Idempotents, topologies and ideals

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Weakly almost periodic representations

(G, τ_G) – topological group,

Representation on a Banach space $\mathcal{X}$: strong operator continuous homomorphism $\pi : G \to \mathrm{Is}(\mathcal{X})$ (invertible isometries on $\mathcal{X}$).
Weak operator continuous $\Rightarrow$ strong operator continuous if

- π unitary on Hilbert (folklore), $\mathcal{X}$ reflexive [Megrelishvili '98];
- τ_G locally compact [Johnson '74].

Defintion

A representation $\pi : G \to \mathrm{Is}(\mathcal{X})$ is called weakly almost periodic (w.a.p.) if $\overline{\pi(G)\xi}^{w}$ is weakly compact for each ξ in $\mathcal{X}$.

Equivalently, $\overline{\pi(G)}^{wot}$ is weak operator compact in $\mathcal{B}(\mathcal{X})$.

Eg. $\mathcal{X}$ reflexive: any π weakly almost periodic

Weakly almost periodic part

$\pi : G \to \mathrm{Is}(\mathcal{X})$ representation:

$$\mathcal{X}^{\pi}_{\mathcal{W}} = \{\xi : \overline{\pi(G)\xi}^{w} \text{ is weakly compact}\}$$

is a closed subspace of $\mathcal{X}$.

In particular, in bounded continuous functions $\mathcal{CB}(G)$, let

$$\mathcal{LUC}(G) = \{f \in \mathcal{CB}(G) : s \mapsto f(s^{-1}\cdot) \text{ continuous}\}$$
$$\mathcal{W}(G) = \mathcal{LUC}(G)_{\mathcal{W}} \text{ (translation-invariant C*-subalgebra)}$$

(really $\mathcal{W}(G) = \mathcal{CB}(G)_{\mathcal{W}}$, also $\mathcal{W}(G) = \mathcal{RUC}(G)_{\mathcal{W}}$ with right translations), and we call the latter space that of <u>weakly almost periodic</u> functions.

A decomposition theorem

Theorem [Jacobs '54, Dye '65, Bergelson-Rosenblatt '88]

$\pi : G \to \mathrm{Is}(\mathcal{X})$ w.a.p. representation (unitary on Hilbert space)
Then $\mathcal{X}$ decomposes as two π-invariant (hence reducing) subspaces

$$\mathcal{X} = \mathcal{X}^{\pi}_{ret} \oplus \mathcal{X}^{\pi}_{wm}$$

where

$$\mathcal{X}^{\pi}_{ret} = \{\xi \in \mathcal{X} : \xi \in \overline{\pi(G)\eta}^{w} \text{ whenever } \eta \in \overline{\pi(G)\xi}^{w}\}$$
$$\mathcal{X}^{\pi}_{wm} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(G)\xi}^{w}\}$$

are spaces of "return" and "weakly mixing" vectors.

Decomposition of functions

Theorem [Eberlein '56, de Leeuw-Glicksberg '61]

$$\mathcal{W}(G) = \mathcal{AP}(G) \oplus \mathcal{W}_0(G)$$

where

$$\mathcal{AP}(G) = \{u \in \mathcal{C}_b(G) : \overline{\{f(s^{-1}\cdot) : s \in G\}}^{\|\cdot\|_\infty} \text{ compact}\}$$
$$\mathcal{W}_0(G) = \{u \in \mathcal{W}(G) : m(|u|) = 0\} \lhd \mathcal{W}(G)$$

for the unique invariant mean m on $\mathcal{W}(G)$, with $\mathcal{W}_0(G) \lhd \mathcal{W}(G)$.

Here, $\mathcal{W}_0(G) = \mathcal{W}(G)_{wm}$.

Semitopological compactifications

$G^{\mathcal{W}}$ – Gelfand spectrum of $\mathcal{W}(G)$, $\varepsilon^{\mathcal{W}} : G \to G^{\mathcal{W}}$ evaluation map

Proposition (folklore)

• $G^{\mathcal{W}}$ semigroup: unique extension of multiplication from dense subgroup $\varepsilon^{\mathcal{W}}(G)$.
• $G^{\mathcal{W}}$ <u>semitopological</u>: $s \mapsto st$, $t \mapsto st$ each continuous.
• Universal property: $h : G \to S$ continuous homo'm into compact semitop'l semigroup S, then $\mathcal{C}(S) \circ h \subseteq \mathcal{W}(G)$, which induces restriction $\rho : G^{\mathcal{W}} \to S$ with

$$\begin{tikzcd} & G^{\mathcal{W}} \arrow[d, dashed, "\rho"] \\ G \arrow[ur, "\varepsilon^{\mathcal{W}}"] \arrow[r, "h"'] & S \end{tikzcd}$$

i.e. $\rho \circ \varepsilon^{\mathcal{W}} = h$.

Topologies

Definition: weakly almost periodic topologies

$$\mathcal{T}(G) = \{\tau \subseteq \tau_G : (G,\tau) \text{ top'l group with } \tau = \sigma(G, \mathcal{W}^\tau(G))\}$$

where $\mathcal{W}^\tau(G) = \mathcal{W}(G) \cap \mathcal{C}_b^\tau(G)$ (τ-continuous elements)

Assumption: $\tau_G \in \mathcal{T}(G)$ and is Hausdorff
Warning: not all elements of $\mathcal{T}(G)$ are Hausdorff
e.g. $\tau_{triv} = \{\varnothing, G\}$, often $\tau_{ap} = \sigma(G, \mathcal{AP}(G))$
Important subsets:

$$\mathcal{T}_{lc}(G) = \{\sigma(G,\{h\}) \,|\, (H,\tau_H) \text{ loc. compact, } h : G \to H \text{ cts. homo'm}\}$$

$$\mathcal{T}_u(G) = \{\sigma(G,\{\pi\}) \,|\, \pi : G \to (\mathrm{Un}(\mathcal{H}), wot) \text{ cts. unitary rep'n}\}$$

$$\mathcal{T}(G) = \{\sigma(G,\{\pi\}) \,|\, \pi : G \to (\mathrm{Is}(\mathcal{X}), wot) \text{ cts. rep'n, } \mathcal{X} \text{ reflexive}\}$$

[Stern '94, Megrelishvili '98]

$$\mathcal{T}_{lc}(G) \subseteq \mathcal{T}_u(G) \subseteq \mathcal{T}(G)$$

On the scope of the classes of topologies

- [Teleman '57] G any topological group: $\varepsilon^{\mathcal{LUC}} : G \to G^{\mathcal{LUC}}$ (spectrum of $\mathcal{LUC}(G)$) homeomorphic embedding onto its range, $G^{\mathcal{LUC}}$ left topological semigroup.
- [Megrelishvili '01] $\mathcal{T}(\mathrm{Homeo}^+[0,1]) = \{\tau_{triv}\}$.
- [Ferri-Galindo '07] $G = (c_0, +)$ (norm topology), $\tau_G \notin \mathcal{T}(G)$.
- [Megrelishvili '02] $G = (L^4[0,1], +)$ (norm topology): $\tau_G \in \mathcal{T}(G) \setminus \mathcal{T}_u(G)$.
- After [Schoenberg '38], $G = (\ell^1, +)$ (norm topology): $\tau_G \in \mathcal{T}_u(G)$ as $e^{-\|\cdot\|_1{}^2} \in \mathrm{P}(G)$.

Conclusions: say $G = \mathbb{Z}^{\oplus\mathbb{N}} = (\mathbb{Z}^2)^{\oplus\mathbb{N}} \hookrightarrow \mathbb{R}^{\oplus\mathbb{N}}$
(i) $\mathcal{T}_{lc}(G) \subsetneq \mathcal{T}_u(G) \subsetneq \mathcal{T}(G) \subsetneq \{\text{group topologies}\}$.
(ii) Quotient groups of unitarizable groups may not be unitarizable.

- [Mayer '97] $G = N \rtimes R$ (certain connected Lie), $\mathcal{T}_{lc}(G) = \mathcal{T}(G)$.

Co-Cauchy/co-compact topologies, after [Ruppert '90]

$\tau \in \mathcal{T}(G)$
$G^{\mathcal{W}^\tau}$ – Gelfand spectrum of $\mathcal{W}^\tau(G)$, $\varepsilon^{\mathcal{W}^\tau} : G \to G^{\mathcal{W}^\tau}$ – evaluation

Completion: $G_\tau = G^{\mathcal{W}^\tau}(\varepsilon^{\mathcal{W}^\tau}(e_G))$ – intrinsic group at identity
G_τ complete w.r.t. 2-sided unifomity

$$\tau \subseteq \tau' \text{ in } \mathcal{T}(G) \;\Rightarrow\; \mathcal{W}^\tau(G) \subseteq \mathcal{W}^{\tau'}(G), \text{ induces } \rho_\tau^{\tau'} : G^{\mathcal{W}^{\tau'}} \to G^{\mathcal{W}^\tau}$$
$$\Rightarrow\; \eta_\tau^{\tau'} = \rho_\tau^{\tau'}|_{G_{\tau'}} : G_{\tau'} \to G_\tau \text{ cts. homo'm, dense range}$$

Lemma (after [Ruppert '90] in abelian case)

For $\tau \subseteq \tau'$ in $\mathcal{T}(G)$ TFAE
(co-compact) $\eta_\tau^{\tau'} : G_{\tau'} \to G_\tau$ open with $\ker \eta_\tau^{\tau'}$ compact
(co-Cauchy) each τ-Cauchy filter admits a τ'-Cauchy refinement

Write $\tau \subseteq_c \tau'$, in this case.

Idempotents

Definition

$\mathrm{ZE}(G^{\mathcal{W}}) = \{e \in G^{\mathcal{W}} : e^2 = e \;\&\; e\varepsilon^{\mathcal{W}}(s) = \varepsilon^{\mathcal{W}}(s)e \;\; \forall s \in G\}.$

In $\mathrm{ZE}(G^{\mathcal{W}})$: $e \leq e' \Leftrightarrow ee' = e$

A Galois connection

Theorem (after [Ruppert '90]; he covers abelian case)

There are maps $T : \mathrm{ZE}(G^{\mathcal{W}}) \to \mathcal{T}(G)$ and $E : \mathcal{T}(G) \to \mathrm{ZE}(G^{\mathcal{W}})$ s.t.

$$T(e) \subseteq T(e') \text{ if } e \leq e'$$
$$E(\tau) \leq E(\tau') \text{ if } \tau \subseteq \tau'$$
$$E(\tau) = E(\tau') \text{ if } \tau \subseteq_c \tau'$$
$$E \circ T = \mathrm{id}_{\mathrm{ZE}(G^{\mathcal{W}})} \text{ and } \tau \subseteq_c T \circ E(\tau).$$

Thus, if $\overline{\mathcal{T}}(G) = T(\mathrm{ZE}(G^{\mathcal{W}}))$, then $T \circ E|_{\overline{\mathcal{T}}(G)} = \mathrm{id}_{\overline{\mathcal{T}}(G)}$.

- (E, T) is a Galois connection for p.o. sets $(\mathcal{T}(G), \mathrm{ZE}(G^{\mathcal{W}}))$.
- $T \circ E : \mathcal{T}(G) \to \overline{\mathcal{T}}(G)$ is a closure operator.

Idea of proof

- Definition of T. For $e \in \mathrm{ZE}(G^{\mathcal{W}})$ let

$$T(e) = \sigma(G, \{s \mapsto e\varepsilon^{\mathcal{W}}(s) \in G^{\mathcal{W}}(e)\}).$$

- Defintion of E. For $\tau \in \mathcal{T}(G)$ let $\rho_\tau : G^{\mathcal{W}} \to G^{\mathcal{W}^\tau}$, given by restriction to $\mathcal{W}^\tau(G) \subseteq \mathcal{W}(G)$, and

$$S_\tau = \rho_\tau^{-1}(\{\varepsilon^{\mathcal{W}^\tau}(e_G)\}) \subseteq G^{\mathcal{W}}$$

which is a closed subsemigroup. [Ruppert's Book '90]: the minimal ideal $K(S_\tau)$ of S_τ is unique and is a group, with identity $E(\tau)$. I.e.

$$E(\tau) = \min \mathrm{E}(S_\tau) \quad \Rightarrow \quad E(\tau) \in \mathrm{ZE}(G^{\mathcal{W}}).$$

Picture of $G^{\mathcal{W}}$

If $\tau \in \overline{\mathcal{T}}(G)$ then

$$G^{\mathcal{W}^\tau} \cong E(\tau)G^{\mathcal{W}} \text{ (compression of } G^{\mathcal{W}}\text{)}$$
$$G_\tau = G^{\mathcal{W}}(E(\tau)) \text{ (intrinsic group at } E(\tau)\text{)}$$

Further, if $\tau \in \mathcal{T}(G)$

$$K_\tau = K(S_\tau) \cong \ker \eta_\tau^{T \circ E(\tau)}$$

is centric in $G^{\mathcal{W}}$, and letting m_{K_τ} be normalized Haar measure we have in convolution on $\mathcal{W}(G)^* \cong \mathrm{M}(G^{\mathcal{W}})$ that $m_{K_\tau} \leq E(\tau)$ and

$$G^{\mathcal{W}^\tau} \cong E(\tau)G^{\mathcal{W}}/K_\tau \cong m_{K_\tau} * G^{\mathcal{W}} \text{ (averaged over } K_\tau\text{)}$$
$$G_\tau = G^{\mathcal{W}}(E(\tau))/K_\tau \cong m_{K_\tau} * G^{\mathcal{W}}(E(\tau))$$

Ideals

Definition

An ideal $\mathcal{J}$ of $\mathcal{W}(G)$ is called an Eberlein-de Leeuw-Glicksberg (E-dL-G) ideal provided

- $\mathcal{J}$ is translation invariant; and
- $\mathcal{J}$ admits a linear complement $\mathcal{A}$, a C*-subalgebra of $\mathcal{W}(G)$.

Main Theorem on Ideals

(i) Let $\tau \in \overline{\mathcal{T}}(G)$, then $\mathcal{W}^\tau(G) = E(\tau) \cdot \mathcal{W}(G)$ and

$$\mathcal{I}(\tau) = \{u \in \mathcal{W}(G) : E(\tau) \cdot u = 0\}$$

is an E-dL-G ideal. Further

$$\mathcal{W}(G) = \mathcal{W}^\tau(G) \oplus \mathcal{I}(\tau).$$

(ii) Any E-dL-G ideal of $\mathcal{W}(G)$ is of the form $\mathcal{I}(\tau)$, as above.

Some decompositions

Lemma

Given $\tau \in \overline{\mathcal{T}}(G)$, $e_G \in U \in \tau$, $\varepsilon > 0$ and $u_1, \ldots, u_n$ in $\mathcal{I}(\tau)$, there is $s \in U$ s.t. $|u_j(s)| < \varepsilon$ for $j = 1, \ldots, n$.

Theorem

Given a w.a.p. rep'n $\pi : G \to \mathrm{Is}(\mathcal{X})$, $\tau \in \overline{\mathcal{T}}(G)$, the spaces

$$\mathcal{X}^{\pi}_{\tau} = \{\xi \in \mathcal{X} : \pi(\cdot)\xi \text{ is } \tau\text{-continuous}\}$$
$$\mathcal{X}^{\pi}_{\tau\perp} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(U)\xi}^{w} \text{ for each } e \in U \in \tau\}$$

are π-reducing with $\quad \mathcal{X} = \mathcal{X}^{\pi}_{\tau} \oplus \mathcal{X}^{\pi}_{\tau\perp}.$

Corollary (refinement of Jacobs, Dye, Bergelson-Rosenblatt)

$$\mathcal{X}^{\pi}_{wm} = \mathcal{X}^{\pi}_{\tau_{ap}\perp} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(U)\xi}^{w} \text{ for each } e \in U \in \tau_{ap}\}$$

Some more decompositions

If $\tau \in \mathcal{T}(G) \setminus \overline{\mathcal{T}}(G)$, we can average π over K_τ to get:

Theorem

Given a w.a.p. rep'n $\pi : G \to \mathrm{Is}(\mathcal{X})$ the space

$$\mathcal{X}^\pi_\tau = \{\xi \in \mathcal{X} : \pi(\cdot)\xi \text{ is } \tau\text{-continuous}\}$$

is π-reducing.

Examples

- [Segal-von Neumann '50] If $\pi : G_d \to \mathrm{Is}(\mathcal{X})$ is w.a.p., then $\mathcal{X}^\pi_{\tau_G}$ is reducing in $\mathcal{X}$; e.g. $\mathcal{W}(G)$ reducing in $\mathcal{W}(G_d)$.
- (after [Lau-Losert '90]) If $N \lhd G$ (and is closed)

$$\mathcal{X}^\pi_{\tau_{G:N}} = \{\xi \in \mathcal{X} : \pi(n)\xi = \xi \text{ for } n \text{ in } N\}$$

is π-reducing in $\mathcal{X}$, where $\tau_{G:N} = \sigma(G, \mathcal{W}(G/N) \circ q_N)$.

Unitarizable topologies

$$\mathcal{T}_u(G) = \{\tau \in \mathcal{T}(G) : \tau = \sigma(G, \mathrm{P}^\tau(G))\}$$

where $\mathrm{P}^\tau(G) = \{u \in \mathcal{C}_b^\tau(G) : u \text{ positive definite}\}$.

Let $\varpi_\tau = \bigoplus_{u \in \mathrm{P}^\tau(G)} \pi_u$ (GNS), so $\sigma(G, \mathrm{P}^\tau(G)) = \sigma(G, \{\varpi_\tau\})$.

Assume: $\tau_G \in \mathcal{T}_u(G)$.

Let $\varpi = \varpi_{\tau_G}$. $G^\varpi = \overline{\varpi(G)}^{wot}$ is a semitopological semigroup.

Theorem (Galois connection, revisited)

There are two order preserving maps

$$P : \mathcal{T}_u(G) \to \mathrm{ZE}(G^\varpi), \quad T_u : \mathrm{ZE}(G^\varpi) \to \mathcal{T}_u(G)$$

so $\tau \subseteq_c T_u \circ P(\tau)$ for each τ in $\mathcal{T}_u(G)$.

Let $\overline{\mathcal{T}}_u(G) = T_u \circ P(\mathcal{T}_u(G))$.

E-dL-G ideals in Fourier-Steiltjes algebras

$\mathrm{B}(G) = \mathrm{span}\mathrm{P}(G)$, $\mathrm{B}(G)^* \cong \mathrm{W}^*(G) = \varpi(G)''$.

$$\varpi(s) \mapsto \varpi(s) \otimes \varpi(s) \text{ extends to } \mathrm{W}^*(G) \to \mathrm{W}^*(G)\overline{\otimes}\mathrm{W}^*(G).$$

Preadjoint makes $\mathrm{B}(G)$ Banach algebra of continuous functions on G; see also [Lau-Ludwig '12].

Theorem

If $\tau \in \overline{\mathcal{T}}_u(G)$ then

$$\mathrm{B}^\tau(G) := P(\tau) \cdot \mathrm{B}(G) = \{u \in \mathrm{B}(G) : u \text{ is } \tau\text{-continuous}\}$$
$$\mathrm{I}(\tau) := (I - P(\tau)) \cdot \mathrm{B}(G) \lhd \mathrm{B}(G).$$

Moreover

$$\mathrm{B}(G) = \mathrm{B}^\tau(G) \oplus_{\ell^1} \mathrm{I}(\tau)$$

is the direct sum of a translation-invariant subalgebra and a translation invariant ideal.

Application: Operator amenability of $\mathrm{B}(G)$

Operator amenability … is a certain "averaging property" for a Banach algebra with cooperative operator space structure.

G – locally compact group: $\mathrm{L}^1(G)$, $\mathrm{A}(G)$ group & Fourier algebras

$$
\begin{tikzcd}
\mathrm{L}^1(G) \arrow[r, squiggly, "\text{dual}"] \arrow[d, hook] \arrow[drr] & \mathrm{L}^\infty(G) & \mathrm{VN}(G) & \mathrm{A}(G) \arrow[l, squiggly, "\text{dual}"'] \arrow[d, hook] \arrow[dll] \\
\mathrm{M}(G) \arrow[urr] & \mathrm{C}_0(G) \arrow[l, squiggly, "\text{dual}"] & \mathrm{C}^*(G) \arrow[r, squiggly, "\text{dual}"'] & \mathrm{B}(G) \arrow[ull]
\end{tikzcd}
$$

G convolution

"$\widehat{G}$" functions

What is know/expected for l.c. G

Theorem

For locally comapct G, TFAE:
(i) G is amenable;
(ii) [Johnson] $\mathrm{L}^1(G)$ is (operator) amenable; and
(iii) [Ruan] $\mathrm{A}(G)$ is operator amenable.

Theorem [Dales, Ghahramani & Helemskiĭ]

For locally compact G:

$$\mathrm{M}(G) \text{ is (operator) amenable } \Leftrightarrow G \text{ is discrete and amenable.}$$

Naïve conjecture

$$\mathrm{B}(G) \text{ is operator amenable } \Leftrightarrow G \text{ is compact.}$$

On operator amenability of $\mathrm{B}(G)$

Theorem [Runde-S.]

$\mathrm{B}(\mathbb{Q}_p \rtimes \mathbb{O}_p^\times)$ is operator amenable.

Theorem

$\mathrm{B}(G)$ operator amenable $\Rightarrow |\overline{\mathcal{T}}_u(G)| < \infty$.

Eg. $G = \mathbb{Q}_p \rtimes \mathbb{O}_p^\times$: $\overline{\mathcal{T}}_u(G) = \{\tau_{ap}, \tau_G\}$.

Theorem

If G locally compact and connected, then

$$\mathrm{B}(G) \text{ is operator amenable } \Leftrightarrow \ G \text{ is compact.}$$

Shameless advertising

N.S., Weakly almost periodic topologies, idempotents and ideals, *Indiana U. Math. J.* (accepted), 33 pp, http://www.iumj.indiana.edu/IUMJ/Preprints/8668.pdf

N.S., On operator amenability of Fourier-Stieltjes algebras, *Bull. Sci. Math*, 158 (2020), 102823, 16 pp

The University of Waterloo acknowledges that much of our work takes place on the traditional territory of the Neutral, Anishinaabeg and Haudenosaunee peoples. Our main campus is situated on the Haldimand Tract, the land granted to the Six Nations that includes six miles on each side of the Grand River. Our active work toward reconciliation takes place across our campuses through research, learning, teaching, and community building, and is centralized within the Office of Indigenous Relations

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Thank-you!