Normalizers & Approximate Units for Inclusions of C^*-Algebras

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Functional Analysis and Operator Algebras in Athens
April 8, 2022

Inclusions

Here are the objects of interest in today's talk.

Definition

An *inclusion* is a pair of C^*-algebras $(\mathcal{C}, \mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{D}$ abelian.

Definition

The inclusion $(\mathcal{C}, \mathcal{D})$

- is a *MASA inclusion* if $\mathcal{D}$ is a MASA in $\mathcal{C}$;
- has the *approximate unit property (AUP)* if $\mathcal{D}$ contains an approximate unit for $\mathcal{C}$.
- is *regular* if the set of *normalizers*
$$\mathcal{N}(\mathcal{C}, \mathcal{D}) := \{v \in \mathcal{C} : v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}\}$$
has dense span in $\mathcal{C}$;
- is *singular* if $\mathcal{N}(\mathcal{C}, \mathcal{D}) = \mathcal{D}$.

Cartan Inclusions

Among the nicest inclusions are Cartan inclusions.

Definition (Renault)

$(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion if

- $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion;
- $\exists$ a faithful conditional expectation $\mathbb{E} : \mathcal{C} \to \mathcal{D}$; and
- $(\mathcal{C}, \mathcal{D})$ has the AUP.

- Renault introduced these as a C^*-analog of Cartan MASA in W^*-algebras (Feldman-Moore); Cartan incln's extend theory of C^*-diagonals (Kumjian). Cartan inclusions have a groupoid model which makes them "fancy matrix algebras".
- Renault included the AUP condition in definition b/c the groupoid models have it & it appears needed due to examples similar to those I'll discuss now.

Some Examples With & Without AUP

Examples

Let $\mathcal{H} = L^2(\mathbb{T})$ with o.n. basis $\{\zeta_n\}_{n\in\mathbb{Z}}$ where $\zeta_n(z) = z^n$, put

$$\mathcal{M} := \{\text{multiplication op's on } \mathcal{B}(\mathcal{H}) \text{ by } f \in C(\mathbb{T}) : f(1) = 0\}$$

$$\mathcal{D}_a := \overline{\text{span}}\{\zeta_n\zeta_n^* : n \in \mathbb{Z}\} \quad \text{(the subscript in } \mathcal{D}_a \text{ denotes "atomic").}$$

Then $(\mathcal{M} + \mathcal{K}(\mathcal{H}), \mathcal{D}_a)$ and $(\mathcal{M} + \mathcal{K}(\mathcal{H}), \mathcal{M})$ are MASA inclusions, but

1. $(\mathcal{M} + \mathcal{K}(\mathcal{H}), \mathcal{D}_a)$ does not have AUP and is neither regular nor singular,
2. $(\mathcal{M} + \mathcal{K}(\mathcal{H}), \mathcal{M})$ has the AUP and is singular.

A modification of (2) gives a singular MASA incl'n w/o AUP:

Example: A Singular MASA Inclusion Without the AUP

With $\mathcal{M}$ and $\mathcal{H}$ as above, let $\{\xi_k : k \in \mathbb{N}\} \subseteq \mathcal{H} \setminus \{0\}$ be dense in $\mathcal{H}$.

Put $p_n = \operatorname{proj} \mathbb{C}\xi_n$, set $P = \bigoplus_{n\in\mathbb{N}} p_n$ and for $T \in \mathcal{M}$, let $\tilde{T} = \bigoplus_{n\in\mathbb{N}} T$. Let

$$\mathcal{D} := \{\tilde{T} : T \in \mathcal{M}\} \quad \text{and} \quad \mathcal{C} := C^*(\{P\} \cup \mathcal{D}).$$

Fact

$(\mathcal{C}, \mathcal{D})$ *is a singular MASA inclusion without the AUP.*

(If $(\tilde{u}_\lambda)$ is an a.u. for $\mathcal{D}$, then

$$\|P - \tilde{u}_\lambda P\| = \sup_n \|(I - u_\lambda)p_n\| = \|I - u_\lambda\| = 1$$

b/c $u_\lambda(1) = 0$.)

Characterizing the AUP for Regular Incl'ns

Proposition

Let $(\mathcal{C}, \mathcal{D})$ *be an inclusion.*

1. $(\mathcal{C}, \mathcal{D})$ *has AUP* $\Rightarrow \forall\, v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), v^*v \in \mathcal{D}$.
2. *If* $(\mathcal{C}, \mathcal{D})$ *is regular &* $v^*v \in \mathcal{D}\, \forall\, v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, *then* $(\mathcal{C}, \mathcal{D})$ *has AUP.*

So: a reg. incl'n $(\mathcal{C}, \mathcal{D})$ *has AUP* $\Leftrightarrow v^*v \in \mathcal{D}\, \forall\, v \in \mathcal{N}(C, \mathcal{D})$.

The proof is easy:

(1) If (u_λ) an a.u. for $\mathcal{D}$ & an a.u. for $\mathcal{C}$, then

$$v^*v = \lim_\lambda v^*u_\lambda v \in \mathcal{D}.$$

(2) Let (u_λ) be an a.u. for $\mathcal{D}$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. As $v^*v \in \mathcal{D}$, get

$$\|vu_\lambda - v\|^2 = \|(u_\lambda v^* - v^*)(vu_\lambda - v)\| \to 0.$$

By regularity, (u_λ) an a.u. for $\mathcal{C}$. $\square$

A Commutation Result

For a general incl'n $(\mathcal{C}, \mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, v^*v may not belong to $\mathcal{D}$ (e.g. when $\mathcal{D}$ is a proper ideal of $\mathcal{C} = C_0(\mathbb{R})$). However,

Proposition (The Commutation Prop'n)

Let $(\mathcal{C}, \mathcal{D})$ be any inclusion, $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Then for every $d \in \mathcal{D}$,

$$v^*vd = dv^*v \in \mathcal{D} \quad \textit{and} \quad vv^*d = dvv^* \in \mathcal{D}.$$

Also, if ρ_1, ρ_2 are states on $\mathcal{C}$ such that $\rho_1|_{\mathcal{D}} = \rho_2|_{\mathcal{D}} \in \hat{\mathcal{D}}$, then

$$\rho_1(v^*v) = \rho_2(v^*v) \quad \& \quad \rho_1(vv^*) = \rho_2(vv^*).$$

I'll sketch the proof, then give a number of consequences.
Proof uses only standard operator theory.

Sketch of Proof

WLOG assume $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ & let $h = v^*v$. Since $h \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\forall d \in \mathcal{D}$,

$$(d^*hd)^2 = d^*hd\, d^*hd \in \mathcal{D} \Rightarrow d^*hd \in \mathcal{D}.$$

For (u_λ) an approx unit for $\mathcal{D}$,

$$Q := \text{sot-}\lim u_\lambda = \text{proj}\, \overline{\mathcal{D}\mathcal{H}} \quad \& \quad Qd = dQ = d.$$

Gives $QhQ = \lim u_\lambda h u_\lambda \in \mathcal{D}'$. Also $QhQ^\perp = 0$ b/c

$$Q^\perp hQhQ^\perp = \text{sot-}\lim Q^\perp (hu_\lambda h) Q^\perp = 0.$$

Thus, $Qh = hQ = QhQ \in \mathcal{D}'$. Then

$$dh = d(Qh) = (Qh)d = hd.$$

- For $0 \leq f \in \mathcal{D}$, $(fh)^2 = f^2h^2 = hf^2h \in \mathcal{D}$, so $fh \in \mathcal{D}$.
- Last statement follows from an application of C.S. inequality:

$$\rho \in \mathcal{S}(\mathcal{C})\ \&\ \rho|_{\mathcal{D}} \in \hat{\mathcal{D}} \Rightarrow \forall\, d \in \mathcal{D},\ \rho(dx) = \rho(d)\rho(x).$$

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Consequence: "Non-commutative Compatifications"

For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, in general $(v, 0) \notin \mathcal{N}(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$. But

Corollary

Let $(\mathcal{C}, \mathcal{D})$ be a MASA incl'n.

1. *For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $(v, 0) \in \mathcal{N}(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ (b/c $v^* v \in \mathcal{D}$)*
2. *Suppose further $(\mathcal{C}, \mathcal{D})$ has AUP, $\mathcal{B}$ unital and $\mathcal{C} \trianglelefteq \mathcal{B}$ is essential. Let $\mathcal{D}_{\mathcal{B}} := M(\mathcal{D}) \cap \mathcal{B}$.*
   Then $(\mathcal{B}, \mathcal{D}_{\mathcal{B}})$ is a MASA incl'n and $\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{B}, \mathcal{D}_{\mathcal{B}})$.

A method for constructing singular MASA inclusions:

Fact (Unital case is in Exel-P-Zarikian, non-unital case uses part (1) of Corollary.)

Suppose $(\mathcal{C}, \mathcal{D})$ a MASA inclusion & $J \trianglelefteq \mathcal{C}$ with $J \cap \mathcal{D} = (0)$. Then $(\mathcal{D} + J, \mathcal{D})$ is a singular MASA inclusion.

Consequence: Dynamical Objects—partial automorphism

Let $(\mathcal{C}, \mathcal{D})$ be any inclusion and fix $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$.

The Partial Automorphism Associated to v

Let B be an AW^*-algebra with $\mathcal{C} \subseteq B$ and let $v = u|v| = |v^*|u$ be the polar decomposition of v in B. Then

- $\overline{vv^*\mathcal{D}}$ and $\overline{v^*v\mathcal{D}}$ are ideals in $\mathcal{D}$ and
- the map $vv^*d \mapsto v^*dv$ uniquely extends to a $*$-isomorphism $\theta_v : \overline{vv^*\mathcal{D}} \to \overline{v^*v\mathcal{D}}$ such that for each $h \in \overline{vv^*\mathcal{D}}$,

$$v\theta_v(h) = hv \quad \text{and} \quad u^*hu = \theta_v(h).$$

Consequence: Dynamical Objects—partial homeomorphism

Dually, have

The Partial Homeomorphism Associated to v

The sets $\operatorname{dom} v := \{\sigma \in \hat{\mathcal{D}} : \sigma(v^* v\mathcal{D}) \neq 0\}$ *and* $\operatorname{range} v := \{\sigma \in \hat{\mathcal{D}} : \sigma(vv^*\mathcal{D}) \neq 0\}$ *are open subsets of* $\hat{\mathcal{D}}$ *and* $\exists$ *a homeomorphism* $\beta_v : \operatorname{dom} v \to \operatorname{range} v$ *such that for every* $h \in \overline{vv^*\mathcal{D}}$ *and* $\sigma \in \operatorname{dom} v$,

$$\beta_v(\sigma)(h) = \sigma(\theta_v(h)).$$

For $\sigma \in \operatorname{dom} v$, *define* $\sigma(v^* v) := \rho(v^* v)$, *where* ρ *is any extension of* σ *to a state on* $\mathcal{C}$. *Then* $\sigma(v^* v) \neq 0$ *and for* $d \in \mathcal{D}$,

$$\beta_v(\sigma)(d) = \frac{\sigma(v^* dv)}{\sigma(v^* v)}.$$

Consequence: Reg. MASA Incl'ns have AUP

Corollary

If $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, then $(\mathcal{C}, \mathcal{D})$ has the AUP.

Proof.

For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, commutation prop'n gives $v^*v \in \mathcal{D}' \cap \mathcal{C}$, so $v^*v \in \mathcal{D}$ (b/c $\mathcal{D}$ a MASA).
By AUP characterization, $(\mathcal{C}, \mathcal{D})$ has AUP. $\square$

Remark: If $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion with $\mathcal{C}$ unital $\exists!$ u.c.p. $\Delta : \mathcal{C} \to I(\mathcal{D})$ with $\Delta|_{\mathcal{D}} = id|_{\mathcal{D}}$ (Δ is pseudo-expectation).
When $\Delta(\mathcal{C}) \subseteq \mathcal{D}$, Δ is a cond. expectation.
For each

$$(\clubsuit, \heartsuit) \in \left\{ \begin{matrix} \text{cond. expectation} \\ \text{not cond. expectation} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{faithful} \\ \text{not faithful} \end{matrix} \right\}$$

$\exists$ a regular MASA inclusion $(\mathcal{C}, \mathcal{D})$ such that Δ has property $\clubsuit$ and $\heartsuit$.

Simplified Definition of Cartan Inclusions

Since regular MASA inclusions have the AUP, we get:

Simplified Definition of Cartan Inclusion

$(\mathcal{C}, \mathcal{D})$ *is a* *Cartan inclusion* *if*

- $(\mathcal{C}, \mathcal{D})$ *is a regular MASA inclusion; and*
- $\exists$ *a faithful cond. expect.* $\mathbb{E} : \mathcal{C} \to \mathcal{D}$.

What about C^*-Diagonals?

Definition (Kumjian)

A normalizer $v \in N(\mathcal{C}, \mathcal{D})$ is free if $v^2 = 0$.

$$N_f(\mathcal{C}, \mathcal{D}) := \{\text{free normalizers}\}$$

Definition

An inclusion $(\mathcal{C}, \mathcal{D})$ *satisfies Kumjian's Conditions* if:

(I) $\exists$ a faithful conditional expectation $\mathbb{E} : \mathcal{C} \to \mathcal{D}$; and

(II) $\ker \mathbb{E} = \overline{\text{span}} N_f(\mathcal{C}, \mathcal{D})$.

Definition (Kumjian)

The incl'n $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal if when

$\mathcal{C}$ unital, $(\mathcal{C}, \mathcal{D})$ satisfies Kumjian's conditions;

$\mathcal{C}$ non-unital, the unitization $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ satisfies Kumjian's conditions.

Unital C^*-Diagonals & Extension Property

In unital setting, C^*-diagonals are Cartan inclusions with extension property:

Fact A ($\Rightarrow$ due to Kumjian, converse due to ?)

When $\mathcal{C}$ UNITAL, $(\mathcal{C}, \mathcal{D})$ *a* C^**-diagonal* $\Leftrightarrow$ $(\mathcal{C}, \mathcal{D})$ *is Cartan & has extension property, i.e.* $\forall\, \sigma \in \hat{\mathcal{D}}$, $\exists!\, \sigma' \in \mathcal{S}(\mathcal{C})$ *with* $\sigma'|_{\mathcal{D}} = \sigma$.

Having Fact A in the non-unital context would lead to streamlined def'n of C^*-diagonals.

Consequence: Free Normalizers

Corollary

Suppose $v \in N_f(\mathcal{C}, \mathcal{D})$ and ρ a state on $\mathcal{C}$ s.t. $\rho|_{\mathcal{D}} \in \hat{\mathcal{D}}$. Then

$$\rho(v) = 0.$$

Proof.

Let $d \in \mathcal{D}$ such that $\rho(d) = 1$. Since $v^*vd, dvv^* \in \mathcal{D}$

$$\rho(v^*v)\rho(vv^*) = \rho(dv^*v)\rho(vv^*d) = \rho(dv^*vvv^*d) = 0.$$

By Cauchy-Schwartz,

$$|\rho(v)|^2 \leq \min\{\rho(v^*v), \rho(vv^*)\} = 0.$$

$\square$

Consequence: Characterizations of Kumjian's Conditions

Corollary on $N_f(\mathcal{C}, \mathcal{D})$ leads to:

Proposition

Suppose $\mathcal{C}$ not unital. TFAE

1. $(\mathcal{C}, \mathcal{D})$ *satisfies Kumjian's conditions.*
2. $(\mathcal{C}, \mathcal{D})$ *is a Cartan inclusion such that every pure state of $\mathcal{D}$ has a unique extension to a state on $\mathcal{C}$.*
3. $(\mathcal{C}, \mathcal{D})$ *is a Cartan inclusion such that every pure state of $\mathcal{D}$ has a unique extension to a state on $\mathcal{C}$ and no pure state of $\mathcal{C}$ annihilates $\mathcal{D}$ (e.g. has Archbold-Bunce-Gregson's E.P.).*
4. $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ *is a Cartan inclusion such that every pure state of $\mathcal{D}$ extends uniquely to a state on $\mathcal{C}$.*
5. $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ *satisfies Kumjian's conditions.*

On the Definition of C^*-Diagonal

Fact A for non-unital case is (1)$\Leftrightarrow$ (2) in previous prop'n, so restating, we have:

Streamlined Definition of C^*-Diagonal

Whether unital or not, $(\mathcal{C}, \mathcal{D})$ a C^*-diagonal if

1. $(\mathcal{C}, \mathcal{D})$ *satisfies Kumjian's conditions; or equivalently,*
2. $(\mathcal{C}, \mathcal{D})$ *is a Cartan inclusion such that every pure state on $\mathcal{D}$ extends uniquely to a state on $\mathcal{C}$.*

Consequence: Unitizations of C^*-Diagonals

(1)$\Leftrightarrow$(5) of previous Proposition gives,

Fact

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion with $\mathcal{C}$ non-unital. Then $(\mathcal{C}, \mathcal{D})$ a C^*-diagonal $\Leftrightarrow$ $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ is a C^*-diagonal.

What about Cartan inclusions? NOT TRUE!

Example ($(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ Cartan $\not\Rightarrow$ $(\mathcal{C}, \mathcal{D})$ Cartan)

Let $\mathcal{C} = C^*(S)$ (Toeplitz Alg), $\mathcal{D} = C^*(\{S^n S^{*n} \cup \{I\})$ & $q : \mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{K} = C(\mathbb{T})$. For $\lambda \in \mathbb{T}$, let $\tau_\lambda(x) = q(x)(\lambda)$, note τ_λ multiplicative. Put

$$\mathcal{C}_\lambda = \ker \tau_\lambda, \quad \mathcal{D}_\lambda = \ker \tau_\lambda \cap \mathcal{D} = \mathcal{C}_\lambda \cap \mathcal{D}.$$

Then $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ a MASA incl'n, but doesn't have AUP (b/c $S - \lambda I \notin \mathcal{K}$), so $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ not regular. Thus $(\mathcal{C}_\lambda, \mathcal{D}_\lambda)$ not Cartan, but $(\tilde{\mathcal{C}}_\lambda, \tilde{\mathcal{D}}_\lambda) \simeq (\mathcal{C}, \mathcal{D})$ is Cartan.

Consequence: Unitization of Cartan Inclusions

Adding the hypothesis of regularity we get:

Fact

Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion with $\mathcal{C}$ non-unital. Then $(\mathcal{C}, \mathcal{D})$ a Cartan inclusion $\Leftrightarrow$ $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ a Cartan inclusion.

Sketch of proof.

$(\Leftarrow)$: Let $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ be Cartan with cond. expect $\mathbb{E} : \tilde{\mathcal{C}} \to \tilde{D}$.

- Then $(\mathcal{C}, \mathcal{D})$ a regular MASA incl'n, so has AUP
- As $\mathbb{E}$ is a $\mathcal{D}$-module map, AUP gives $\mathbb{E}(\mathcal{C}) = \mathcal{D}$.
- Define $E = \mathbb{E}|_{\mathcal{C}}$ to get faithful cond. expect. of $\mathcal{C}$ onto $\mathcal{D}$.

$(\Rightarrow)$: This is routine. $\square$

Consequence: Intermediate Subalgebras

Intermediate algebras (maybe nonselfadjoint) have AUP:

Corollary

Suppose $(\mathcal{C}, \mathcal{D})$ *a reg. MASA incl'n, and* $\mathcal{A}$ *a norm-closed alg with* $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$*. Then* $(\mathcal{A}, \mathcal{D})$ *has AUP.*

A Problem: Intermediate Subalgebras

Question

When is an incl'n $(\mathcal{C}, \mathcal{D})$ intermediate to a regular MASA inclusion (i.e. $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{B}$, where a $(\mathcal{B}, \mathcal{D})$ reg. MASA incl'n)?

A unital, regular MASA inclusion $(\mathcal{B}, \mathcal{D})$ has the unique pseudo-expectation property (i.e. $\exists!$ u.c.p. map $E : \mathcal{B} \to I(\mathcal{D})$ with $E|_{\mathcal{D}} = \mathrm{id}$). This passes to intermediate inclusions, so we get

Necessary Conditions

- $(\mathcal{C}, \mathcal{D})$ *has the AUP*
- $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ *has the unique pseudo-expectation property*

An Interesting Example

Example (Based on an example in Exel-P-Zarikian)

Fix $n \geq 3$, $\Gamma = SL_n(\mathbb{Z})$ acting on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by matrix multiplication, μ normalized Haar meas on $\mathbb{T}^n$. On $\mathcal{H} = L^2(\mathbb{T}^n, \mu)$, let $(U_s f)(t) = f(s^{-1} \cdot t)$ & let

$$\mathcal{D} := \{\text{multiplication op's by } f \in C(\mathbb{T}^n)\} \text{ \& } \mathcal{C} := C^*(\mathcal{D}, \{U_s\}_{s\in\Gamma}).$$

Then

- $(\mathcal{C}, \mathcal{D})$ a reg. MASA incl'n (mostly b/c action of Γ on $\mathbb{T}^n$ is top. free);
- Γ has prop. (T) $\Rightarrow \mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}$ (Chau-Lau-Rosenblatt).

So $\mathcal{D} \subseteq \mathcal{D} + \mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}$, i.e.
$(\mathcal{D} + \mathcal{K}(\mathcal{H}), \mathcal{D})$ a singular MASA incl'n intermediate to a reg. MASA incl'n.

Also, $\exists$ free action of Γ on Cantor set admitting inv. measure (Elek, '21) $\Rightarrow \exists$ incl'n $(\mathcal{C}_\kappa, \mathcal{D}_\kappa)$ having ext. property & $\mathcal{K} \subseteq \ker \mathbb{E}$.

A Related Example

Minor modifications to previous example give:

Example (E-P-Z)

Let $\mathfrak{D} \subseteq \mathcal{B}(\mathcal{H})$ be a non-atomic MASA. Set $\mathfrak{C}_0 = \operatorname{span} \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathfrak{D})$ and $\mathfrak{C} = \overline{\operatorname{span}} \, \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathfrak{D})$. Then

1. $(\mathfrak{C}, \mathfrak{D})$ has the pure state extension property;
2. $\mathfrak{C}_0 \cap \mathcal{K}(\mathcal{H}) = (0)$; but
3. $\mathcal{K}(\mathcal{H}) \subseteq \mathfrak{C}$.

Thus: $(\mathfrak{D} + \mathcal{K}(\mathcal{H}), \mathfrak{D})$ is a singular MASA inclusion intermediate to a regular MASA inclusion $(\mathfrak{C}, \mathfrak{D})$ having extension prop.

Remark: Item (3) affirmatively answers a question raised by Paulsen & Katavolos in *On ranges of bimodule projections* Canad. Math. Bull. 2005.

A Test Question

Do these kinds of examples work more generally?

Test Question

If $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ *&* $\mathcal{D}''$ *a non-atomic MASA, when is* $(\mathcal{D} + \mathcal{K}(\mathcal{H}), \mathcal{D})$ *an intermediate inclusion? (*$\mathcal{D}$ *not assumed unital.)*

A Concrete Case: If $C(\mathbb{T})$ *acts on* $L^2(\mathbb{T})$ *as multiplication op's, and* $\mathcal{D} \trianglelefteq C(\mathbb{T})$ *is an essential ideal, is* $(\mathcal{D} + \mathcal{K}(L^2(\mathbb{T})), \mathcal{D})$ *an intermediate incl'n?*

Note: The necessary conditions hold in test question setting.

No MASA in $\mathcal{B}(\mathcal{H})$ is Intermediate

Theorem

Suppose $\dim \mathcal{H} = \aleph_0$ *&* $\mathcal{D}$ *is a MASA in* $\mathcal{B}(\mathcal{H})$. *Then* $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ *is not intermediate to a regular MASA inclusion.*

When $\mathcal{D}$ HAS A CONTINUOUS PART, there are multiple cond. expectations (= psuedo-expectations b/c $\mathcal{D}$ is injective) of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{D}$, so $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ not intermediate.

When $\mathcal{D}$ an ATOMIC MASA, one shows:

- ♠ $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ is not regular;

1. If $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ is intermediate to a reg. MASA incl'n, then it is intermediate to a C^*-diagonal;
2. If $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ intermediate to a C^*-diag $(\mathcal{C}, \mathcal{D})$, then $\mathcal{K}(\mathcal{H})$ an essential ideal of $\mathcal{C}$, which forces $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{C} \subseteq M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$.

So $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ intermediate to a reg. MASA incl'n $\Rightarrow$ $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ regular, contradicting ♠.

Comments on ♠

Turns out that ♠ follows from work of Katavolos-Paulsen.

Here's my very different alternate proof: let $q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be quotient map. Assuming $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ is regular, then

- $(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}), q(\mathcal{D}))$ is a regular MASA inclusion;
- the cond. expect. $\tilde{\mathbb{E}} : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to q(\mathcal{D})$ is not faithful (if $P = \bigoplus_n \xi_n \xi_n^*$ where $\xi_n = \left(\sum_{j=1}^n e_n\right)/n$, then $\mathbb{E}(P) = 0$).

So $\mathcal{L} := \{\dot{T} \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) : \tilde{\mathbb{E}}(\dot{T}^* \dot{T}) = 0\}$ is a non-zero ideal, contradicting simplicity of Calkin alg.

Problem

Find $T \in \mathcal{B}(\mathcal{H}) \setminus \overline{\text{span}}\mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D})$.

Some Behavior of Non-Commutative Compatifications

Desirable properties (e.g. extension property, regularity) may or may not be preserved under non-comm. compactification.

Example

Let (e_n) be usual o.n. basis for $\mathcal{H} := \ell^2(\mathbb{N})$. Put

$$\mathcal{D} = \overline{\operatorname{span}}\{e_n e_n^* : n \in \mathbb{N}\} \quad \& \quad \mathcal{C} := \mathcal{K}(\mathcal{H}),$$

so $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. The following "compacifications" of $(\mathcal{C}, \mathcal{D})$ exhibit differing behaviors:

1. $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ is a C^*-diagonal.
2. Let S be unilat. shift. Then $\mathcal{C} \trianglelefteq C^*(S)$ is essential, and $(C^*(S), \tilde{D})$ is Cartan, but not a C^*-diagonal (EP fails)
3. $(M(\mathcal{C}), M(\mathcal{D})) = (\mathcal{B}(\mathcal{H}), \ell^\infty)$ has EP (by recent spectacular solution of Kadison-Singer), but is not regular.

More Questions

The previous example suggests the following

Question

Suppose $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal, with $\mathcal{C}$ not unital.

1. *When is $(M(\mathcal{C}), M(\mathcal{D}))$ regular?*
2. *Must $(M(\mathcal{C}), M(\mathcal{D}))$ have the extension property?*

THANK YOU!