of $\mathbb{R}^d \rtimes H$ is *irreducible* and admits an *admissible vector*.

Note: Irreducibly admissible dilation groups are characterized by action of H on $\widehat{\mathbb{R}^d}$.

Assumption

H is irreducibly admissible.

So,

- π has admissible vectors (or wavelets), i.e.

$$\exists\ \phi \in L^2(\mathbb{R}^d) \text{ such that } \mathrm{Im}(\mathcal{W}_\phi) \subseteq L^2(G).$$

- With such admissible ϕ, we have the isometry

$$\mathcal{W}_\phi : L^2(\mathbb{R}^d) \to L^2(G),\ f \mapsto \langle f, \pi(\cdot)\phi \rangle.$$

Action of H on the dual space

- $\widehat{\mathbb{R}^d} = \{\chi_b : b \in \mathbb{R}^d\}$, where $\chi_b(x) = \exp(2\pi i\, b^T x)$.
- Right action of H on $\widehat{\mathbb{R}^d}$ is defined as $\chi_b \cdot h = \chi_{h^T b}$.

Theorem [Bernier-Taylor '96, Führ '96, '10]

$$H \text{ irreducibly admissible } \Leftrightarrow \begin{cases} \text{(i) } \exists! \text{ open orbit } \mathcal{O} = H^T \xi_0, \\ \text{(ii) } H_{\xi_0} = \left\{ h \in H : h^T \xi_0 = \xi_0 \right\} \text{ is compact.} \end{cases}$$

Action of H on the dual space

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Theorem [Bernier-Taylor '96, Führ '96, '10]

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Consequences: For $\mathcal{O} \subseteq \mathbb{R}^d$ as above,

- $\mathbf{0} \notin \mathcal{O}$. $\mathcal{O}$ conull. $\forall \xi \in \mathcal{O}$, $\mathbb{R}^* \xi \subset \mathcal{O}$.
- $\mathcal{O}$ is a homogeneous H-space, so $H/H_{\xi_0} \simeq \mathcal{O}$ homeomorphic.
- The action of H on $\mathcal{O}$ is proper.
  $\forall$ cpt $K \subset \mathcal{O}$, $H_K := \{(h, \xi) \in H \times \mathcal{O}, (h^T \xi, \xi) \in K \times K\}$ is cpt.
- ϕ : Schwartz function and $\widehat{\phi}$ cptly supported $\rightsquigarrow$ admissible vector.

Recall: Let $G = \mathbb{R}^d \rtimes H$, where

- H irreducibly admissible,
- $\phi \in L^2(\mathbb{R}^d)$ admissible vector.

The CWT $\mathcal{W}_\phi : L^2(\mathbb{R}^d) \to L^2(G),\ f \mapsto \langle f, \pi(\cdot)\phi \rangle$ is an isometry.

$$\boxed{\mathcal{W}_\phi f(y,h) = \langle f, \pi(y,h)\phi \rangle.}$$

Wavelet reconstruction formula

For every $f \in L^2(\mathbb{R}^d)$,

$$f = \int_{\mathbb{R}^d} \int_H (\mathcal{W}_\phi f)(y,h) \cdot (\pi(y,h)\phi) \frac{dh}{|\det(h)|} dy,$$

interpreted in the weak sense.

Cor. $\{\pi(y,h)\phi\}_{y \in \mathbb{R}^n, h \in H}$ is a continuous frame.

Singularities/smoothness of tempered distributions

- Schwartz norms: $|\psi|_N := \max_{\alpha\in\mathbb{N}_0^d, |\alpha|\le N} \sup_{z\in\mathbb{R}^d} (1+|z|)^N |\partial^\alpha \psi(z)|$.
- Schwartz space: $\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) : |\psi|_N < \infty \ \forall N \in \mathbb{N}_0 \right\}$.
- Fourier transform $\mathcal{F} : \psi \in \mathcal{S}(\mathbb{R}^d) \to \widehat{\psi} \in \mathcal{S}(\mathbb{R}^d)$ is cts.
- Tempered distrib: $\mathcal{S}'(\mathbb{R}^d)$.
- $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $\widehat{T}(\psi) = T(\widehat{\psi})$.

Note: If $u \in \mathcal{S}'$ is compactly supported, then $\widehat{u}$ is a smooth function.

Schwartz's Paley-Weiner Theorem

Take a compactly supported $u \in \mathcal{S}'$. TFAE:

- u is a *smooth function*.
- $\widehat{u}$ is *fast decreasing* on $\mathbb{R}^d$, i.e.

$$\forall N \ \exists C_N \ \forall \xi \in \mathbb{R}^d, \ |\widehat{u}(\xi)| \le C_N (1+|\xi|)^{-N}.$$

Examples of smoothness/singularity

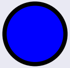
Point singularity

$\delta_{(0,0)} : f \mapsto f(0,0)$
$S = \{(0,0)\}$

Linear singularity

$L : f \mapsto \int_{\mathbb{R}} f(x,0)dx$
$S = \{(x,0) : x \in \mathbb{R}\}$

Curve singularity



$B : f \mapsto \int_{B_1} f(x,y)dxdy$
$S = \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}$

Classical wavelet transform identifies location of singularities.

For a "good" wavelet ϕ, $\mathcal{W}_\phi(F)$ decays rapidly near x as $a \to 0$ iff F is smooth at x.

Wavefront sets

Recall: Compactly supported u is smooth iff $\widehat{u}$ is fast decreasing, i.e.

$$\forall N \ \exists C_N \ \forall \xi \in \mathbb{R}^d, \ |\widehat{u}(\xi)| \leq C_N(1+|\xi|)^{-N}.$$

Fix $u \in \mathcal{S}'(\mathbb{R}^d)$ and $N \in \mathbb{N}$.

Smoothness of order N at x_0 in direction ξ_0

$(x_0, \xi_0) \in \mathbb{R}^d \times S^{d-1}$ is an N-regular directed point of u if

- $\exists \ \varphi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \equiv 1$ around x_0,
- $\exists$ open $\xi_0 \in W \subset S^{d-1}$,
- $\exists$ constant $C_N > 0$,

such that $|\widehat{\varphi u}(\xi)| \leq C_N(1+|\xi|)^{-N}$, $\forall \ \xi$ in the cone of W.

N-wavefront set of u

$WF^N(u) = \{(x,\xi)$ which are not N-regular directed points of $u\}$.

Definition

$WF(u) = \Big\{(x,\xi) : \text{ not } N\text{-regular directed point for some } N\Big\}.$

Example

- $WF(\delta) = (0,0) \times [0,2\pi) = \{((0,0),t) : t \in [0,2\pi)\}.$
- $WF(L) = \{((0,y),0) : \ y \in \mathbb{R}\}.$
- $WF(B) = \{((\cos\theta, \sin\theta),\theta) : \ \theta \in [0,2\pi)\}.$

Question: Can we recognize wavefront sets using decay behavior of $\mathcal{W}_\phi$?

Example (Candès-Donoho, '03)

- Curvelet transform $\Gamma(f)(a, b, \theta)$ is a directional transform.
- Does not have affine structure.

Curvelet identifies (location, direction) of singularities.

$\Gamma(f)$ decays rapidly near (x_0, θ_0) iff $(x_0, \theta_0) \in WF(f)^c$.

Example (Bernier-Taylor '96, Kutyniok-Labate '09)

- $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{\frac{1}{2}} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$.
- π quasi-regular rep of $\mathbb{R}^2 \rtimes H$, $\phi \in L^2(\mathbb{R}^2)$ admissible.
- $\mathcal{S}_\phi(f)(x, a, b) = \langle f, \pi(x, a, b)\phi \rangle$, $x \in \mathbb{R}^2, a > 0, b \in \mathbb{R}$.

[Kutyniok-Labate '09]: Shearlets can resolve wavefront sets.

Generalization

Setup:

- $G = \mathbb{R}^d \rtimes H$, where H is irreducibly admissible.
- $\mathcal{W}_\psi f(x,h) = \langle f, \pi(x,h)\psi \rangle$, for $f \in L^2(\mathbb{R}^d)$.
- Assume $\psi \in \mathcal{S}(\mathbb{R}^d)$. For $u \in \mathcal{S}'(\mathbb{R}^d)$, we define

$$\mathcal{W}_\psi u(x,h) = \langle u \mid \pi(x,h)\psi \rangle.$$

Goal

Give criteria for $WF^N(u)$ in terms of CWT decay, e.g.

$$(x,\xi) \notin WF^N(u) \quad \Leftrightarrow \quad \exists \text{ nbhd } U \ni x \text{ s.t. } \forall\, y \in U \ \forall\, h \in K \quad |\mathcal{W}_\psi u(y,h)| \le C\|h\|^N,$$

for suitable $K \subset H$ depending on ψ and ξ.

Recall: (x_0, ξ_0) is an N-regular directed point of u if $\forall\, \xi$ near ξ_0,

$$|\widehat{\varphi u}(\xi)| \le C_N(1+|\xi|)^{-N}.$$

Goal: criteria for $WF^N(u)$ in terms of CWT

$$(x,\xi_0) \notin WF^N(u) \quad \Leftrightarrow \quad \exists \text{ nbhd } U \ni x \text{ s.t. } \forall y \in U \ \forall h \in K \\ |\mathcal{W}_\psi u(y,h)| \le C\|h\|^N,$$

for suitable subset $K \subset H$ is depending on ψ and ξ_0.

- $[\pi(x,h)f](y) = |\det(h)|^{-1/2} \cdot f\left(h^{-1}(y-x)\right).$
- $[\mathcal{F}(\pi(x,h)f)](\xi) = |\det(h)|^{1/2} \cdot e^{-2\pi i\langle x,\xi\rangle} \cdot \widehat{f}\left(h^T\xi\right).$

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d}\int_H \left(W_\psi u\right)(y,h) \cdot \left(\mathcal{F}\left[\pi(y,h)\psi\right]\right)(\xi) \frac{\mathrm{d}h}{|\det(h)|}\,\mathrm{d}y.$$

So,

- $K \sim \left\{h \in H : \operatorname{supp}(\widehat{\psi}(h^T\cdot) \subseteq \text{ nbhd of } \xi_0\right\}.$
- K depends on $\operatorname{supp}(\psi)$ and nbhd of ξ_0.

Resolution of Wavefront set

1. Let $V \subseteq \mathcal{O}$ be open and precompact.
2. Assume that dual action is V-microlocally admissible.
3. Take any ψ admissible with $\mathrm{supp}(\widehat{\psi}) \subset V$. (similar to shearlets)

[Fell-Führ-Voigtlaender, '16]

Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $(x, \xi) \in \mathbb{R}^d \times (\mathcal{O} \cap S^{d-1})$.

(i) Suppose $(x, \xi) \in WF(u)^c$. Then,
$\exists U \ni x \;\exists R > 0 \;\exists W \ni \xi$ s.t. for all ψ as above,

$$\forall N \in \mathbb{N} \;\forall y \in U \;\forall h \in K_i \;|\mathcal{W}_\psi u(y, h)| \leq \|h\|^N.$$

(ii) Suppose $U \ni x \;\exists R > 0 \;\exists W \ni \xi$ s.t. for all ψ as above,

$$\forall N \in \mathbb{N} \;\forall y \in U \;\forall h \in K_o \;|\mathcal{W}_\psi u(y, h)| \leq \|h\|^N.$$

Then $(x, \xi) \in WF(u)^c$.

Near-characterization of WF^N; Case 1

1. Let $V \subseteq \mathcal{O}$ be open and precompact.
2. Dual action is V-microlocally admissible with $\alpha_1, \alpha_2 > 0$.
3. Take any admissible ψ with $\mathrm{supp}(\widehat{\psi}) \subset V$. (localized in frequency domain)

Theorem 1 [Führ-G.]

Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $(x, \xi) \in \mathbb{R}^d \times (\mathcal{O} \cap S^{d-1})$.

(i) Suppose $(x, \xi) \in WF^N(u)^c$. Then,
$\exists U \ni x \ \exists R > 0 \ \exists W \ni \xi$ s.t. for all ψ as above,

$$\exists C > 0 \ \ \forall\, y \in U \ \ \forall\, h \in K_i \ \ |\mathcal{W}_\psi u(y, h)| \le C \|h\|^{N - \alpha_1 d/2}.$$

(ii) Suppose $\exists U \ni x \ \exists R > 0 \ \exists W \ni \xi$ s.t. for all ψ as above,

$$\exists C > 0, \forall\, y \in U \ \ \forall\, h \in K_o \ \ |\mathcal{W}_\psi u(y, h)| \le C \|h\|^{\alpha_1 N + \frac{3}{2}\alpha_1 d + \alpha_2}.$$

Then $(x, \xi) \in WF^N(u)^c$.

Closer look at conditions on ψ

Kutyniok-Labate, '09

If $\operatorname{supp}(\widehat{\psi}) \subseteq$ compact canonical wedge then $\mathcal{S}_\psi$ resolves WF.

Grohs, '11

If ψ has ∞-ly many vanishing moments then $\mathcal{S}_\psi$ resolves WF.

Corollary: If $\psi \in C_c(\mathbb{R}^d)$ then $\mathcal{S}_\psi$ resolves WF.

Vanishing moments in higher dimensions

$\psi \in L^1(\mathbb{R}^d)$ has vanishing moments in $\mathcal{O}^c$ of order r if

1. $\partial^\alpha \widehat{\psi}$ are continuous for all $|\alpha| \leq r$,
2. $\partial^\alpha \widehat{\psi}$ are vanishing on $\mathcal{O}^c$ for all $|\alpha| < r$.

Near-characterization of WF^N; Case 2

1. Let $V \subseteq \mathcal{O}$ open and precompact.
2. Dual action is V-microlocally admissible with $\alpha_1, \alpha_2 > 0$.
3. Take any admissible $\psi \in C_c(\mathbb{R}^d)$ (localized in space domain) with vanishing moments of order r on $\mathcal{O}^c$.

Theorem 2 [Führ-G.]

Let r be "big enough". For every $u \in \mathcal{S}'(\mathbb{R}^d)$,

(i) If $(x_0, \xi_0) \in WF^N(u)^c$ then $\exists\ U \ni x_0,\ R > 0,\ W \ni \xi_0$, s.t.

$$\forall \psi\ \exists C > 0\ \ \forall y \in U\ \ \forall h \in K_i\ |\mathcal{W}_\psi u(y,h)| \leq C\|h\|^{N - \alpha_1 d/2}.$$

(ii) If $\exists\ U \ni x_0,\ R > 0,\ W \ni \xi_0$, s.t. for all ψ as above,

$$\exists C > 0, \forall y \in U\ \ \forall h \in K_o\ |\mathcal{W}_\psi u(y,h)| \leq C\|h\|^{\alpha_1 N + \frac{3}{2}\alpha_1 d + \alpha_2},$$

then $(x_0, \xi_0) \in WF^N(u)^c$.

Sketch of proof

- From CWT reconstruction formula,

$$\mathcal{W}_{\psi_1} u = \mathcal{W}_{\psi_2} u * \mathcal{W}_{\psi_1} \psi_2.$$

- If ψ_1 and ψ_2 have r-vanishing moments, for large r, then

$$\left|\mathcal{W}_{\psi_1}\psi_2(x,h)\right| \leq C(1+|x|)^{-\beta_1}(1+\|h\|)^{-\beta_2}(1+\|h^{-1}\|)^{-\beta_3},$$

where C depends on ψ_1, ψ_2, and does not depend on x, h.

- Take ψ_2 with compact frequency support. Use Theorem 1.

Thank you very much!