CP-Semigroups and dilations, subproduct systems and superproduct systems

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(joint work with Michael Skeide)

Technion

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Bhat's dilation theorem

Theorem (Bhat*, 1996)

Let $T = (T_t)_{t \geq 0}$ *be a Markov-semigroup on* $\mathcal{B}(H)$. *Then there exists a Hilbert space* K *containing* H, *and a unital E-semigroup* $\vartheta = (\vartheta_t)_{t \geq 0}$ *on* $\mathcal{B}(K)$, *such that*

$$T_t(A) = P_H \vartheta_t(A) P_H \quad , \qquad \textit{for all } t \geq 0 \textit{ and } A \in \mathcal{B}(H)$$

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$$\begin{tikzcd}
\mathcal{B}(K) \arrow[r, "\vartheta_t"] & \mathcal{B}(K) \arrow[d, "P_H \bullet P_H"] \\
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Definition

A **dilation** of T is a triple $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A}$ is a C*-algebra, $\vartheta = (\vartheta_s)_{s \in \mathbb{S}}$ is an E-semigroup, and $p \in \mathcal{A}$ is a projection, such that $\mathcal{B} = p\mathcal{A}p$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in \mathcal{B}, s \in \mathbb{S}$$

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Arveson, Bhat, Bhat-Skeide, Markiewicz, Muhly-Solel, Powers, SeLegue, S., S.-Solel, Skeide, Solel, Vernik,...

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$$T_s(pap) = p\vartheta_s(a)p \quad \text{for all } a \in \mathcal{A}, s \in \mathbb{S}$$

Questions

1. **Find necessary & sufficient conditions for existence of dilation.**
2. **Fix k. Does every Markov semigroup over $\mathbb{N}^k$ have a dilation?**

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Tensor product $E \odot F$: obtained from $E \otimes_{alg} F$ by inner product

$$\langle x \otimes y, x' \otimes y' \rangle := \langle y, \langle x, x' \rangle y' \rangle$$

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$w_{s,t}$ **is an isometry!**

Subproduct systems and product systems

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$$\begin{tikzcd}
\mathcal{E}_{r+s+t} \arrow[r] \arrow[d] & \mathcal{E}_r \odot \mathcal{E}_{s+t} \arrow[d] \\
\mathcal{E}_{r+s} \odot \mathcal{E}_t \arrow[r] & \mathcal{E}_r \odot \mathcal{E}_s \odot \mathcal{E}_t
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A **subproduct system** is a family $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-correspondences, where $\mathcal{E}_0 = \mathcal{B}$, together with a family $\{w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \odot \mathcal{E}_t\}$ of isometric bimodule maps, which iterate co-associatively, i.e., the following diagram is commutative ($\forall r, s, t$):

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\mathcal{E}_{r+s+t} \arrow[r] \arrow[d] & \mathcal{E}_r \odot \mathcal{E}_{s+t} \arrow[d] \\
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Definition

A family $\{\xi_s \in \mathcal{E}_s\}_{s\in\mathbb{S}}$ is called a **unit** if $w_{s,t}\xi_{s+t} = \xi_s \odot \xi_t$ for all s, t.

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Example (full tensor product). Let E be a Hilbert space. Define

$$E_n = E^{\otimes n}$$

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Quiz: What is the CP-semigroup that $\mathcal{E}^{\circledS}$ is its GNS system?

Reminder: The GNS representation $(\mathcal{E}, \xi)$ of a CP map

Let $T : \mathcal{B} \to \mathcal{B}$ be a CP map. Then there exists a unique C*-correspondence $\mathcal{E}$ over $\mathcal{B}$, and a vector $\xi \in \mathcal{E}$, such that

$$\mathrm{span}\, \overline{\mathcal{B}\xi\mathcal{B}} = \mathcal{E}$$

and

$$T(b) = \langle \xi, b\xi \rangle \quad \text{for all } b \in \mathcal{B}$$

Construction: on $\mathcal{E}_0 = \mathcal{B} \otimes_{alg} \mathcal{B}$ put inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* T(a^* c) d$$

and bimodule operation

$$a(x \otimes y)b = ax \otimes yb$$

Complete the quotient, and put $\xi = 1 \otimes 1$. This works:

$$\langle \xi, b\xi \rangle = \langle 1 \otimes 1, b \otimes 1 \rangle = 1^* T(1^* b) 1 = T(b)$$

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Question: what about non strict or non-full $\mathcal{A} \neq \mathcal{B}^a(E)$?

Sufficient condition for dilation

Theorem (Reformulated)

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Corollary (Bhat-Skeide 2000)

Let T be a UCP map on a C-algebra $\mathcal{B}$. Then there exists a triple $(\mathcal{A}, \vartheta, p)$ such that ϑ is a unital $*$-endomorphism and*

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The converse direction

We saw above: a sufficient condition for the existence of a dilation for a unital CP-semigroup T is that its GNS subproduct system embeds into a product system.

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What about the converse direction?

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A: sometimes, but not always.

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$(E_s)_{s \in \mathbb{S}}$ is a **superproduct system** (but not always a product system)

Superproduct systems

Definition

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A **product system** is a superproduct system in which $v_{s,t}$ are all unitaries.

Recap

Subproduct system:	$\mathcal{E}_s \odot \mathcal{E}_t \supseteq \mathcal{E}_{s+t}$
Product system:	$E_s \odot E_t = E_{s+t}$
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T on $\mathcal{B}$ a CP-semigroup $\rightsquigarrow$ subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences (the **GNS subproduct system**) and a unit $(\xi_s)_{s\in\mathbb{S}}$ such that

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- If T has a dilation $(\mathcal{A}, \vartheta, p)$, then the GNS subproduct system must **embed into a superproduct system.**

Dilations and superproduct systems

Theorem (S.-Skeide 2022)

Let $T = (T_s)_{s \in \mathbb{S}}$ be a Markov semigroup on a von Neumann algebra $\mathcal{B}$.

- *A sufficient condition for T to a have a full and strict dilation, is that the GNS subproduct system of T embeds into a* **product** *system.*
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Corollary (S.-Skeide)

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Proof

We built an example of a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a **superproduct** system.

Example

Subproduct system of Hilbert spaces that does not embed into a superproduct system

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QUIZ: What's missing?

Every* subproduct system gives rise to a CP-semigroup

But the subproduct system isn't recovered as the GNS system

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QUESTION: $\mathcal{F}^{\circledcirc}$ is certainly related to $\mathcal{E}^{\circledcirc}$ but how?

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ME subproduct system associated with a CP-semigroup

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Theorem

If T is strict then $\mathcal{E}^{\odot}$ with $\mathcal{E}_s = E^* \odot \mathcal{F}_s \odot E$ is a subproduct system, the subproduct system of $\mathcal{B}$-correspondences associated with T.

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Corollary

For every adjointable subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s \in \mathbb{S}}$ over $\mathcal{B}$ there is a strict CP-semigroup T on $\mathcal{B}^a(E)$ for some $\mathcal{B}$-correspondence E such that $\mathcal{E}^{\circledcirc}$ is the subproduct system of $\mathcal{B}$-correspondences associated with T.

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Return to example: $\mathcal{E}^{\otimes}$ a subPS that does not embed into superPS

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Return to example: $\mathcal{E}^{\otimes}$ a subPS that does not embed into superPS
$\mathcal{E}^{\otimes}$ Morita equivalent to the GNS system $\mathcal{F}^{\otimes}$ of a CP-semigroup T
$\mathcal{E}^{\otimes}$ does not embed $\Rightarrow$ $\mathcal{F}^{\otimes}$ does not embed $\Rightarrow$ T has no dilation

Dilations and superproduct systems: open problems

Theorem (S.-Skeide 2022)

Let $T = (T_s)_{s \in \mathbb{S}}$ be a Markov semigroup on a von Neumann algebra $\mathcal{B}$.

- *A sufficient condition for T to a have a dilation, is that the GNS subproduct system of T embeds into a* **product** *system.*
- *A necessary condition for T to have a dilation, is that the GNS subproduct system of T embeds into a* **superproduct** *system.*

Open problems:

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- Does embeddability of the GNS system into a superproduct system guarantee the existence of a dilation?
- Does the existence of a dilation imply the existence of a strict full dilation? (full and strict dilation $\cong$ embed into PS)

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Open problems:

- Does embeddability of the GNS system into a superproduct system guarantee the existence of a dilation?
- Does the existence of a dilation imply the existence of a strict full dilation? (full and strict dilation $\cong$ embed into PS)
- Can every superproduct system be embedded into a product system?

Thank you!