Crossed products of operator spaces and approximation properties

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Fejér's theorem

If $f \in L^\infty(\mathbb{T})$, with Fourier series $S_N(f)(t) = \sum_{n=-N}^{N} \widehat{f}(n)e^{int}$, then

$$\frac{1}{N}\sum_{n=0}^{N-1} S_N(f) = F_N * f \xrightarrow{w^*} f,$$

where $F_N(t) = \dfrac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ikt}$ is Fejér's kernel.
Under the Fourier transform, the sequence $(\widehat{F_N})$ forms a b.a.i. for the Fourier algebra $A(\mathbb{Z}) \simeq L^1(\mathbb{T})$ and

$$\widehat{F_N} \cdot x \xrightarrow{w^*} x \quad \forall x \in L(\mathbb{Z}) \simeq L^\infty(\mathbb{T}),$$

where

$$u \cdot \lambda_n = u(n)\lambda_n, \quad u \in A(\mathbb{Z}).$$

Note that $L(\mathbb{Z}) = \mathbb{C} \rtimes_\iota \mathbb{Z}$ where ι is the trivial $\mathbb{Z}$-action on $\mathbb{C}$.
What about non-trivial crossed products by non-abelian groups?

Fejér-type approximations in general

Let G be a locally compact group acting on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ and $(\pi, \widetilde{\lambda})$ be the associated covariant representation on $H \otimes L^2(G) \simeq L^2(G, H)$:

$$(\widetilde{\lambda}_t \xi)(s) = \xi(t^{-1}s), \quad (\pi(a)\xi)(s) = (s^{-1} \cdot a)\xi(s), \quad s, t \in G,\ \xi \in L^2(G, H),\ a \in M.$$

There is a canonical $A(G)$-module action on $M \rtimes G = \overline{\text{span}}^{\text{w}^*} \{\widetilde{\lambda}(G)\pi(M)\}$ given by

$$u \cdot (\widetilde{\lambda}_s \pi(a)) = u(s)\widetilde{\lambda}_s \pi(a), \quad u \in A(G),\ s \in G,\ a \in M.$$

It holds that $x \in \{u \cdot x : u \in A(G)\}''$ for any $x \in M \rtimes G$.

Questions

1. Is it true that $x \in \overline{A(G) \cdot x}^{w^*}$ for all $x \in M \rtimes G$?
2. Does $x \in \overline{\text{span}}^{\text{w}^*} \{\widetilde{\lambda}_s(u \cdot x) :\ s \in G,\ u \in A(G)\}$ for any $x \in M \rtimes G$?
3. Is there a net $u_i \in A(G)$ such that $u_i \cdot x \xrightarrow{w^*} x$ for any $x \in M \rtimes G$?

Interestingly, such approximation problems can be interpreted functorially by extending the crossed product functor $(\cdot) \rtimes G$ to the category of dual operator spaces.

Tensor products

A dual operator space is a w*-closed subspace of $\mathcal{B}(H)$ for some Hilbert space H.
Let $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$ be dual operator spaces.

- Spatial tensor product:

$$X \bar{\otimes} Y = \overline{\text{span}}^{w^*} \{x \otimes y : x \in X, y \in Y\} \subseteq \mathcal{B}(H \otimes K),$$

  where $(x \otimes y)(h \otimes k) = (xh) \otimes (yk)$, for $h \in H,\ k \in K$.

- Fubini tensor product:

$$X \bar{\otimes}_{\mathcal{F}} Y = \{x \in \mathcal{B}(H \otimes K) : (\omega \otimes \text{id})(x) \in Y \text{ and } (\text{id} \otimes \phi)(x) \in X,$$

$$\forall\, \omega \in \mathcal{B}(H)_*,\ \phi \in \mathcal{B}(K)_*\}.$$

- $X \bar{\otimes} Y \subseteq X \bar{\otimes}_{\mathcal{F}} Y \simeq \left(X_* \widehat{\otimes} Y_*\right)^*$.
- We say that Y has property S_σ if $X \bar{\otimes} Y = X \bar{\otimes}_{\mathcal{F}} Y$ for any dual operator space X.
- (Kraus) Every *injective* von Neumann algebra M (e.g. of type I) has property S_σ.
- (Tomiyama) If M and N are von Neumann algebras, then $M \bar{\otimes} N = M \bar{\otimes}_{\mathcal{F}} N$.

Hopf-von Neumann algebras

A Hopf-von Neumann algebra (HvNa) is a pair (M, Δ), where M is a von Neumann algebra and $\Delta\colon M \to M\overline{\otimes}M$ is a comultiplication, that is a normal unital *-injection which is coassociative:

$$(\Delta \otimes \mathrm{id}_M) \circ \Delta = (\mathrm{id}_M \otimes \Delta) \circ \Delta$$

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M\overline{\otimes}M \arrow[d, "\Delta\otimes\mathrm{id}_M"] \\ M\overline{\otimes}M \arrow[r, "\mathrm{id}_M\otimes\Delta"] & M\overline{\otimes}M\overline{\otimes}M \end{tikzcd}$$

Let G be a locally compact group with left Haar measure.

- $L^\infty(G)$ (regarded as a von Neumann algebra acting on $L^2(G)$ by multiplication) is a HvNa with comultiplication $\alpha_G\colon L^\infty(G) \to L^\infty(G \times G) \simeq L^\infty(G)\overline{\otimes}L^\infty(G)$,

$$\alpha_G(f)(s,t) = f(ts), \quad s,t \in G,\ f \in L^\infty(G).$$

- Let $\lambda\colon G \to \mathcal{B}(L^2(G))$ be the left regular representation of G:

$$\lambda_s\xi(t) = \xi(s^{-1}t), \qquad \xi \in L^2(G).$$

  The left group von Neumann algebra $L(G) := \lambda(G)'' \subseteq \mathcal{B}(L^2(G))$, is also a HvNa with comultiplication $\delta_G\colon L(G) \to L(G)\overline{\otimes}L(G)$,

$$\delta_G(\lambda_s) = \lambda_s \otimes \lambda_s, \quad s \in G.$$

Comodules

Let (M, Δ) be a HvNa. An M-comodule is a pair (X, α), where X is a dual operator space and $\alpha\colon X \to X \overline{\otimes}_{\mathcal{F}} M$ is an M-action on X, i.e. a w*-continuous complete isometry which is coassociative over Δ:

$$(\alpha \otimes \mathrm{id}_M) \circ \alpha = (\mathrm{id}_X \otimes \Delta) \circ \alpha$$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & X \overline{\otimes}_{\mathcal{F}} M \arrow[d, "\alpha \otimes \mathrm{id}_M"] \\ X \overline{\otimes}_{\mathcal{F}} M \arrow[r, "\mathrm{id}_X \otimes \Delta"] & X \overline{\otimes}_{\mathcal{F}} M \overline{\otimes}_{\mathcal{F}} M \end{tikzcd}$$

The fixed point space of X is the subspace

$$X^{\alpha} = \{x \in X : \alpha(x) = x \otimes 1_M\}.$$

An M-subcomodule of X is w*-closed subspace $Y \subseteq X$ such that $\alpha(Y) \subseteq Y \overline{\otimes}_{\mathcal{F}} M$, i.e. $(Y, \alpha|_Y)$ is an M-comodule.

An M-comodule (iso)morphism between M-comodules (X, α) and (Z, β) is a w*-continuous complete contraction (surjective complete isometry) $\phi\colon X \to Z$, such that

$$\beta \circ \phi = (\phi \otimes \mathrm{id}_M) \circ \alpha.$$

Why comodules?

There is a bijective correspondence

$$\{G\text{-actions on dual op. spaces}\} \longleftrightarrow \{L^\infty(G)\text{-comodules}\}$$

Namely, an $L^\infty(G)$-action α on X corresponds to the G-action $\gamma\colon G \to \mathrm{Aut}^{\mathrm{ci}}_{\mathrm{w}^*}(X)$,

$$\gamma_s = \alpha^{-1} \circ (\mathrm{id}_X \otimes \mathrm{Ad}\lambda_s) \circ \alpha, \; s \in G.$$

Conversely, a G-action γ on X defines a unique $L^\infty(G)$-action α such that

$$\langle \alpha(x), \omega \otimes h \rangle = \int_G \langle \gamma_s^{-1}(x), \omega \rangle h(s)\, \mathrm{d}s, \; x \in X, \; \omega \in X_*, \; h \in L^1(G),$$

using that $X \overline{\otimes} L^\infty(G) \simeq \left(X_* \widehat{\otimes} L^1(G)\right)^*$.
If G is abelian, then $L(G) \simeq L^\infty(\widehat{G})$ where $\widehat{G}$ is the dual group and $\widehat{G}$-actions correspond to $L(G)$-comodules.
For non-abelian G the dual $\widehat{G}$ is not a group. However, we can use $L(G)$-comodules as a substitute for $\widehat{G}$-actions.

Comodules vs modules

Let (M, Δ) be a HvNa. The predual M_* becomes a Banach algebra with respect to the product

$$\omega\phi = (\omega \otimes \phi) \circ \Delta, \qquad \omega, \phi \in M_*.$$

Also, every M-comodule (X, α) is an M_*-module:

$$\omega \cdot x := (\mathrm{id}_X \otimes \omega)(\alpha(x)), \qquad \omega \in M_*,\ x \in X.$$

Remark

The product induced on $L^1(G) \simeq L^\infty(G)_*$ by the comultiplication α_G is given by

$$(hk)(t) = (k * h)(t) = \int_G k(s)h(s^{-1}t)ds, \quad k, h \in L^1(G),\ t \in G.$$

The product induced on the Fourier algebra $A(G) \simeq L(G)_*$ by the comultiplication δ_G coincides with the pointwise product of functions

$$(uv)(s) = u(s)v(s), \quad s \in G,\ u, v \in A(G).$$

Non-degeneracy and saturation

Let (M, Δ) be a HvNa acting on the Hilbert space K and let (X, α) be an M-comodule with $X \subseteq \mathcal{B}(H)$. We say that X is non-degenerate if

$$X \overline{\otimes} \mathcal{B}(K) = \overline{\operatorname{span}}^{w^*} \{(1_H \otimes b)\alpha(x) : \ b \in \mathcal{B}(K), \ x \in X\}.$$

The saturation space of (X, α) is the space

$$\operatorname{Sat}(X, \alpha) := \{y \in X \overline{\otimes}_{\mathcal{F}} M : \ (\mathrm{id}_X \otimes \Delta)(y) = (\alpha \otimes \mathrm{id}_M)(y)\}.$$

Obviously, $\alpha(X) \subseteq \operatorname{Sat}(X, \alpha)$. If $\alpha(X) = \operatorname{Sat}(X, \alpha)$, we say that (X, α) is saturated.

Proposition

(i) *If (X, α) is a non-degenerate M-comodule, then $X = \overline{\operatorname{span}}^{w^*} \{M_* \cdot X\}$;*
(ii) *If every M-comodule is non-degenerate, then every M-comodule is saturated;*
(iii) *If $M = L(G)$, then the converses of (i) and (ii) hold;*
(iv) *$L^\infty(G)$-comodules are always non-degenerate and saturated.*

Saturation & approximation properties

Let M be a von Neumann algebra. We say that a net $\Phi_i \in CB_\sigma(M)$ (i.e. the space of completely bounded normal maps on M) converges to the map $\Phi \in CB_\sigma(M)$ in the stable point-w*-topology if

$$(\mathrm{id}_{\mathcal{B}(\ell^2)} \otimes \Phi_i)(x) \xrightarrow{w^*} (\mathrm{id}_{\mathcal{B}(\ell^2)} \otimes \Phi)(x) \text{ for all } x \in \mathcal{B}(\ell^2)\overline{\otimes} M.$$

Proposition

For a HvNa (M, Δ) the following conditions are equivalent:

(a) *Every M-comodule is saturated;*

(b) *For any M-comodule (X, α), $x \in \overline{M_* \cdot x}^{\mathrm{w}^*}$ for all $x \in X$;*

(c) *There exists a net $\{\omega_i\} \subseteq M_*$, such that $\omega_i \cdot x \longrightarrow x$ in the w*-topology for any M-comodule X and any $x \in X$;*

(d) *There exists a net $\{\omega_i\} \subseteq M_*$, such that the net $\{(\mathrm{id}_M \otimes \omega_i) \circ \Delta\} \subseteq CB_\sigma(M)$ converges to the identity map id_M in the stable point-w*-topology.*

The AP

Definition (Haagerup-Kraus)

A locally compact group G has the approximation property (AP) if there is a net $\{u_i\}$ in the Fourier algebra $A(G)$ such that $u_i \to \mathbf{1}$ in the $\sigma(M_{cb}A(G), Q(G))$-topology.

Every $u \in A(G) \simeq L(G)_*$ defines a map $M_u \in CB_\sigma(L(G))$ with $M_u(\lambda_s) = u(s)\lambda_s$, namely

$$M_u = (\mathrm{id} \otimes u) \circ \delta_G : L(G) \xrightarrow{\delta_G} L(G)\overline{\otimes}L(G) \xrightarrow{\mathrm{id}\otimes u} L(G)$$

In other words, $M_u(x) = u \cdot x,\ x \in L(G)$ (the canonical $A(G)$-module action on $L(G)$).

Theorem (Haagerup-Kraus, 1993)

A locally compact group G has the AP if and only if there exists a net $\{u_i\}_{i\in I}$ in $A(G)$, such that $M_{u_i} \longrightarrow \mathrm{id}_{L(G)}$ in the stable point-w-topology.*

The AP (continued)

Proposition

For a locally compact group G the following are equivalent

1. *G has the AP;*
2. *Every $L(G)$-comodule is saturated;*
3. *Every $L(G)$-comodule is non-degenerate;*
4. *All saturated $L(G)$-comodules are non-degenerate;*
5. *For any $L(G)$-comodule (Y, δ) and any $y \in Y$, we have $y \in \overline{A(G) \cdot y}^{w^*}$;*
6. *There exists a net $\{u_i\}_{i \in I}$ in $A(G)$, such that $u_i \cdot y \xrightarrow{w^*} y$ for any $L(G)$-comodule (Y, δ) and any $y \in Y$.*

Spatial crossed product

For an $L^\infty(G)$-comodule (X, α) with $X \subseteq \mathcal{B}(H)$ the spatial crossed product of X by α is the w*-closed $\mathbb{C}1_H\overline{\otimes}L(G)$-submodule of $X\overline{\otimes}\mathcal{B}(L^2(G))$ generated by $\alpha(X)$. That is the space

$$X\overline{\rtimes}_\alpha G = \overline{\text{span}}^{\text{w}^*}\{(1_H \otimes \lambda_s)\alpha(x) : \ s \in G, \ x \in X\}.$$

Remark

- If α is trivial, i.e. $\alpha(x) = x \otimes 1$ for all $x \in X$, then $X\overline{\rtimes}_\alpha G = X\overline{\otimes}L(G)$.
- Also, if X is a von Neumann algebra and α is a unital *-homomorphism induced by a G-action γ, then from the covariance relations

$$\alpha(\gamma_s(x)) = (1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_s^{-1}) \qquad s \in G, \ x \in X$$

it follows that $X\overline{\rtimes}_\alpha G = (\alpha(X) \cup (\mathbb{C}1\overline{\otimes}L(G)))''$, i.e. the usual von Neumann algebra crossed product.

Fubini crossed product

Let

$$\sigma(a \otimes b) = b \otimes a, \qquad a, b \in \mathcal{B}(L^2(G)),$$

$$U_G f(s,t) = \Delta_G(t)^{1/2} f(st, t), \qquad f \in L^2(G \times G),\ s, t \in G.$$

For an $L^\infty(G)$-comodule (X, α) the map $\widetilde{\alpha} \colon X \overline{\otimes} \mathcal{B}(L^2(G)) \to X \overline{\otimes} \mathcal{B}(L^2(G)) \overline{\otimes} L^\infty(G)$, defined as the following composition:

$$\widetilde{\alpha} := (\mathrm{id}_X \otimes \mathrm{Ad} U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{\mathcal{B}(L^2(G))}),$$

is an $L^\infty(G)$-action on $X \overline{\otimes} \mathcal{B}(L^2(G))$.

$$\begin{tikzcd}
X \overline{\otimes} \mathcal{B}(L^2(G)) \arrow[r, "\alpha \otimes \mathrm{id}_{\mathcal{B}(L^2(G))}"] \arrow[rrd, dashed, "\widetilde{\alpha}"'] & X \overline{\otimes} L^\infty(G) \overline{\otimes} \mathcal{B}(L^2(G)) \arrow[r, "\mathrm{id}_X \otimes \sigma"] & X \overline{\otimes} \mathcal{B}(L^2(G)) \overline{\otimes} L^\infty(G) \arrow[d, "\mathrm{id}_X \otimes \mathrm{Ad} U_G^*"] \\
 & & X \overline{\otimes} \mathcal{B}(L^2(G)) \overline{\otimes} L^\infty(G)
\end{tikzcd}$$

The Fubini crossed product of X by α is the fixed point space

$$X \rtimes_\alpha^{\mathcal{F}} G = \left(X \overline{\otimes} \mathcal{B}(L^2(G))\right)^{\widetilde{\alpha}} = \{y \in X \overline{\otimes} \mathcal{B}(L^2(G)) :\ \widetilde{\alpha}(y) = y \otimes 1\}.$$

Alternative definition of $X \rtimes_\alpha^{\mathcal{F}} G$

Note that if the $L^\infty(G)$-action α on X corresponds to a G-action $\gamma\colon G \to \mathrm{Aut}^{\mathrm{ci}}_{\mathrm{w}*}(X)$, then $\widetilde{\alpha}$ is the $L^\infty(G)$-action on $X \overline{\otimes} \mathcal{B}(L^2(G))$ which corresponds to the G-action $s \in G \mapsto \gamma_s \otimes \mathrm{Ad}\rho_s$, where ρ is the right regular representation

$$\rho_s \xi(t) = \Delta_G(s)^{1/2} \xi(ts), \qquad \xi \in L^2(G).$$

Thus we have

$$X \rtimes_\alpha^{\mathcal{F}} G = \{y \in X \overline{\otimes} \mathcal{B}(L^2(G)) : \ (\gamma_s \otimes \mathrm{Ad}\rho_s)(y) = y \ \forall s \in G\}.$$

So, if α is trivial, i.e. $\alpha(x) = x \otimes 1$ for all $x \in X$, then $\gamma_s = \mathrm{id}_X$ for all $s \in G$ and hence $X \rtimes_\alpha^{\mathcal{F}} G = X \overline{\otimes}_{\mathcal{F}} L(G)$.
Moreover, it is easy to verify the following

- $(\mathbb{C}1_H \overline{\otimes} L(G))(X \rtimes_\alpha^{\mathcal{F}} G) \subseteq X \rtimes_\alpha^{\mathcal{F}} G$;
- $\alpha(X) \subseteq X \rtimes_\alpha^{\mathcal{F}} G$

and therefore

$$X \overline{\rtimes}_\alpha G \subseteq X \rtimes_\alpha^{\mathcal{F}} G.$$

$X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$???

1. (Digernes-Takesaki, 1975) If X is a von Neumann algebra and α is an $L^\infty(G)$- action on X which is a unital *-homomorphism, then $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$.
2. (Salmi-Skalski, 2015) If α is an $L^\infty(G)$-action on a (non-degenerately represented) W*-TRO X such that α is a (non-degenerate) TRO-morphism, then $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$.
3. (Crann-Neufang, 2019) If G has the AP, then $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ for any $L^\infty(G)$-comodule (X, α). For inner amenable (e.g. discrete) G, the converse is also true.

Remark

- $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ for any $L^\infty(G)$-comodule $(X, \alpha) \implies L(G)$ has property S_σ. Indeed, if α is the trivial action, then $X\overline{\rtimes}_\alpha G = X\overline{\otimes}L(G)$ and $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\otimes}_{\mathcal{F}}L(G)$.
- For inner amenable G, $L(G)$ has property $S_\sigma \iff G$ has the AP (Crann).
- Counterexample: $L(SL_3(\mathbb{Z}))$ does not have property S_σ (Lafforgue-de la Salle).

Crossed products of $L(G)$-comodules

For an $L(G)$-comodule (Y, δ) with $Y \subseteq \mathcal{B}(H)$ one can define the spatial crossed product of Y by δ

$$Y \overline{\ltimes}_{\delta} G = \overline{\text{span}}^{\text{w}^*} \{(1_H \otimes f)\, \delta(y) : \ f \in L^{\infty}(G), \ y \in Y\}$$

as well as the Fubini crossed product

$$Y \ltimes_{\delta}^{\mathcal{F}} G = \left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right)^{\widetilde{\delta}},$$

where $\widetilde{\delta} \colon Y \overline{\otimes} \mathcal{B}(L^2(G)) \to (Y \overline{\otimes} \mathcal{B}(L^2(G))) \overline{\otimes}_{\mathcal{F}} L(G)$ is the $L(G)$-action

$$\widetilde{\delta} = (\text{id}_Y \otimes \text{Ad} W_G) \circ (\text{id}_Y \otimes \sigma) \circ (\delta \otimes \text{id}_{\mathcal{B}(L^2(G))})$$

and

$$W_G \xi(s,t) = \xi(s, st), \quad \xi \in L^2(G \times G), \ s, t \in G.$$

Dual actions

1. For an $L^\infty(G)$-comodule (X,α) with $X \subseteq \mathcal{B}(H)$, the map
$$\widehat{\alpha}(x) = (1_H \otimes W_G^*)(x \otimes 1_{L^2(G)})(1_H \otimes W_G), \quad x \in X \rtimes_\alpha^{\mathcal{F}} G,$$
is an $L(G)$-action on $X \rtimes_\alpha^{\mathcal{F}} G$ called the dual of α.
Moreover,
$$\widehat{\alpha}\,(X \overline{\rtimes}_\alpha G) \subseteq (X \overline{\rtimes}_\alpha G)\,\overline{\otimes}_{\mathcal{F}} L(G).$$
That is $X \overline{\rtimes}_\alpha G$ is an $L(G)$-subcomodule of $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$.
2. For an $L(G)$-comodule (Y,δ) with $Y \subseteq \mathcal{B}(K)$, the map
$$\widehat{\delta}(x) = (1_K \otimes U_G^*)(x \otimes 1_{L^2(G)})(1_K \otimes U_G), \quad x \in Y \ltimes_\delta^{\mathcal{F}} G$$
is an $L^\infty(G)$-action on $Y \ltimes_\delta^{\mathcal{F}} G$ called the dual of δ.
Note that $\widehat{\delta}$ corresponds to the G-action $G \ni s \mapsto \mathrm{id}_Y \otimes \mathrm{Ad}\rho_s$ and
$$\widehat{\delta}\,(Y \overline{\ltimes}_\delta G) \subseteq (Y \overline{\ltimes}_\delta G)\,\overline{\otimes} L^\infty(G),$$
i.e. $Y \overline{\ltimes}_\delta G$ is an $L^\infty(G)$-subcomodule of $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$.

$L^\infty(G)$-comodules are saturated and non-degenerate

Proposition

For any $L^\infty(G)$-comodule (X, α) we have $\alpha(X) = \mathrm{Sat}(X, \alpha) = \left(X \rtimes_\alpha^{\mathcal{F}} G\right)^{\widehat{\alpha}} = (X \overline{\rtimes}_\alpha G)^{\widehat{\alpha}}$.

On the other hand, for an $L(G)$-comodule (Y, δ), the $L^\infty(G)$-comodule $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$ is non-degenerate. Using this we obtain the following:

Theorem

Let G be any locally compact group. For any $L(G)$-comodule (Y, δ), we have $Y \ltimes_\delta^{\mathcal{F}} G = Y \overline{\ltimes}_\delta G$.

Therefore we can simply write $Y \ltimes_\delta G$ instead of $Y \ltimes_\delta^{\mathcal{F}} G$ or $Y \overline{\ltimes}_\delta G$.
Also, for any $L(G)$-comodule (Y, δ), we have $\mathrm{Sat}(Y, \delta) = (Y \ltimes_\delta G)^{\widehat{\delta}}$.

Takesaki-duality for $L^\infty(G)$-comodules

Theorem

For any $L^\infty(G)$-comodule (X,α) we have

- $\left(X \rtimes_\alpha^{\mathcal{F}} G\right) \ltimes_{\widehat{\alpha}} G = (X \overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G \simeq X \overline{\otimes} \mathcal{B}(L^2(G))$;
- $\mathrm{Sat}(X \overline{\rtimes}_\alpha G, \widehat{\alpha}) = \mathrm{Sat}(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha}) = \widehat{\alpha}\left(X \rtimes_\alpha^{\mathcal{F}} G\right)$;
- $X \rtimes_\alpha^{\mathcal{F}} G = \{y \in X \overline{\otimes} \mathcal{B}(L^2(G)) : A(G) \cdot y \subseteq X \overline{\rtimes}_\alpha G\}$;
- $X \overline{\rtimes}_\alpha G = \overline{\mathrm{span}}^{w^*} \{A(G) \cdot (X \rtimes_\alpha^{\mathcal{F}} G)\}$;
- $(X \overline{\rtimes}_\alpha G, \widehat{\alpha})$ *is the largest non-degenerate* $L(G)$*-subcomodule of* $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ *and* $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ *is the smallest saturated* $L(G)$*-comodule containing* $X \overline{\rtimes}_\alpha G$ *as a subcomodule.*

In particular, $X \overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G \iff (X \overline{\rtimes}_\alpha G, \widehat{\alpha})$ *is saturated* $\iff (X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ *is non-degenerate.*

Note: The isomorphism in the first statement is an $L^\infty(G)$-comodule isomorphism for the actions $\widehat{\widehat{\alpha}}$ and $\widetilde{\alpha}$.

Takesaki-duality for $L(G)$-comodules

The same ideas apply in the case of $L(G)$-comodules:

Proposition

For any $L(G)$-comodule (Y, δ), the map

$$\phi\colon Y\overline{\otimes}\mathcal{B}(L^2(G)) \to Y\overline{\otimes}\mathcal{B}(L^2(G))\overline{\otimes}\mathcal{B}(L^2(G))$$

$$\phi = (\mathrm{id}_Y \otimes \mathrm{Ad}W) \circ (\delta \otimes \mathrm{id}_{\mathcal{B}(L^2(G))}),$$

where $W\xi(s,t) = \Delta_G(t)^{-1/2}\xi(s, st^{-1})$, $s, t \in G$, $\xi \in L^2(G \times G)$, is a w-continuous complete isometry such that*

1. *(Y, δ) is non-degenerate if and only if $\phi\left(Y\overline{\otimes}\mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G)\,\overline{\rtimes}_{\widehat{\delta}}G$;*
2. *(Y, δ) is saturated if and only if $\phi\left(Y\overline{\otimes}\mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G) \rtimes_{\widehat{\delta}}^{\mathcal{F}} G$.*

Note: In any case, ϕ is an $L(G)$-comodule isomorphism for the actions $\widetilde{\delta}$ and $\widehat{\widehat{\delta}}$.

A functorial characterization of the AP

Let us summarize:

- G has the AP iff every saturated $L(G)$-comodule is non-degenerate;
- (Y, δ) is a saturated $L(G)$-comodule iff $Y \overline{\otimes} \mathcal{B}(L^2(G)) \simeq (Y \ltimes_\delta G) \rtimes_{\hat{\delta}}^{\mathcal{F}} G$;
- (Y, δ) is a non-degenerate $L(G)$-comodule iff $Y \overline{\otimes} \mathcal{B}(L^2(G)) \simeq (Y \ltimes_\delta G) \overline{\rtimes}_{\hat{\delta}} G$

Theorem

For a locally compact group G the following conditions are equivalent:

(a) *G has the AP;*

(b) *$X \rtimes_\alpha^{\mathcal{F}} G = X \overline{\rtimes}_\alpha G$ for any $L^\infty(G)$-comodule (X, α);*

(c) *$(Y \ltimes_\delta G) \rtimes_{\hat{\delta}}^{\mathcal{F}} G = (Y \ltimes_\delta G) \overline{\rtimes}_{\hat{\delta}} G$ for any $L(G)$-comodule (Y, δ).*

Condition (H)

Using Takesaki-duality we also get the following:

Theorem

Let (Y, δ) *be a saturated* $L(G)$*-comodule. The following are equivalent:*

(i) $(Z \ltimes_\delta G)\overline{\rtimes}_{\hat{\delta}} G = (Z \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\hat{\delta}} G$ *for any* $L(G)$*-subcomodule* Z *of* Y*;*

(ii) $y \in \overline{A(G) \cdot y}^{w^*}$ *for any* $y \in Y$*.*

In the case where $Y = L(G)$ and $\delta = \delta_G$ we have the next:

Corollary

The following conditions are equivalent:

(i) *For any* $u \in A(G)$*, we have* $u \in \overline{A(G)u}^{\|\cdot\|}$ *(Ditkin's property at* ∞*);*

(ii) $y \in \overline{A(G) \cdot y}^{w^*}$ *for any* $y \in L(G)$ *(Eymard's condition (H));*

(iii) $\mathrm{Bim}_{L^\infty(G)}(J^\perp)\overline{\rtimes}_{\mathrm{Ad}_\rho} G = \mathrm{Bim}_{L^\infty(G)}(J^\perp) \rtimes^{\mathcal{F}}_{\mathrm{Ad}_\rho} G$ *for any closed ideal* $J \subseteq A(G)$*.*

The 'dual' version of condition (H)

Theorem

Let (X, α) be an $L^\infty(G)$-comodule. The following are equivalent:

(i) $Z\overline{\rtimes}_\alpha G = Z \rtimes_\alpha^{\mathcal{F}} G$ *for any* $L^\infty(G)$*-subcomodule* Z *of* X*;*

(ii) $y \in \overline{\mathrm{span}}^{w^*}\{(1 \otimes \lambda_s)(u \cdot y) : s \in G,\ u \in A(G)\}$ *for any* $y \in X \rtimes_\alpha^{\mathcal{F}} G$.

Taking $X = L^\infty(G)$ and $\alpha = \alpha_G \longleftrightarrow \mathrm{Ad}\lambda$, we get:

Corollary

The following are equivalent:

(i) $Z\overline{\rtimes}_{\mathrm{Ad}\lambda} G = Z \rtimes_{\mathrm{Ad}\lambda}^{\mathcal{F}} G$ *for any left-translation invariant w*-closed subspace* Z *of* $L^\infty(G)$*;*

(ii) *For any* $x \in \mathcal{B}(L^2(G))$*, it holds that*

$$x \in \overline{\mathrm{span}}^{w^*}\{\lambda_s(u \cdot x) : s \in G,\ u \in A(G)\} \quad \textit{(dual condition (H))}$$

where $u \cdot (\lambda_s f) = u(s)\lambda_s f$ *for* $s \in G,\ f \in L^\infty(G),\ u \in A(G)$.

A Fejér-property for $\mathcal{B}(L^2(G))$

Definition

We say that G has the Fejér-property if there exists a net $(u_i) \subseteq A(G)$ with $u_i \cdot x \xrightarrow{w^*} x$ for any $x \in \mathcal{B}(L^2(G))$.

Clearly, if G has the AP, then it has the Fejér-property and the latter implies both condition (H) and dual condition (H).

Proposition

- *If G has the Fejér-property, then $X \overline{\rtimes}_\alpha G$ is $\sigma(X \overline{\otimes} \mathcal{B}(L^2(G)), X_* \otimes \mathcal{B}(L^2(G))_*)$-dense in $X \rtimes_\alpha^{\mathcal{F}} G$, for any $L^\infty(G)$-comodule (X, α).*
- *In particular, G has the AP if and only if G has the Fejér-property and $X \overline{\rtimes}_\alpha G$ is $\sigma(X \overline{\otimes} \mathcal{B}(L^2(G)), X_* \otimes \mathcal{B}(L^2(G))_*)$-closed for any $L^\infty(G)$-comodule (X, α).*

Some open problems

- Condition (H): $x \in \overline{A(G) \cdot x}^{w^*}\ \forall x \in L(G)$;
- Dual condition (H): $x \in \mathrm{Bim}_{L(G)}\{A(G) \cdot x\}\ \forall x \in \mathcal{B}(L^2(G))$;
- Fejér-property: $\exists (u_i) \subseteq A(G),\ \forall x \in \mathcal{B}(L^2(G)),\ x = w^*\text{-}\lim_i u_i \cdot x$.

Questions

- Are there any groups (e.g. $SL_3(\mathbb{Z})$?) failing either condition (H) or dual condition (H)? What about the Fejér-property?
- Is any of them equivalent to the AP?
- Can we find a group G with the Fejér-property admitting some spatial crossed product $X \overline{\rtimes}_\alpha G$ which is not $X_* \otimes \mathcal{B}(L^2(G))_*$-closed (thus failing the AP)?
- Any connection with exactness?

Thank you for your attention!