The Lifting Property for C^*-algebras

Gilles Pisier
Texas A&M University
and
Université Paris-Sorbonne

. . . . . .

Functional Analysis and Operator Algebras
Athens, March 11 2022

Let me start by setting the stage
by defining the **lifting problems**
we wish to discuss:

Let C a C^*-algebra. Let $C/\mathcal{I}$ be a quotient C^*-algebra.
Let A be another C^*-algebra.

LIFTING

Global Lifting Problem :

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "\widehat{u}"] \arrow[r, "u"'] & C/\mathcal{I} \end{tikzcd}$$

Local Lifting Problem :

$$\begin{tikzcd} & & C \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "u^E"] \arrow[rr, bend right, "u_{|E}"'] & A \arrow[r, "u"'] & C/\mathcal{I} \end{tikzcd}$$

Discussion: contractions, positive contractions, global case open

vNa: $C^{**} = \mathcal{I}^{**} \oplus (C/\mathcal{I})^{**}$

Let A be a unital C^*-algebra.

Definition

A (*separable*) has the lifting property (LP in short) if $\forall C/\mathcal{I}$, $\forall u : A \to C/\mathcal{I}$ u.c.p. $\exists \widehat{u} : A \to C$ u.c.p. lifting u

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "\widehat{u}"] \arrow[r, "u"] & C/\mathcal{I} \end{tikzcd}$$

(There is also a non-separable variant)

Definition

A has the local lifting property (LLP in short) if $\forall C/\mathcal{I}$ $\forall u : A \to C/\mathcal{I}$ u.c.p. u is **locally liftable** i.e. $\forall E \subset A$ f.d. op. sys. $u_{|E} : E \to C/\mathcal{I}$ admits a u.c.p. lifting $u^E : E \to C$.

$$\begin{tikzcd} & & C \arrow[d, "q"] \\ E \arrow[urr, "u^E"] \arrow[r, hook] \arrow[rr, bend right, "u_{|E}"'] & A \arrow[r, "u"] & C/\mathcal{I} \end{tikzcd}$$

In the general case,
we say A has LP (resp. LLP) if its unitization does

From Local to Global ?

Open Problem (Kirchberg 1993) :

$$\text{LLP} \Rightarrow \text{LP ?}$$

(in the separable case)

Partial Motivation :

If the Connes embedding problem has a positive [1] solution then (Kirchberg) the LLP implies the LP

[1] **A recent paper entitled MIP* = RE posted on arxiv in Jan. 2020 by Ji, Natarajan, Vidick, Wright, and Yuen contains a negative solution**

Examples of C^*-algebras with LP

• Nuclear C^*-algebras (Choi-Effros 1977)
(Typically: $C^*(G)$ for G amenable countable discrete group)

• $C^*(\mathbb{F}_N)$ where $\mathbb{F}_N$ is a free group $(2 \leq N \leq \infty)$
(Kirchberg, 1994)

Both have the LP

Remark (digression): If $C_\lambda(G)$ is QWEP (no counterexample known) then

$$C_\lambda(G) \text{ LLP } \Leftrightarrow G \text{ amenable}$$

In particular $C_\lambda(\mathbb{F}_N)$ fails LLP for $N \geq 2$

CLASSICAL FACT: Any separable unital A can be written as

$$A = C^*(\mathbb{F}_\infty)/\mathcal{I} \text{ for some ideal } \mathcal{I}$$

Therefore it suffices to consider the lifting problem for

$$C = C^*(\mathbb{F}_\infty) \text{ and } u = Id : A \to C/\mathcal{I}$$

Definition (Equivalent definition)

A has the LP if any unital $*$-homomorphism $u : A \to C/\mathcal{I}$ is **liftable**, i.e. admits a u.c.p. lifting $\widehat{u} : A \to C$.

Definition (Equivalent definition)

A separable C^*-algebra A has the LLP if any unital $*$-homomorphism $u : A \to C/\mathcal{I}$ is **locally liftable**, i.e. for any $E \subset A$ f.d. operator system the restriction $u_{|E} : E \to C/\mathcal{I}$ admits a u.c.p. lifting $u^E : E \to C$.

Let us say that a discrete group G has LP if $C^*(G)$ does.
Note:

$$\{amenable\} \cup \{\text{free groups}\} \subset \{LP\}$$

$\rightarrow$ not so easy to find counterexamples !

Open Problem Does $\mathbb{F}_2 \times \mathbb{F}_2$ (or a product of free groups) have the LP or the LLP ?

Counter-examples to LP: Property (T)

Among $C^*(G)$ for G discrete group (reduced case easier)
All counterexamples are **Kazhdan property (T) groups**
Ozawa (PAMS 2004): $\exists G$ with $C^*(G)$ failing LP
Thom (2010) produced an explicit example with $C^*(G)$ failing LLP
Ioana, Spaas and Wiersma (2020) showed

Theorem (ISW 2020)

For $G = SL(n, \mathbb{Z})$ for $n \geq 3$ $C^*(G)$ fails LLP.

Still open for general (T) groups
They also showed:

Theorem (ISW 2020)

If $C^*(G)$ has (T) and is NOT finitely presented
then $C^*(G)$ fails LP.
Moreover: There are uncountably many such groups

Tensor products of C^*-algebras (background)

Originating in Japan in the late 1950's
Turumaru, then
Takesaki (1958), Guichardet (1965),
Lance (1973)
Choi-Effros, Kirchberg (1976-7)
Effros-Lance (1977)
Archbold-Batty (1980) Effros-Haagerup (1985)
Kirchberg (1993) $\cdots$

Minimal and a maximal tensor product denoted respectively by

$$A \otimes_{\min} B \text{ and } A \otimes_{\max} B.$$

These are obtained by completing the algebraic $A \otimes B$ with respect to the minimal and maximal C^*-norms $\| \ \|_{\min}$ or $\| \ \|_{\max}$
When $A \subset B(H)$ and $B \subset B(K)$ then

$$\forall t \in A \otimes B \quad \|t\|_{\min} = \|t\|_{B(H \otimes_2 K)} \text{ "spatial norm"}$$

$$\forall t \in A \otimes B \quad \|t\|_{\max} = \sup\{\|\pi \cdot \sigma(t)\|_{B(\mathcal{H})} \mid \pi, \sigma \text{ with commuting ranges}\}$$

where sup runs over all $\mathcal{H}$'s and all pairs (π, σ) of $*$-homomorphisms

$$\begin{tikzcd}
 & B(\mathcal{H}) & \\
A \arrow[ur, "\pi"] & & B \arrow[ul, "\sigma"']
\end{tikzcd}$$

$$
\begin{tikzcd}
 & B(\mathcal{H}) & \\
A \arrow[ur, "\pi"] & & B \arrow[ul, "\sigma"']
\end{tikzcd}
$$

Effros-Lance 1977:
If A is a vNa : $(t \in A \otimes B)$

$$\|t\|_{\mathrm{nor}} = \sup\{\|\pi\cdot\sigma(t)\|_{B(\mathcal{H})} \mid \pi \text{ normal }, \sigma \text{ with commuting ranges}\}$$

If A and B are both vNa :

$$\|t\|_{\mathrm{bin}} = \sup\{\|\pi\cdot\sigma(t)\|_{B(\mathcal{H})} \mid \pi, \sigma \text{ both normal with commuting ranges}\}$$

$$(A^{**} \otimes_{\mathrm{bin}} B^{**}) \subset (A \otimes_{\max} B)^{**} \text{ isometrically}$$

Tensor products of C^*-algebras (background)

Let A, C be C^*-algebras
for any C^*-norm $\| \ \|$ on $A \otimes C$ (algebraic)

$$\| \ \|_{\min} \leq \| \ \| \leq \| \ \|_{\max}$$

After Completions: $A \otimes_{\min} C$ and $A \otimes_{\max} C$
Def: A is nuclear if $A \otimes_{\min} C = A \otimes_{\max} C$ for any C

$$A \otimes_{\max} [C/\mathcal{I}] = [A \otimes_{\max} C]/[A \otimes_{\max} \mathcal{I}] \qquad \text{“projectivity”}$$

$$\forall B \subset C \qquad A \otimes_{\min} B \subset A \otimes_{\min} C \qquad \text{“injectivity”}$$

Tensor products of C^*-algebras (background)

Let A, C be C^*-algebras
for any C^*-norm $\| \ \|$ on $A \otimes C$ (algebraic)

$$\| \ \|_{\min} \leq \| \ \| \leq \| \ \|_{\max}$$

After Completions: $A \otimes_{\min} C$ and $A \otimes_{\max} C$
A is nuclear if $A \otimes_{\min} C = A \otimes_{\max} C$ for any C

$$A \otimes_{\min} [C/\mathcal{I}] \neq [A \otimes_{\min} C]/[A \otimes_{\min} \mathcal{I}]$$

$$\forall B \subset C \qquad A \otimes_{\max} B \not\subset A \otimes_{\max} C$$

Kirchberg's characterization of LLP

Theorem

Let A be a C^*-algebra

$$A \text{ has LLP } \Leftrightarrow A \otimes_{\min} B(\ell_2) = A \otimes_{\max} B(\ell_2)$$

$$A \text{ has LLP } \Leftrightarrow A \otimes_{\min} \mathbb{B} = A \otimes_{\max} \mathbb{B}$$

where

$$\mathbb{B} = (\oplus \sum_{n \geq 1} M_n)_\infty.$$

(often denoted $\prod_{n\geq 1} M_n$ in C^*-literature)

Stability properties

LP and LLP are stable under (finite) direct sums (easy)

LP and LLP are stable under extensions

LLP stable under (maximal) free products of arbitrary family (P. 1996)

LP stable under (maximal) free products of arbitrary family (Boca 1996, easy by Boca 1991)
Indeed, if A_i $(i \in I)$ is such that id_{A_i} is liftable up in $C_i = C(\mathbb{F}_\infty)$

$$\begin{tikzcd} & C_i \arrow[d, "q_i"] \\ A_i \arrow[ur, "u_i"] \arrow[r, "id_{A_i}"'] & A_i \end{tikzcd} \qquad \begin{tikzcd} & *_{i\in I} C_i \arrow[d, "*_{i\in I} q_i"] \\ *_{i\in I} A_i \arrow[ur, "*_{i\in I} u_i"] \arrow[r, "id"'] & *_{i\in I} A_i \end{tikzcd}$$

Boca 1991: u_i u.c.p. $\Rightarrow$ $*_{i\in I} u_i$ u.c.p.

$$id_A : A \xrightarrow{u} C \xrightarrow{v} A$$

If id_A factors through C with decomposable maps u, v then

$$C\ LP \text{ (resp. LLP)} \Rightarrow\ A\ LP \text{ (resp. LLP)}$$

$$* * * * * * * * * * * * * * * *$$

LLP stable under closure of union of arbitrary nested family
$A = \overline{\cup A_i}$
(easy using $\mathbb{B} \otimes_{\min} A = \mathbb{B} \otimes_{\max} A$)
$\rightarrow$**analogue unclear for LP**

New Characterization of LP for a C^* -algebra A

For any family $(D_i)_{i\in I}$ of C^*-algebras and any $t \in \ell_\infty(\{D_i\}) \otimes A$ we have

$$(*) \qquad \|t\|_{\ell_\infty(\{D_i\})\otimes_{\max} A} \le \sup_{i\in I} \|t_i\|_{D_i\otimes_{\max} A}$$

where $t_i = (p_i \otimes Id_A)(t)$ with $p_i : \ell_\infty(\{D_i\}) \to D_i$ denoting the i-th coordinate projection.
In other words we have a natural isometric embedding

$$(*) \qquad \ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\}).$$

More precisely $\ell_\infty(\{D_i\}) \otimes_{\max} A$ can be identified with the closure of $\ell_\infty(\{D_i\}) \otimes A$ (algebraic tensor product) in $\ell_\infty(\{D_i \otimes_{\max} A\})$.

Main result: $LP \Leftrightarrow (*)$

Remark: (*) is always true for the min-norm

Notation

Let $E \subset A$ (closed **linear** subspace), A, D C^*-algebras
We denote (abusively)

$$D \otimes_{\max} E = \overline{D \otimes E}^{\| \ \|_{\max}} \subset D \otimes_{\max} A$$

or

$$E \otimes_{\max} D = \overline{E \otimes D}^{\| \ \|_{\max}} \subset A \otimes_{\max} D$$

equipped with induced max-norm

$$(*) \qquad \ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\}).$$

Proposition

LP ⇒ (*)

Sketch of Proof

Using $A = C^*(\mathbb{F}_\infty)/\mathcal{I}$: Reduce to the case when $A = C^*(\mathbb{F}_\infty)$.

Let

$$E_N = \mathrm{span}[1, U_1, \cdots, U_{N-1}] \subset A = C^*(\mathbb{F}_\infty)$$

$(U_0 = 1)$

Enough to show for any N and all $t \in \ell_\infty(\{D_i\}) \otimes_{\max} E_N$

$$\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E_N} \leq \sup_{i \in I} \|t_i\|_{\{D_i \otimes_{\max} E_N\}}.$$

Preliminary fact: $\forall D$, $\forall (x_j)_{0\le j\le n} \in D^{N+1}$

$$\|\sum x_j \otimes U_j\|_{D\otimes_{\max} E_N} = \inf\{\|\sum a_j^* a_j\|^{1/2}\|\sum b_j^* b_j\|^{1/2} \mid x_j = a_j^* b_j\}$$

Then assume

$$\sup_{i\in I} \|\sum x_j(i) \otimes U_j\|_{D_i \otimes_{\max} E_N} < 1$$

$$\Rightarrow x_j(i) = a_j(i)^* b_j(i)$$

with

$$\|\sum a_j(i)^* a_j(i)\|^{1/2}\|\sum b_j(i)^* b_j(i)\|^{1/2} < 1$$

now $x_j = a_j^* b_j$ with $a_j = (a_j(i))$ and $b_j = (b_j(i))$

$$\Rightarrow \|\sum (x_j(i))_{i\in I} \otimes U_j\|_{\ell_\infty(\{D_i\})\otimes_{\max} E_N} \le \|\sum a_j^* a_j\|^{1/2}\|\sum b_j^* b_j\|^{1/2} < 1$$

QED

Main tool: Maximally bounded maps

Let $E \subset A$ be an operator space (A a C^*-algebra) Let D be another C^*-algebra. We denote (abusively)

$$D \otimes_{\max} E = \overline{D \otimes E}^{\| \ \|_{\max}} \subset D \otimes_{\max} A$$

Definition

$u : E \to C$ is called maximally bounded if for any C^*-algebra D

$$\|u\|_{mb} := \|Id_D \otimes u : D \otimes_{\max} E \to D \otimes_{\max} C\| < \infty$$

We denote by $MB(E, C)$ the normed space of such u's

Similar definition for maximally positive for E operator system

Decomposable maps

$$u : A \to C$$

$$u = u_1 - u_2 + i(u_3 - u_4)$$

$$u_j \in CP(A, C) \quad \forall j = 1, 2, 3, 4$$

Haagerup (1985) :
Assuming $v(a^*) = v(a)^* \ \forall a$

$$\|v : A \to B\|_{dec} = \inf\{\|v_1 + v_2\| \mid v = v_1 - v_2, v_1, v_2 \in CP(A, B)\}$$

Kirchberg (unpublished) $\|u\|_{mb} = \|i_C u\|_{dec}$

Characterization of MB maps

Theorem

Let $E \subset A$ be an operator subspace, $u : E \to C$

$$\|u\|_{mb} = \inf \|\tilde{u}\|_{dec},$$

where the infimum runs over all maps $\tilde{u} : A \to C^{**}$ such that $\tilde{u}_{|E} = i_C u$ (infimum attained),
where $i_C : C \to C^{**}$ is canonical inclusion

$$\begin{tikzcd}
A \arrow[rrd, dashed, "\tilde{u}"] & & \\
E \arrow[u, hook] \arrow[r, "u"'] & C \arrow[r, hook, "i_C"'] & C^{**}
\end{tikzcd}$$

Main result

Theorem

The following are equivalent:

- (i) *A has the LP*
- (ii) *A satisfies (*) (for any (D_i))*
- (iii) $\forall E \subset A$ *f.d.* $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ *contractively*
- (iv) $\forall D \quad D^{**} \otimes_{\max} A \subset (D \otimes_{\max} A)^{**}$ *isometrically*
- (v) $\forall M$ *vNa* $\quad M \otimes_{\max} A = M \otimes_{\mathrm{nor}} A$ *isometrically*
- (vi) *For any family $(D_i)_{i \in I}$ of C^*-algebras and any ultrafilter on I we have a natural isometric embedding*

$$[\prod_{i \in I} D_i / \mathcal{U}] \otimes_{\max} A \subset \prod_{i \in I} [D_i \otimes_{\max} A] / \mathcal{U}.$$

Equivalent properties

(i) A has the LP

(ii) A satisfies (*) (for any (D_i))

(iii) $\forall E \subset A$ f.d. $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ contractively

Proof: main new point is (ii) $\Rightarrow$ (iii) .
We set
$MB(E, C)^* = C^* \otimes_\alpha E$ (recall $\dim(E) < \infty$)
Then (*) implies a property of α that leads to (iii)

Equivalent properties

(i) A has the LP

(ii) A satisfies (*) (for any (D_i))

(iii) $\forall E \subset A$ f.d. $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ contractively

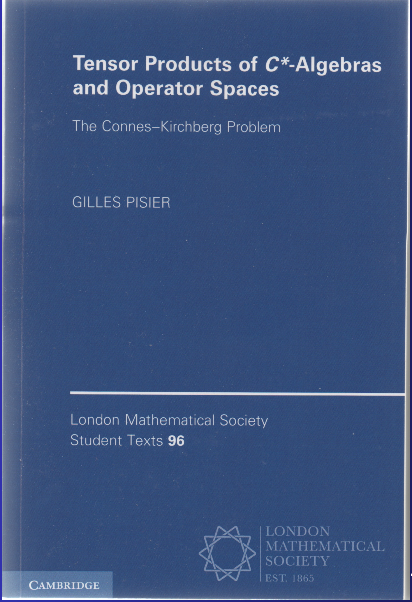
For (iii) $\Rightarrow$ (i) the proof uses more conventional tools mainly

$$C^{**} \simeq [C/\mathcal{I}]^{**} \oplus \mathcal{I}^{**}$$

Another formulation of

(iii) $\forall E \subset A$ f.d. $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ contractively is :

(iii)' For any $u \in MB(A, C^{**})$ with $\|u\|_{mb} \leq 1$ there is a net $u_i \in MB(A, C)$ with $\|u_i\|_{mb} \leq 1$ such that $u_i \to u$ pointwise weak*


Tensor Products of C*-Algebras and Operator Spaces
The Connes–Kirchberg Problem
GILLES PISIER
London Mathematical Society Student Texts 96
LONDON MATHEMATICAL SOCIETY
EST. 1865
CAMBRIDGE

Thank you !