

Quantum path spaces, correspondences, and quantum Cuntz-Krieger Algebras

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Cuntz-Krieger algebras.

Input: A - $n \times n$ matrix with entries in $\{0, 1\}$

$A = [A(v,w)]$ = adjacency matrix of a simple directed graph.

$$G = \begin{cases} V = \text{vertices}, |V| = n \\ E \subseteq V \times V = \text{edges}, (v,w) \in E \Leftrightarrow A(v,w) = 1. \end{cases}$$

Output: universal $\overset{\text{unital}}{\curvearrowleft} C^*$ -algebra $\mathcal{O}_A = \mathcal{O}_G$

Cuntz-Krieger algebra

$$\mathcal{O}_A = C^*(S_v, v \in V \mid \underbrace{(CK1) - (CK3)}_{\text{Cuntz-Krieger Relations}})$$

($\zeta \kappa 1$) S_v is a partial isometry $\forall v \in V$
 $(S_v S_v^* S_v = S_v)$

($\zeta \kappa 2$) $\forall v \in V, S_v^* S_v = \sum_{w \in V} S_w S_w^* A(w, v)$

($\zeta \kappa 3$) $1_{\mathcal{O}_A} = \sum_{v \in V} S_v S_v^*$.

$\{\mathcal{O}_A\}_A$ — rich class, tractable, (include "inert" graph $C^*-algebras$)

A = "all 1's matrix" $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & - & - & - \\ \vdots & & & \end{pmatrix} \Leftrightarrow G = \text{complete graph on } n \text{ vertices}$

$\sim \mathcal{O}_A = \mathcal{O}_n = \text{(unital -algebra on } n \text{-generators)} = K_n$

Generalize the construction $\begin{array}{ccc} A & \longrightarrow & \mathcal{O}_A \\ \downarrow & & \\ \text{q. adjacency matrices} & & \\ (\Leftrightarrow \text{q. graphs}) & & \end{array}$

Quantum Graphs (via q. adjacency matrices)

See M. Dawes'

recent arxiv paper.

$|V|=n \rightarrow$ Consider $B = C(V)$

f.d. C^* -algebra

$A \in M_n(\{0,1\}) \rightarrow$ linear map $A: B \rightarrow B$

} quat.
these

$\rightarrow A$ is completely positive

$\rightarrow A$ is a Schur-idempotent

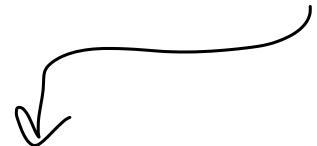
B-fd. C^* -algebra (not nec. comm.) = "CCV"

A q. adjacency matrix on B ,

$A: B \rightarrow B$ linear, sp.

+ A is a quantum Schur idempotent. ?

Fix a "nice" faithful state $\psi: B \rightarrow \mathbb{C}$



" δ -form", a state $\psi: B \rightarrow \mathbb{C}$ is a δ -form if

$m: B \otimes B \rightarrow B$ (mult.)

$m^*: B \rightarrow B \otimes B$ (Hilb. space dual wrt. $L^2(B, \psi)$)

m^* is a multiple of an isometry.

That is $\exists \delta > 0$ such that

$$mm^* = \delta^2 \text{id}_B$$

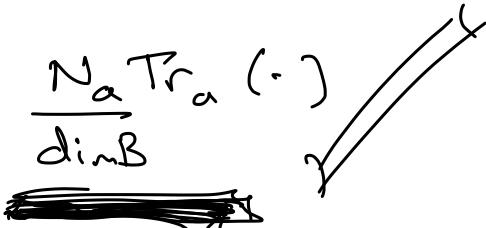
↓

If B is abelian $\exists!$ unique δ -form on B
 \iff "uniform measure on $\sqrt{\delta}$ "
 $\delta^2 = \dim B < n$

If B is arbitrary, but ψ is a tracial δ -form
then it's unique.

$$B = \bigoplus_{a=1}^n M_{N_a}(\mathbb{C}) , \quad (\delta^2 = \dim B)$$

ψ = "Plancharel" trace

$$= \sum_a \frac{N_a \text{Tr}_a(\cdot)}{\dim B}$$


Fact: $B = C(V)$, $A : B \rightarrow B$

A = adj. matrix

$$\iff m(A \otimes A) m^* = \boxed{nA}$$

$m^* : B \rightarrow B \otimes B$ (wrt. ψ = uniform trace on B)

Definition : Fix B fd C^* -alg., $\psi - \delta$ form on B .

$(B, \psi) = \text{"measured q. space"}$

A · q. adjacency matrix on B ;,

is CP linear map $A: B \rightarrow B$,

$$m(A \otimes A)m^* = S^2 A$$

Call triple

(B, ψ, A) = a quantum graph

" A is Schur idempotent
wrt. (B, ψ) ".

Examples: (1) B -abelian = $C(V) = \mathbb{C}^n$
get usual $A \in M_n(\{0,1\})$ ✓

(2) Fix (B, ψ) ,

$A = \text{id}_B$ - q. adj. matrix

(B, ψ, A) = Trivial q-graph ✓

(3) A - *-automorphism (homomorphism) of B

(A cp., $m(A \otimes A) = A \circ m$)

↓

$$m(A \otimes A)m^* = A \circ m m^* = S^2 A$$

(4) $A = S^2 \psi(\cdot) 1_B$ check gives q. adj. matrix!

q. analogue of $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$

$\rightarrow K(B, \gamma) = (B, \gamma, A)$ = q. complete graph
 \uparrow
as above

Goal: Construct analogue of

$$A \longrightarrow \mathcal{O}_A$$

where $A: B \rightarrow B$ q. adj. matrix.

Classical: $v \longmapsto S_v \in \mathcal{O}_A$ + Relations.
 \uparrow
 \vee

Quantum: Linearize this!

e_v = canonical projection in $C(V)$
 $\left(\begin{matrix} "S: B \rightarrow \mathcal{O}_A \\ n \end{matrix} \right) \quad , \quad S_v = S(e_v) \times \text{const.}$
 (W)

Definition ($\bar{\omega} \in V^W$): $(B, \gamma_A) = G$ q-graph
A quantum cutz- V -regular family in $B(H)$ is
a linear map.

$S: B \rightarrow B(H)$, such that:

Let $\mu : B(H) \otimes B(H) \rightarrow B(H)$
multiplication.

$$(\text{QCK}^{\#}) \quad \mu(I \otimes \mu)(S \otimes S^*) (I \otimes m^*) m^* = S$$

As $\boxed{S^{(a)} S^{(a)*} f^{(a)}} = S^{(a)} \in M_{N_a}(B(H))$

$$(\text{QCK}_2) \quad \mu(S \otimes S)m^* = \mu(S \otimes S^*)m^* A$$

$$S^{(a)*} S^{(a)} = \sum_b S^{(b)} S^{(b)*} \overset{?}{=} A_a$$

$$(\text{QCK}_3) \quad \mu(S \otimes S^*)m^*(I_B) = S^{-2} I_{B(H)}$$

Note: $\boxed{S^*(\alpha) = S(\alpha^*)^*}$ $\overset{(\text{by def})}{\text{if}} \left(\bigoplus_a S^{(a)} S^{(a)*} \right)$

Rnk: $B = \text{abelian} = CK^{\#}$ $= 1$

the $\text{QCK}^{\#} = CK^{\#}$

here: $S(n_{\mathcal{E}_V}) = S_V \oplus \mathcal{O}_A$

$$B = \bigoplus_a M_{N_a}, \exists \text{ adopted matrix units}$$

$$c_{\alpha i j}^{(a)} = \underline{f_{i j}^{(a)}} \in M_{N_a}$$

get: $S^{(a)} = [S_{i j}^{(a)}], S_{i j}^{(a)} = S(f_{i j}^{(a)})$

Big question: \mathcal{O}_A (classical case)

are dynamical objects!

$$\mathcal{O}_A = C(X_A) \rtimes_{\alpha, \mathbb{N}} N \quad (\text{Exel crossed product})$$

Current direction : Is there an analogue of this in quantum case ??

Partial Results : $K(B, \psi)$ = complete case.

Fact : $S^2 \in N (\tilde{\omega} \in \mathbb{E}vw)$

$\mathcal{O}_{K(B, \psi)}$ = universal C^* -alg. of QCH - $K(B, \psi)$
family

$$\cong \mathcal{O}_{\dim B}$$

$$\mathcal{O}_G \longrightarrow \begin{array}{l} \text{Cuntz-Pimsner algebra} \\ \text{associated to} \\ \text{a Cuntz } C^* \text{-corresp.} \\ \text{ass. to } G \end{array} \quad \mathcal{O}_{E_G}$$

$$G = K(B, \psi)$$

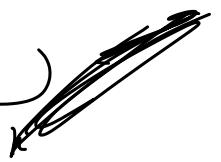
$$\mathcal{O}_{E_G} = \mathcal{O}_{\dim B} = \text{Cuntz algebra}$$

$$\forall B, \psi, \quad \mathcal{O}_{E_G} \cong \mathcal{O}_{\dim B} = \underline{C(X_A)} \rtimes \mathbb{N}$$



NC. C^* -dynamical
System

$$C(X_A) = B^{\otimes \mathbb{N}}$$



$C(X_A) = ?$ in general.