Cartan subalgebras in groupoid C^*-algebras

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Acknowledgment, and Call to Action

I want to start by acknowledging that the University of Montana is in the aboriginal territories of the Salish and Kalispel people. I honor the path they have always shown us in caring for this place for the generations to come.

I would also like to acknowledge and honor the history of all of the Indigenous people who have called Montana home.

Recognizing that such acknowledgments are not enough, I remind myself and others to keep working to combat injustices, past and present.

Motivation

Thanks to the Gelfand-Naimark Theorem, if B an Abelian C^*-algebra, $B \cong C_0(X)$.

Abelian C^*-algebras can be studied using algebra, analysis, and topology.

If A not Abelian but $B \subseteq A$ is large and Abelian, we can say a lot about A by just studying B.

"Large Abelian" for us means Cartan subalgebra.

Cartan subalgebras in C^*-algebras

Definition

Let A be a C^*-algebra. A subalgebra $B \subseteq A$ is Cartan if

1. B is maximal Abelian in A
2. There is a faithful conditional expectation $\Phi : A \to B$
$$\Phi(bab') = b\Phi(a)b' \;\forall\; b, b' \in B; \qquad \Phi \text{ linear, contractive}$$
3. $N(B) := \{n \in A : n^*bn, nbn^* \in B \forall b \in B\}$ densely spans A
4. B contains an approximate identity for A.

Why important? Existence of a Cartan subalgebra is connected to the classification program [BL17, Li19].
In many situations (cf. [MM14, BCW17, BNR$^+$16, NR12]), an (iso)morphism of Cartan subalgebras lifts to one of the larger algebra.
A Cartan subalgebra gives a dynamical (groupoid) model for A [Ren08].

Renault's Theorem

Theorem (Ren08)

If $B \subseteq A$ is a Cartan subalgebra, then there is a unique topologically principal étale groupoid $\mathcal{G}$ and a twist Σ over $\mathcal{G}$ such that $B \cong C_0(\mathcal{G}^{(0)})$ and $A \cong C_r^*(\mathcal{G}; \Sigma)$.

$\mathcal{G}$ is often called the Weyl groupoid and Σ is the Weyl twist of the pair (A, B).

Renault's result builds on earlier work by Kumjian [Kum86], which connected principal groupoids with a stricter version of Cartan subalgebras called C^*-diagonals.

Our research question

Non-principal groupoids give rise to C^*-algebras too!

And sometimes they even have Cartan subalgebras.

In that case, we have two groupoid models for the C^*-algebra: the original one, and the Weyl groupoid.

What's the relationship between these two groupoids?

What are groupoids?

A groupoid $\mathcal{G}$ is a generalization of a group, where every element has an inverse but multiplication isn't always defined. Write

$$\mathcal{G}^{(2)} = \{(g,h) \in \mathcal{G} \times \mathcal{G} \text{ composable}\}$$

for the set where we can multiply.

Example

1. Groups. $G^{(2)} = G \times G$.
2. Vector bundle $V \xrightarrow{\pi} M$: $V^{(2)} = \{(v,w) : \pi(v) = \pi(w)\}$; operations are pointwise addition in $\pi^{-1}(x) \cong \mathbb{C}^n$.
3. Fundamental groupoid $\Pi(X)$ of a space X: homotopy classes of $f : [0,1] \to X$. $\Pi(X)^{(2)} = \{([f],[g]) : f(1) = g(0)\}$; $[f]^{-1} = [t \mapsto f(1-t)]$
4. Transformation groups: Given an action α of a group G on a space X, $X \rtimes_\alpha G = X \times G$ as a space;

$$(X \rtimes_\alpha G)^{(2)} = \{((x,g)(y,h)) : \alpha_h(y) = x\},$$

Group C^*-algebras

Recall: for a (discrete) group G, $C_r^*(G)$ is generated by the left-regular representation $\{\lambda_g\}_{g\in G}$ of G on $\ell^2(G)$: $C_r^*(G)$ is the smallest norm-closed $*$-subalgebra of $B(\ell^2(G))$ containing $\{\lambda_g : g \in G\}$.

What's the left-regular representation? On basis vectors,

$$\lambda_g(\delta_h) = \delta_{gh}.$$

Equivalently, if $f \in \ell^2(G)$, $\lambda_g(f)(h) = f(g^{-1}h)$.

Twisted group C^*-algebras

We also have projective representations of G, associated to a 2-cocycle $c : G \times G \to \mathbb{T}$.

$$c(g,h)c(gh,k) = c(g,hk)c(h,k).$$

$C_r^*(G,c)$ is generated by $\{\lambda_g^c\}_{g\in G}$, where

$$\lambda_g^c(\delta_h) = c(g,h)\delta_{gh}.$$

Equivalently, for a function $f \in \ell^2(G)$,

$$\lambda_g^c f(h) = f(g^{-1}h)c(g,g^{-1}h).$$

(The 2-cocycle condition guarantees that $\lambda_g^c\lambda_h^c = c(g,h)\lambda_{gh}^c$.)

Motivation for Groupoid C^*-algebras

If $\mathcal{G}$ is a topological groupoid, in that multiplication and inversion are continuous, we can form $C^*_r(\mathcal{G}) \subseteq B(L^2(\mathcal{G}, \mu))$.

$C^*_r(\mathcal{G})$ isn't generated by a unitary representation of $\mathcal{G}$, generally; however, if $\mathcal{G} = G$ is a group, $C^*_r(\mathcal{G}) = C^*_r(G)$.

$$C^*_r(X \rtimes G) = C_0(X) \rtimes_r G.$$

So: $C^*_r(\mathcal{G})$ is a good substitute for unitary representations of a groupoid $\mathcal{G}$.

$\mathcal{G}^{(0)} = \{gg^{-1} : g \in \mathcal{G}\}$ "space of units". $C_0(\mathcal{G}^{(0)}) \subseteq C^*_r(\mathcal{G})$ is an Abelian subalgebra.
If $\mathcal{G}$ is topologically principal then $C_0(\mathcal{G}^{(0)})$ is a Cartan subalgebra.

Given a locally compact Hausdorff groupoid $\mathcal{G}$, represent $C_c(\mathcal{G}) \ni f$ on $L^2(\mathcal{G}, \mu) \ni \xi$ by

$$\pi(f)\xi(\gamma) = \int_{\eta:(\gamma,\eta)\in\mathcal{G}^{(2)}} f(\gamma\eta)\xi(\eta^{-1})d\mu(\eta).$$

Check: If $\mathcal{G}$ is a group, $\xi = \delta_g, f = \chi_h$, what is $\pi(f)\xi$?

Given γ, need $\eta = \gamma^{-1}h$ and $\eta^{-1} = g$. So $\gamma = hg$, and

$$\pi(\chi_h)\delta_g = \delta_{hg}.$$

Note that $\pi(f) \in B(L^2(\mathcal{G}, \mu))$ since integration is linear and f was assumed compactly supported.

Groupoid C^*-algebras

Check: $\pi(f)\pi(g) = \pi(fg)$, where

$$fg(\gamma) = \int_{\eta:(\gamma,\eta)\in\mathcal{G}^{(2)}} f(\gamma\eta)g(\eta^{-1})d\mu(\eta).$$

In general we don't have $fg = gf$, but if $f, g \in C_c(\mathcal{G}^{(0)})$ we do.

Proposition

$C_c(\mathcal{G}^{(0)})$ is an Abelian subalgebra of $C_c(\mathcal{G})$.

$\mathcal{G}^{(0)}$ is the space of units in that if $u \in \mathcal{G}^{(0)}$ and $(u, \gamma) \in \mathcal{G}^{(2)}$, then $u\gamma = \gamma$. Similarly, if $(\eta, u) \in \mathcal{G}^{(2)}$ then $\eta u = \eta$.

Proof: If $f, g \in C_c(\mathcal{G}^{(0)})$ then

$$fg(\gamma) = \int_{\eta\in\mathcal{G}^{(0)}} f(\gamma\eta)g(\eta^{-1})\, d\mu(\eta)$$

is nonzero only when $\gamma = \eta = \eta^{-1}$ is a unit.

That is, $fg(u) = f(u)g(u)$; pointwise multiplication is Abelian. $\square$

Similarly, $\pi(f)^* = \pi(f^*)$ where $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

Define $C^*(\mathcal{G}) = \overline{\pi(C_c(\mathcal{G}))} \subset B(L^2(G, \mu))$

Twisted groupoid C^*-algebras

We can also build C^*-algebras from projective representations of groupoids.

A 2-cocycle on $\mathcal{G}$ is a function $c : \mathcal{G}^{(2)} \to \mathbb{T}$ such that

$$c(g, hk)c(h, k) = c(gh, k)c(g, h) \,\forall\, g, h, k \in \mathcal{G} \text{ with } (g, h), (h, k) \in \mathcal{G}^{(2)}.$$

- If we take $c(g, h) = 1$ for all $(g, h) \in \mathcal{G}^{(2)}$ we get the trivial 2-cocycle. $C_r^*(\mathcal{G}, c) = C_r^*(\mathcal{G})$.
- Usually we assume that if $u \in \mathcal{G}^{(0)}$ then $c(g, u) = c(u, g) = 1$ for all g.

Twisted groupoid C^*-algebras

Given a locally compact Hausdorff groupoid $\mathcal{G}$ with a 2-cocycle c, represent $C_c(\mathcal{G}, c) \ni f$ on $L^2(\mathcal{G}, \mu) \ni \xi$ by

$$\pi_c(f)\xi(\gamma) = \int_{\eta:(\gamma,\eta)\in\mathcal{G}^{(2)}} f(\gamma\eta)\xi(\eta^{-1})c(\gamma\eta, \eta^{-1})d\mu(\eta).$$

Check: if $\mathcal{G} = G$, $\pi_c(\chi_h)\delta_g = c(h, g)\delta_{hg}$, so $\pi_c(\chi_h) = \lambda_h^c$.

We have $\pi_c(f)\pi_c(g) = \pi_c(f \cdot_c g)$, where

$$f \cdot_c g(\gamma) = \int_{\eta:(\gamma,\eta)\in\mathcal{G}^{(2)}} f(\gamma\eta)g(\eta^{-1})c(\gamma\eta, \eta^{-1})d\mu(\eta).$$

Again, $C_c(\mathcal{G}^{(0)})$ is an Abelian subalgebra.

Similarly, $\pi_c(f)^* = \pi_c(f^*)$ where $f^*(\gamma) = \overline{f(\gamma^{-1})c(\gamma, \gamma^{-1})}$.

The reduced twisted groupoid C^*-algebra is $C_r^*(\mathcal{G}, c) = \overline{\pi_c(C_c(\mathcal{G}, c))} \subseteq B(L^2(G, \mu))$.

Theorem (Ren08)

If $B \subseteq A$ is a Cartan subalgebra, then there is a unique topologically principal étale groupoid $\mathcal{G}$ and a twist Σ over $\mathcal{G}$ such that $B \cong C_0(\mathcal{G}^{(0)})$ and $A \cong C_r^*(\mathcal{G}; \Sigma)$.

$\mathcal{G}$ is étale if $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms; for a group $\mathcal{G}$, or group action $X \ltimes \mathcal{G}$, this means $\mathcal{G}$ is discrete.

$\mathcal{G}$ is topologically principal if points with trivial isotropy are dense in $\mathcal{G}^{(0)}$.

$$\overline{\{u \in \mathcal{G}^{(0)} : \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma) = u\} = \{u\}\}} = \mathcal{G}^{(0)}.$$

If $\mathcal{G}$ is a (nontrivial) group, $\mathcal{G}^{(0)} = \{e\}$, so $\mathcal{G}$ is never topologically principal.

Our motivating question: Given a non-principal group(oid) $\mathcal{G}$ with a 2-cocycle c, when can we find Cartan subalgebras in $C^*_r(\mathcal{G}, c)$?

In that case, we have two group(oid) models for the C^*-algebra. What's the relationship between $(\mathcal{G}, c)$ and the Weyl groupoid/twist $(\mathcal{H}, \Sigma)$?

Example

Fix $\theta \in \mathbb{R}$. Let $G = \mathbb{Z}^2, c_\theta((m,n),(j,k)) = e^{2\pi i\theta(nj)}$. Then $C^*_r(\mathbb{Z}^2, c_\theta) = A_\theta$ is the noncommutative torus.
We can also view $A_\theta = C^*_r(\mathbb{T} \rtimes_\theta \mathbb{Z})$, where $\theta_n(z) = e^{2\pi i n\theta} z$. If $\theta \notin \mathbb{Q}$, $\mathbb{T} \rtimes_\theta \mathbb{Z}$ is topologically principal, so Kumjian/Renault theory says $C(\mathbb{T}) \subseteq A_\theta$ is Cartan.
That is, we have a Cartan subalgebra in $C^*_r(\mathbb{Z}^2, c_\theta)$ – even though $\mathbb{Z}^2$ is not (topologically) principal.

Theorem (Duwenig-G-Norton-Reznikoff-Wright)

Let $\mathcal{G}$ be a locally compact Hausdorff, second countable étale groupoid with a 2-cocycle c. Suppose $\mathcal{S}$ is a subgroupoid of $\mathcal{G}$ which is maximal among Abelian subgroupoids of $\mathcal{G}$ on which c is symmetric – $c(s,t) = c(t,s)$ for all $s, t \in \mathcal{S}$. If $\mathcal{S}$ is clopen, normal, and immediately centralizing, then $C_r^*(\mathcal{S}, c)$ is a Cartan subalgebra of $C_r^*(\mathcal{G}, c)$.

Proof sketch.

- $\mathcal{S}$ Abelian, c symmetric on $\mathcal{S} \Rightarrow C_r^*(\mathcal{S}, c)$ Abelian
- Maximality of $\mathcal{S}$ + imm. cent. $\Rightarrow C_r^*(\mathcal{S}, c)$ maximal Abelian.
- $\mathcal{S}$ normal $\Rightarrow$ normalizers of $C_r^*(\mathcal{S}, c)$ generate $C_r^*(\mathcal{G}, c)$.
- $\mathcal{S}$ clopen $\Rightarrow$ conditional expectation.
- Maximality of $\mathcal{S} \Rightarrow \mathcal{G}^{(0)} \subseteq \mathcal{S} \Rightarrow C_r^*(\mathcal{S}, c)$ contains an approximate unit. $\square$

Definition

For $k \geq 1$ and a subgroup S of a group G, denote $C_k(S) = \{g \in G : \forall\, s \in S, \exists\, 1 \leq j \leq k \text{ s.t. } gs^j = s^j g\}$. We say S is immediately centralizing if $C_k(S) = C_1(S)$ for all k.

That is, if g eventually commutes with powers of s for all $s \in S$, then g commutes with S.

Example

1. G Abelian $\Rightarrow$ S immediately centralizing for all $S \leq G$.
2. G has the unique root property [Bau60]: whenever $g^k = h^k$ we have $g = h$.
   If $gs^j = s^j g$, then $(gsg^{-1})^j = s^j$ and so $gsg^{-1} = s$.

Second Theorem

Theorem (Duwenig-G-Norton)

Let $\mathcal{G}, \mathcal{S}, c$ be as in Theorem 1, so that $C_r^*(\mathcal{S}, c) \leq C_r^*(\mathcal{G}, c)$ is Cartan. The associated Weyl groupoid is $\mathcal{H} = \widehat{\mathcal{S}_c} \rtimes \mathcal{G}/\mathcal{S}$. If we have a continuous section $\mathfrak{s} : \mathcal{G}/\mathcal{S} \to \mathcal{G}$, then the Weyl twist Σ is given by a 2-cocycle.

Recall: $(\mathcal{H}, \Sigma)$ built from $(C_r^*(\mathcal{S}, c), C_r^*(\mathcal{G}, c))$. Completions of $C_c(\mathcal{S}, c), C_c(\mathcal{G}, c)$. Tricky: how to describe every element of $\mathcal{H}$ using just info from $(\mathcal{S}, \mathcal{G}, c)$.

$\widehat{\mathcal{S}_c}$ is a bundle over $\mathcal{G}^{(0)} = \mathcal{S}^{(0)}$, with fibres the 1-dimensional projective representations of $\mathcal{S}$.

Σ measures the failure of $\mathfrak{s}$ to be multiplicative.

Γ-Cartan subalgebras

Definition (BFPR)

A C^*-algebra A is topologically graded by a discrete Abelian group Γ if $A = \bigoplus_{g \in \Gamma} A_g$, where A_g is a subspace of A satisfying

- $A_g^* = A_{-g}$
- $A_g A_h \subseteq A_{g+h}$
- We have a faithful conditional expectation $\Psi : A \to A_0$.

A subalgebra $B \subseteq A$ is Γ-Cartan if $B \subseteq A_0$ is a Cartan subalgebra, and $N(B, A)$ densely spans A.

Γ-Cartan pairs also give rise to groupoid models:

Theorem (BFPR)

If B is a Γ-Cartan subalgebra of A, then there is a unique étale groupoid $\mathcal{H}$ and a Γ-graded twist Σ over $\mathcal{H}$ such that $A \cong C_r^*(\mathcal{H}, \Sigma)$ and $B \cong C_0(\mathcal{H}^{(0)})$.

Multiple groupoid models for Γ-Cartan algebras

Theorem (Brown-G)

If $A = C_r^*(\mathcal{G})$ is a groupoid C^*-algebra, and $B \subseteq A$ is a Γ-Cartan subalgebra such that $B = C_r^*(\mathcal{S})$ for an open normal subgroupoid $\mathcal{S} \leq \mathcal{G}$, then $\mathcal{H} \cong \widehat{\mathcal{S}} \rtimes (\mathcal{G}/\mathcal{S})$.

Future work:

- What is the Weyl twist in this case?
- If the section $\mathfrak{s} : \mathcal{G}/\mathcal{S} \to \mathcal{G}$ is "almost" continuous, what can we say about the Weyl twist?
- Not all Cartan subalgebras in groupoid C^*-algebras arise as in our theorems; cf. [BNR+16]. Can we describe the Weyl groupoid in these cases?

Thanks!

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