Extensions, unitarizability, and amenable operator algebras

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[Typos corrected after talk]

Prologue: when "is" a Banach algebra a C^*-algebra?

Notation. If A is a Banach algebra, we write $A \cong \mathrm{C}^*$ if there is a C^*-algebra B and an isomorphism of Banach algebras $A \leftrightarrow B$.

(In this talk, homomorphisms between Banach algebras are always assumed to be continuous, but they need not be contractive. Likewise, isomorphisms of Banach algebras are not necessarily isometric.)

Example 1. *Let $T \in \mathbb{M}_d$ with d distinct eigenvalues. Since T is diagonalizable, the subalgebra $A = \mathrm{alg}(T)$ is isomorphic to $\mathbb{C}^d$. If we give $\mathbb{C}^d$ its natural C^*-norm, then the isomorphism $A \leftrightarrow \mathbb{C}^d$ is usually not isometric. Moreover, without diagonalizing T it is not obvious how to equip A with the "correct" involution.*

In general it seems hard to find "non-trivial" conditions on a given Banach algebra A which ensure $A \cong \mathrm{C}^*$. Necessary conditions are easier.

Lemma 2. *Every unital* C^**-algebra is spanned (as a complex vector space) by its group of unitaries.*

Therefore: if A *is a unital Banach algebra and* $A \cong \mathrm{C}^*$*, there is a bounded subgroup* $G \subset (A)_{\mathsf{inv}}$ *which spans* A*.*

Less well-known is the following result.

Theorem 3. [Cleveland, 1963] *Every injective homomorphism from a* C^**-algebra to a Banach algebra is bounded below.*

Therefore: if $A \cong \mathrm{C}^*$ *and* B *is* **any Banach algebra***, then every bounded HM* $A \to B$ *has closed range.*

We remark that if $A \cong \mathrm{C}^*$ and J is a closed ideal of A, then $J \cong \mathrm{C}^*$ and $A/J \cong \mathrm{C}^*$. It turns out that the converse is false: that is, the property of being $\cong \mathrm{C}^*$ is not preserved by extensions.

Extensions of Banach algebras

By an **extension of Banach algebras**, we mean a short exact sequence

$$0 \to J \xrightarrow{f} A \xrightarrow{g} B \to 0$$

- J, A, B are Banach algebras;
- f is an injective continuous homomorphism;
- g is a surjective continuous homomorphism;
- $\mathrm{im}(f) = \ker(g)$.

Typically J and B are given as “simpler” or “known” examples and we wish to understand which A can arise.

The case we focus on today: $J_d := c_0 \otimes \mathbb{M}_d$ with multiplier algebra $M(J_d) = \ell^\infty \otimes \mathbb{M}_d$.

We also want to consider the **corona algebra** of J_d. This is the quotient algebra $\mathcal{Q}_d := M(J_d)/J_d = (\ell^\infty/c_0) \otimes \mathbb{M}_d$.

The Busby correspondence

Throughout J is a Banach algebra with a BAI, and B is a Banach algebra.

- An extension $J \to A \to B$ gives rise to a continuous homomorphism $\phi : B \to M(J)/J$, called the associated "Busby map" of the extension.
- Given a continuous HM $\phi : B \to M(J)/J$, there is a **pullback algebra**

$$P_\phi := \{(b,m) \colon \phi(b) = q_J(m)\} \subset B \oplus_\infty M(J)$$

  which fits into a natural extension $J \to P_\phi \to B$.
- If $J \to A \to B$ has Busby map ϕ, then $A \cong P_\phi$.

Proposition 4. *Suppose J and B are C^*-algebras and $\phi : B \to M(J)/J$ is a continuous HM. If $s \in (M(J)/J)_{\mathrm{inv}}$ and $\psi := s\phi(\cdot)s^{-1}$ is a $*$-HM, then $P_\phi \cong P_\psi$ and P_ψ is a* **self-adjoint** *subalgebra of $B \oplus_\infty M(J)$.*

The message. Given $J \to A \to B$ where J and B are C^*-algebras: under some extra conditions, $A \cong \mathrm{C}^*$.

An unusual operator algebra

Recall: $J_d = c_0 \otimes \mathbb{M}_d$ and its multiplier algebra is $\ell^\infty \otimes \mathbb{M}_d$.

Theorem 5. [C.–Farah–Ozawa**, 2014)]** *There is a closed unital subalgebra $A \subset \ell^\infty \otimes \mathbb{M}_2$ with $J_2 \subset A$ such that:*

1) $A/J_2 \cong C(X)$ *for some separable non-metrizable X;*

2) *there is no bounded subgroup of $(A)_{\mathrm{inv}}$ whose linear span is dense in A.*

Property 1) implies that A is amenable as a Banach algebra.
Property 2) implies that A is not isomorphic to any C^*-algebra.

Brief comments on the proof. Recall: $\mathcal{Q}_2 = (\ell^\infty/c_0) \otimes \mathbb{M}_2$.

- The example arises from a carefully chosen bounded subgroup $\Gamma \subset (\mathcal{Q}_2)_{\mathrm{inv}}$ which is abelian, locally finite but **uncountable**.
- The key idea/tactic: if A was spanned by a bounded group, this leads (indirectly) to the existence of $s \in (\mathcal{Q}_2)_{\mathrm{inv}}$ such that $s\Gamma s^{-1}$ is contained in $U(\mathcal{Q}_2)$; and Γ is chosen so that no such s exists.

Unitarizable subgroups of unital C^*-algebras

Notation. For D a unital (C^*) algebra $(D)_{\mathsf{inv}}$ = invertible group and $(D^+)_{\mathsf{inv}}$ the positive part.

Lemma 6. *Let G be a subgroup of $(D)_{\mathsf{inv}}$. TFAE:*

1) *there exists $s \in (D)_{\mathsf{inv}}$ such that $sGs^{-1} \subseteq \mathcal{U}(D)$.*
2) *the action $G \curvearrowright (D^+)_{\mathsf{inv}}$, $\alpha_g(h) = ghg^*$, has a fixed point.*

If either of these holds, we say that G is **unitarizable (in D)**.

We list some examples where subgroups are unitarizable.

Example 7. *D arbitrary, G a finite subgroup of $(D)_{\mathsf{inv}}$. Take $h := \sum_{x \in G} xx^*$.*

Example 8. [Dixmier/Day] *D a von Neumann algebra, Γ an* **amenable** *bounded subgroup of $(D)_{\mathsf{inv}}$. Take h to be a weak-star average of xx^* over all x in Γ.*

Example 9. Γ *any bounded subgroup of* $(\mathbb{M}_d)_{\mathsf{inv}}$. *Let* G *be the closure of* Γ *inside* $\mathbb{M}_d$*; then* G *is a compact group and we may take* $h := \int_G xx^*\,dx$.

Example 10. Γ *any bounded subgroup of* $(\ell^\infty(\mathbb{I}) \otimes \mathbb{M}_d)_{\mathsf{inv}}$*: use the previous example in each coordinate.*

Warning. If $D = C(X) \otimes \mathbb{M}_d$ then there can be bounded subgroups of $(D)_{\mathsf{inv}}$ which are not unitarizable in D, even though they are unitarizable inside the larger algebra $\ell^\infty(X) \otimes \mathbb{M}_d$ by the previous example.

(There are even counterexamples isomorphic to $\mathbb{Z}$ inside $C(\mathbb{N}_\infty) \otimes \mathbb{M}_2$.)

Proposition 11. *Let* $\mathcal{Q}_d := (\ell^\infty/c_0) \otimes \mathbb{M}_d$. *Then every bounded* **countable** *subgroup of* $(\mathcal{Q}_d)_{\mathsf{inv}}$ *is unitarizable in* $\mathcal{Q}_d$.

The result may be known to experts but we did not locate an explicit reference in the literature. Details will appear in the PhD thesis of B. Green (Lancaster).

An application to extensions

Using the fact that every unital C^*-algebra is spanned by its unitaries, the previous proposition yields:

Corollary 12. *Let B be a* **separable** C^**-algebra and let $\phi : B \to \mathcal{Q}_d$ be a bounded homomorphism. Then there exists $s \in (\mathcal{Q}_d)_{\mathsf{inv}}$ such that $s\phi(\cdot)s^{-1}$ is a $*$-homomorphism. Hence:*

If $J_d \to A \to B$ is an extension of Banach algebras then $A \cong \mathrm{C}^*$

Reminder. The "CFO example" is an extension $J_2 \to A \to C(X)$ where $A \not\cong \mathrm{C}^*$.

An open problem

Fact (for background context). If X is compact Hausdorff, A is a closed subalgebra of $C(X) \otimes \mathbb{M}_d$, and $A \cong \mathrm{C}^*$, then A is amenable.

From the results of C.–Farah–Ozawa we know that there is an amenable subalgebra of $\ell^\infty \otimes \mathbb{M}_2 = C(\beta\mathbb{N}) \otimes \mathbb{M}_2$ which is not $\cong \mathrm{C}^*$.

Question. If X is metrizable, is every amenable subalgebra of $C(X) \otimes \mathbb{M}_d$ is isomorphic to a C^*-algebra?

This is open even for $X = [0, 1]$ (at time of writing). On the other hand, it follows from results of Gifford (1997, PhD thesis) that if A is an amenable subalgebra of $c_0 \otimes \mathbb{M}_d$ then $A \cong \mathrm{C}^*$.

This suggests trying to attack the question above in cases where X has "lots of isolated points".

Cantor–Bendixson rank

Throughout X is compact Hausdorff and **countable**. By (a version of) the Baire category theorem X must have at least one isolated point.

Definition. Let $\mathrm{Is}(X)$ denote the set of isolated points in X. We define the **Cantor–Bendixson derivative** of X to be the complement $X^{(1)} := X \setminus \mathrm{Is}(X)$.

The higher CB-derivatives are defined recursively by $X^{(n)} := (X^{(n-1)})^{(1)}$, etc. We say that X has **finite CB-rank** if $X^{(n)} = \emptyset$ for some $n \in \mathbb{N}$.

Example 13. *Let $\mathbb{N}_\infty$ be the one-point compactification of $\mathbb{N}$ and let $X = (\mathbb{N}_\infty)^2$.*

Then $\mathrm{Is}(X) = \mathbb{N}^2$, $X^{(1)}$ consists of "two copies of $\mathbb{N}_\infty$ joined at ∞", and $X^{(2)} = \{\infty\}$.

A partial result

The following result forms part of the Lancaster PhD thesis of B. Green (being written up).

Theorem 14. [C.+Green, in preparation] *Let X be countable and compact with finite Cantor–Bendixson rank. Let A be an amenable subalgebra of $C(X) \otimes \mathbb{M}_d$ with the following extra property:*

$$\textit{every bounded HM from } A \textit{ to a } \mathrm{C}^*\textit{-algebra has closed range.} \qquad (\diamondsuit)$$

Then A is isomorphic to a (separable) C^*-algebra.

Comments on the condition $(\diamondsuit)$.

- If $X^{(1)}$ is finite, then we can drop the condition $(\diamondsuit)$.
- If A is a Banach algebra satisfying $(\diamondsuit)$ then so does every quotient of A.
- If $A \cong \mathrm{C}^*$ then it satisfies $(\diamondsuit)$; this follows from the theorem of Cleveland mentioned at the start.

Outline of the proof

We argue by induction on the CB rank of X. (In the base case X is finite, so we are dealing with amenable subalgebras of $\mathbb{M}_n$ for suitable n.)

Recall: $\mathrm{Is}(X)$ is the set of isolated points in X and $X^{(1)} = X \setminus \mathrm{Is}(X)$.

Consider the restriction homomorphism $\rho : C(X) \otimes \mathbb{M}_d \to C(X^{(1)}) \otimes \mathbb{M}_d$, with kernel $C_0(\mathrm{Is}(X)) \otimes \mathbb{M}_d$. We make two observations:

- Since A satisies ($\diamondsuit$), $\rho(A)$ is a closed subalgebra of $C(X^{(1)}) \otimes \mathbb{M}_d$ which also satisfies ($\diamondsuit$). Also $\rho(A)$ is amenable. Hence, by the inductive hypothesis $\rho(A) \cong B$ for some (separable) C^*-algebra B.
- Since A is amenable, results of GIFFORD (1997 PhD thesis) imply that $\ker(\rho) \cap A$ is amenable and isomorphic to a c_0-sum of matrix algebras.

To simplify the exposition, assume $\ker(\rho) \cap A \cong J_d = c_0 \otimes \mathbb{M}_d$. Then A fits into an extension of Banach algebras $J_d \to A \to B$. Since B is separable, we can invoke our earlier results to conclude that $A \cong \mathrm{C}^*$. $\square$