Quantum path spaces, correspondences, and quantum Cuntz-Krieger Algebras

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Cuntz-Krieger algebras:

Input: A – $n \times n$ matrix with entries in $\{0,1\}$

$A = [A(v,w)] \equiv$ adjacency matrix of a simple directed graph.

$$G = \begin{cases} V = \text{vertices}, & |V| = n \\ E \subseteq V \times V = \text{edges}, & (v,w) \in E \iff A(v,w) = 1. \end{cases}$$

Output: universal C^*-algebra (unital) $\mathcal{O}_A = \mathcal{O}_G$

Cuntz-Krieger algebra

$$\mathcal{O}_A = C^*\Big(S_v,\ v \in V \ \Big|\ \underbrace{(CK1) - (CK3)}_{\text{Cuntz-Krieger Relations}} \Big)$$

(CK1) S_v is a partial isometry $\forall v \in V$
$(S_v S_v^* S_v = S_v)$

(CK2) $\forall v \in V$, $S_v^* S_v = \sum_{w \in V} S_w S_w^* A(w,v)$

(CK3) $1_{\mathcal{O}_A} = \sum_{v \in V} S_v S_v^*$.

$\{\mathcal{O}_A\}_A$ — rich class, tractable, (include "most" graph C^*-algebras)

$A =$ "all 1's matrix" $\quad A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & & \end{pmatrix} \Longleftrightarrow G =$ complete graph on n vertices $= K_n$

$\rightsquigarrow \mathcal{O}_A = \mathcal{O}_n =$ Cuntz-algebra on n-generators.

Generalize the construction $A \longrightarrow \mathcal{O}_A$

$\underbrace{A}$ $\to$ q. adjacency matrices
($\Longleftrightarrow$ q. graphs)

Quantum Graphs (via q. adjacency matrices)

$\hookrightarrow$ See M. Daws' recent arxiv paper.

$|V| = n \rightsquigarrow$ Consider $B = C(V)$
f.d. C^*-algebra

$A \in M_n(\{0,1\}) \longrightarrow$ linear map $A : B \to B$

$\longrightarrow$ A is completely positive

$\longrightarrow$ A is a Schur-idempotent

quant. these

B – f.d. C^*-algebra (not nec. comm.) = "$C(V)$"

A q. adjacency matrix ~~on~~ on B,

$A : B \to B$ linear, cp.

+ A is a quantum Schur idempotent. ?

Fix a "nice" faithful state $\psi : B \to \mathbb{C}$

"δ-form", a state $\psi : B \to \mathbb{C}$ is a δ-form if

$m : B \otimes B \longrightarrow B$ (mult.)

$m^* : B \to B \otimes B$ (Hilb. space dual w.r.t. $L^2(B,\psi)$)

m^* is a multiple of an isometry.

That is $\exists\ \delta > 0$ such that

$$\boxed{mm^* = \delta^2 \mathrm{id}_B}$$

If B is abelian $\exists!$ unique δ-form on B

$\Longleftrightarrow$ "uniform measure on V"

$$\delta^2 = \dim B = n$$

If B is arbitrary, but ψ is a tracial δ-form then it's unique.

$$B = \bigoplus_{a=1}^{m} M_{N_a}(\mathbb{C}),$$

$$\left(\delta^2 = \dim B\right)$$

$\psi =$ "Plancharel" trace

$$= \sum_a \frac{N_a \operatorname{Tr}_a(\cdot)}{\dim B}$$

[redacted]

<u>Fact</u>: $B = C(V)$, $A : B \to B$

$A =$ adj. matrix

$$\Longleftrightarrow \boxed{m(A \otimes A) m^* = n A} \qquad \delta^2 = n$$

$m^* : B \to B \otimes B$ (wrt. ψ = uniform trace on B)

<u>Definition</u>: Fix B fd C^*-alg., ψ - δ form on B.

(B,ψ) = "Measured q. space"

A . q. adjacency matrix on B is is CP linear map $A : B \to B$, δ^2

$$\boxed{m(A\otimes A)m^* = \delta^2 A}$$

Call triple (B,ψ,A) = a quantum graph

"A is Schur idempotent wrt. (B,ψ)".

Examples: (1) B - abelian $= C(V) = \mathbb{C}^n$
get usual $A \in M_n(\{0,1\})$ ✓

(2) Fix (B,ψ),

$A = id_B$ - q. adj. matrix

(B,ψ,A) = Trivial q-graph ✓

(3) A - $*$-automorphism (homomorphism) of B

$$\left(A \text{ cp.},\ m(A\otimes A) = A\circ m \right)$$

$$\Downarrow$$

$$m(A\otimes A)m^* = A\circ mm^* = \delta^2 A \quad ✓$$

(4) $A = \delta^2 \psi(\cdot)\, 1_B$ check gives q-adj matrix!

q. analogue of $A = \begin{pmatrix} 1 & 1 & - & - & 1 & 1 \\ 1 & & & & & \\ \vdots & - & - & & 1 & \\ & - & - & - & 1 & \end{pmatrix}$ //.

$\rightsquigarrow K(B,\psi) = (B, \psi, \underset{\substack{\uparrow \\ \text{as above}}}{A}) =$ q. complete graph.

Goal: Construct analogue of

$$A \longrightarrow \mathcal{O}_A$$

where $A : B \to B$ q. adj. matrix.

Classical: $\underset{\substack{\cap \\ V}}{v} \longmapsto S_v \in \mathcal{O}_A$ + Relations.

Quantum: Linearize this!

e_v = canonical projection in $C(V)$

$$S : \underset{\substack{\cap \\ C(V)}}{B} \longrightarrow \mathcal{O}_A, \qquad \underbrace{S_v = S(e_v)} \times \text{const.}$$

Definition ($\overline{\omega}$ EUW): $(B, \psi, A) = G$ q. graph

A quantum Cuntz-Krieger G-family in $B(H)$ is a linear map

$$S : B \longrightarrow B(H),$$ such that:

let $\mu : B(H)\otimes B(H) \to B(H)$ multiplication.

(QCK1) $\mu(1\otimes\mu)(S\otimes S^*\otimes S)(1\otimes m^*)m^* = S$

$\forall a$ $\boxed{S^{(a)}S^{(a)*}S^{(a)} = S^{(a)}}$ $\in M_{N_a}(B(H))$

(QCK2) $\mu(S^*\otimes S)m^* = \mu(S\otimes S^*)m^* A$

$S^{(a)*}S^{(a)} = \sum_b S^{(b)}S^{(b)*}\hat{A}^b_a$

(QCK3) $\mu(S\otimes S^*)m^*(1_B) = \delta^{-2}1_{B(H)}$

$(\delta^2\psi\otimes id)\left(\bigoplus_a S^{(a)}S^{(a)*}\right) = 1$

<u>Note</u>: $\boxed{S^*(x) = S(x^*)^*}$

<u>Rmk</u>: B = abelian $= C(V)$

the $QCK^{\#} = CK^{\#}$

<u>here</u>: $S(ne_v) = S_v \in \mathcal{O}_A$

$B = \bigoplus_a M_{N_a}$, $\exists$ <u>adapted matrix units</u>

$c_{a,i,j}e^{(a)}_{ij} = \underline{f^{(a)}_{ij}} \in M_{N_a}$

<u>get</u>: $S^{(a)} = [S^{(a)}_{ij}]$, $S^{(a)}_{ij} = S(f^{(a)}_{ij})$

<u>Big question</u>: ? $\mathcal{O}_A$ (classical case)

are dynamical objects !

$$\mathcal{O}_A = C(X_A) \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \quad \text{(Exel Crossed product)}$$

<u>Current direction</u> : Is there an analogue of this in quantum case ??

<u>Partial Results</u> : $K(B,\psi)$ = complete case.

Fact : $\boxed{\delta^2 \in \mathbb{N} \ (\tilde{\omega}\ EUW)}$

$$\mathcal{O}_{K(B,\psi)} = \text{universal } C^*\text{-alg. of } QCK\text{-}K(B,\psi) \text{ family}$$

$$\simeq \mathcal{O}_{\dim B}$$

$\mathcal{O}_G \longrightarrow$ Cuntz-Pimsner algebra associated to a certain C^*-corresp. ass. to G $\Big] \ \mathcal{O}_{E_G}$

$G = K(B,\psi)$

$$\mathcal{O}_{E_G} = \mathcal{O}_{\dim B} = \text{Cuntz algebra}$$

$$\forall B, \psi, \quad \mathcal{O}_{E_G} \simeq \mathcal{O}_{\dim B} = C(X_A) \rtimes \mathbb{N}$$

NC. C^*-dynamical system

$$\underbrace{C(X_A) = B^{\otimes \mathbb{N}}}$$

$C(X_A) = ?$ in general.