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Crossed and semicrossed products II

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We will work exclusively with $G = \mathbb{Z}$

We want to incorporate $*$-endomorphisms of a C^*-algebra $\mathcal{Q}$ that might not be injective

The notion of a covariant repn. as introduced in the previous lecture is troublesome. Indeed if $(\mathcal{Q}, \alpha)$ is a C^*-dynamical system with α not injective and $(\pi, U, \mathcal{H})$ a covariant representation, then for $a \in \operatorname{Ker}\alpha$

$$\pi(\alpha(a)) = U\pi(a)U^* = 0 \Rightarrow \pi(a) = 0$$

i.e. we are really obtaining covariant repns for $(\mathcal{Q}/\operatorname{Ker}\alpha, \dot{\alpha})$

Peters in 1981 found the right approach, suprisingly by studying non-selfadjoint operator algebras.
The concept in C^*-algebra theory was formalized later with the introduction of C^*-correspondences.

Let $(\mathcal{Q}, \alpha)$ as above and let $\pi : \mathcal{Q} \to \mathcal{B}(\mathcal{H})$

any representation. Consider

$$\hat{\pi}(a) = \begin{bmatrix} \pi(a) & & & \mathbf{0} \\ & \pi(\alpha(a)) & & \\ \mathbf{0} & & \pi(\alpha^2(a)) & \\ & & & \ddots \end{bmatrix} \qquad (*)$$

$$V = \begin{bmatrix} 0 & 0 & 0 & & \mathbf{0} \\ I & 0 & 0 & & \\ 0 & I & 0 & & \\ & \mathbf{0} & & \ddots & \end{bmatrix}$$

and notice that $\hat{\pi}(a)\hat{V} = \hat{V}\hat{\pi}(\alpha(a))$, $\forall a \in \mathcal{A}$ $(*)$
Furthermore if π is faithful, then so is $\hat{\pi}$
Also unitary covariant representations satisfy $(*)$

$$U\pi(a)U^* = \pi(\alpha(a))$$

and so

$$\pi(a)U^* = U^*\pi(\alpha(a))$$

The above motivate the following definition

An isometric covariant repn. $(\pi, \mathcal{U}, \mathcal{H})$ of the dynamical system $(\mathcal{Q}, \alpha)$ consists of

(i) $\mathcal{H}$ Hilbert space

(ii) $\pi : \mathcal{Q} \longrightarrow B(\mathcal{H})$ $*$-representation

(iii) an isometry $V \in B(\mathcal{H})$

so that

$$\pi(a) V = V \pi(\alpha(a))$$

The representation $(\hat{\pi}, V)$ appearing in $(*)$ is an isometric covariant representation for $(\mathcal{Q}, \alpha)$ (orbit representation)

We can define four universal operator algebras.

$$\left.\begin{array}{l} \mathcal{Q} \rtimes_\alpha^{un} \mathbb{Z} \supseteq \mathcal{Q} \rtimes_\alpha^{un} \mathbb{Z}^+ \\ \mathcal{Q} \rtimes_\alpha^{is} \mathbb{Z} \supseteq \mathcal{Q} \rtimes_\alpha^{is} \mathbb{Z}^+ \end{array}\right\} \text{semicrossed pr}$$

where $\mathcal{Q} \rtimes_\alpha^{un} \mathbb{Z}^+$ (resp. $\mathcal{Q} \rtimes_\alpha^{is} \mathbb{Z}^+$) are the subalgebras of the corresponding C^*-algebras generated by $\mathcal{Q}$ and the universal unitary (resp. isometry)

We saw that $\mathcal{Q} \rtimes_\alpha^{is} \mathbb{Z} \not\cong \mathcal{Q} \rtimes_\alpha^{un.} \mathbb{Z}$
for various reasons

(i) if α is not injective, $\mathcal{Q} \rtimes_\alpha^{un} \mathbb{Z}$ cannot contain a faithful copy of $\mathcal{Q}$

(ii) even if α is a $*$-automorphism of $\mathcal{Q}$, the algebra $\mathcal{Q} \rtimes_\alpha^{is} \mathbb{Z}$ cannot be simple. while in many cases $\mathcal{Q} \rtimes_\alpha^{un} \mathbb{Z}$ ($\cong \mathcal{Q} \rtimes_{\alpha^{-1}} \mathbb{Z}$) is.

The situation for non-selfadjoint algebras is much more pleasant

Wold decomposition for isometries

Let V be an isometry acting on a Hilbert space $\mathcal{H}$.
Let $W_V := [V(\mathcal{H})]^\perp$. Then the subspaces $\{V^n(W_V)\}$ are mutually orthogonal and the restriction of V on the reducing subspace

$$M_V = \bigoplus_{n=0}^{\infty} V^n(W_V) \quad (*)$$

is unitarilly equivalent to a forward shift. Furthermore, restriction of V on the orthogonal complement of $(*)$ is a unitary operator.

Proof.

(i) The subspaces $\{V^n(W_V)\}_{n=0}^{\infty}$ are orthogonal to each other

Indeed if $x, y \in W_V$ and $m > n$, then

$$\langle V^m x, V^n y \rangle = \langle (V^n)^* V^m x, y \rangle$$

$$= \langle (V^n)^* V^n V^{m-n} x, y \rangle$$

$$= \langle V^{m-n} x, y \rangle = 0$$

(ii) The subspace $\bigoplus_{n=0}^{\infty} V^n(W_V)$ is reducing for V

Clearly it is invariant for V. Furthermor all V^n $n \geq 1$, are invariant for V^*

Finally, for $x \in W_V, = [V(\mathcal{H})]^{\perp}$

$$\langle V^* x, V^* x \rangle = \langle x, V V^* x \rangle = 0$$

and so W_V invariant for V^* as well

(Recall that a subspace M is reducing for an oper S iff the orthogonal projection on M commutes with S.)

(iii) $V|_{M_V}$ is a shift.

Define a unitary $\mathcal{U}_V$ on $\sum_{n=0}\oplus V^n(W_V)$ by

$$\mathcal{U}_V\left(\sum_{n=0}^{\infty}\oplus V^n x_n\right) = (x_0, x_1, x_2, \dots), \quad x_n \in$$

Then

$$\mathcal{U}_V^* S\, \mathcal{U}_V = V$$

where

$$S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$$

(Note that S acts on $W_V \oplus W_V \oplus \cdots$ and admits a matrix w.r.t. that decomposition

$$S = \begin{bmatrix} 0 & & & & \mathbf{0} \\ I & 0 & & & \\ & I & 0 & & \\ \mathbf{0} & & I & 0 & \\ & & & & \ddots \end{bmatrix}$$

)

(iv) $V|_{\mathcal{M}_V^\perp}$ is unitary (onto suffices)

If $x \in \mathcal{M}_V^\perp$ with $x \perp V(\mathcal{M}_V^\perp)$ then $x \in \mathcal{M}_V \Rightarrow x$

Wold decomposition for covariant representatio of dynamical systems

Consider a C^*-dynamical system $(\mathcal{Q}, \alpha)$ with α

a unital endomorphism and let $(\pi, V, \mathcal{H})$ be an isomet
covariant representation.

Let W_v, M_v and U_v and S be the Wold decom
for V. We examine the consequences on the representa
π

(i) W_v is an invariant (hence reducing) subspace for

Indeed, since $\pi(a)V = V\pi(\alpha(a))$, $\forall a \in \mathcal{Q}$
$\pi(\mathcal{Q})$ leaves invariant $V(\mathcal{H})$ and therefor its
complement $V(\mathcal{H})^{\perp}$

Let us denote by π_0 the restriction of π on W_v

(ii) $M_v = \sum_{n=0}^{\infty} \oplus V^n W_v$ is invariant by $\pi(\mathcal{Q})$

Therefore the covariant representation (π, V)
splits as

$$(\pi, V) = (\pi|_{M_v^{\perp}}, V|_{M_v^{\perp}}) \oplus (\pi|_{M_v}, V|_{M_v})$$

unitary covariant
representation

Now use the unitary U_v to obtain an
equivalence

$$U_v^*(V|_{M_v})U = S$$

where S is a shift.
Remarkaby on $\pi|_{M_V}$ has a familiar picture under conjugation by U_V
Indeed

$$U_V \pi(a) U_V^* (x_0, x_1, x_2, \ldots) =$$

$$= U_V \pi(a) \left(\sum V^n x_n \right)$$

$$= U_V \left(\sum V^n \pi_0(\alpha^{(n)}(a)) x_n \right)$$

$$= \left(\pi_0(a) x_0,\ \pi_0(\alpha(a)) x_1,\ \pi_0(\alpha^2(a)) x_2, \ldots \right)$$

i.e. as a matrix

$$U_V^* \pi|_{M_V}(a) U_V = \begin{bmatrix} \pi_0(a) & & & 0 \\ & \pi_0(\alpha(a)) & & \\ & & \pi_0(\alpha^2(a)) & \\ 0 & & & \ddots \end{bmatrix}$$

Therefore $(\pi|_{M_V}, V|_{M_V})$ is unitarilly equivalent to one (the only !!) fundamental example of covariant representation, the orbit representati

Our analysis has signifficant consequences

<u>THEOREM</u> (Peters '81) Let $(\mathcal{A}, \alpha)$ be an injective C^*-dynamical system.
Then
$$\mathcal{A} \rtimes_\alpha^{un} \mathbb{Z}^+ \cong \mathcal{A} \rtimes_\alpha^{is} \mathbb{Z}^+$$
isometrically.

<u>Proof</u> We prove the automorphic case
Let u and v be the univeral unitary and isometry in $\mathcal{A} \rtimes_\alpha^{un} \mathbb{Z}^+$ and $\mathcal{A} \rtimes_\alpha^{is} \mathbb{Z}^+$ respectively.
It suffices to show that the map
$$x \equiv \sum_{n=0}^{k} v^n a_n \longmapsto \sum_{n=0}^{k} u^n a_n \equiv x$$
is an isometry. Since
$$\|x\|_{is} = \sup\left\{ \|(\pi \rtimes V)(x)\| \;\middle|\; (\pi, V) \text{ isom. cov. repr} \right.$$
$$\|x\|_{un} = \sup\left\{ \|(\pi \rtimes U)(x)\| \;\middle|\; (\pi, U) \text{ unit. cov. rep} \right.$$

we have that $\|x\|_{un} \leq \|x\|_{is}$ and so the above map is a well defined contraction.

On the other hand, if (π, V) is an isometric covariant repn then it is unitarilly equivalent to the restriction on an invariant subspace of a unitary covariant repn.

Indeed

$$(\pi, V) = \underbrace{\left(\pi|_{M_V^\perp}, V|_{M_V^\perp}\right)}_{\text{unitary covariant representation}} \oplus \left(\pi|_{M_V}, V|_{M_V}\right)$$

and $(\pi|_{M_V}, V|_{M_V})$ is unitarilly equivalent to restriction of

$$\tilde{\pi}(a) = \begin{bmatrix} \ddots & & & & & \\ & \pi(\alpha^{-2}(a)) & & & 0 & \\ & & \pi(\alpha^{-1}(a)) & & & \\ & & & \boxed{\pi(a)} & & \\ & 0 & & & \pi(\alpha(a)) & \\ & & & & & \pi(\alpha^{2}(a)) \end{bmatrix}$$

$$\begin{bmatrix} & & \ddots \end{bmatrix}$$

$$\tilde{S} = \begin{bmatrix} \ddots & & & & & \\ & 0 & & & & \\ & I & 0 & & \mathbf{0} & \\ & & I & \boxed{0} & & \\ \mathbf{0} & & & I & 0 & \\ & & & & I & 0 \\ & & & & & & \ddots \end{bmatrix}$$

on an invariant subspace. Hence

$$\|(\pi \rtimes V)(x)\| = \|(\pi|_{M_V^\perp} \rtimes V|_{M_V^\perp})(x) \oplus (\pi|_{M_V} \rtimes V|_{M_V})(x$$

$$= \max\{ \|(\pi|_{M_V^\perp} \rtimes V|_{M_V^\perp})(x)\| , \|(\pi|_{M_V} \rtimes V|_{M_V})$$

$$\leq \max\{ \|(\pi|_{M_V^\perp} \rtimes V|_{M_V^\perp})(x)\| , \|(\tilde{\pi} \rtimes \tilde{S})(x)\| \}$$

$$\leq \|x\|_{un}$$

and so $\|x\|_{is.} \leq \|x\|_{un}$. ∎

The technique of the above proof gives another important result.

<u>THEOREM</u> The orbit representations suffice t norm the semicrossed product.

<u>Proof</u> (automorphic case) By the previous Theorem

$$\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}^{+} \subseteq \mathfrak{A} \rtimes_{\alpha^{-1}} \mathbb{Z} = \mathfrak{A} \rtimes^{r}_{\alpha^{-1}} \mathbb{Z}$$

and this one is normed by a two-sided orbit representation.

$$\hat{\pi}(a) = \begin{bmatrix} \ddots & & & & & \\ & \pi(\alpha^{-2}(a)) & & & \mathbf{0} & \\ & & \pi(\alpha^{-1}(a)) & & & \\ & & & \boxed{\pi(a)} & & \\ & \mathbf{0} & & & \pi(\alpha^{1}(a)) & \\ & & & & & \pi(\alpha^{2}(a)) \\ & & & & & & \ddots \end{bmatrix}$$

$$U = \begin{bmatrix} \ddots & & & & \\ & 0 & & & \mathbf{0} \\ & I & 0 & & \\ & & I & \boxed{0} & \\ \mathbf{0} & & & I & 0 \end{bmatrix}$$

one sid

Now restrict and approximat $\hat{\pi} \rtimes U$ by orbit representations.

Not clear what the "right" C^*-algebra is for an $*$-endomorphism

It turns out, we need to abstract.

Hilbert C^*-module $(X, \mathfrak{A}, \langle \cdot | \cdot \rangle)$

(i) $\mathfrak{A}$ · C^*-algebra, unital
(ii) $(X, \|\cdot\|)$ is a Banach space and a right $\mathfrak{A}$-module
(iii) $\langle \cdot | \cdot \rangle$ is an $\mathfrak{A}$-valued inner product, i.e.,

$$\langle \xi | \eta a \rangle = \langle \xi | \eta \rangle a$$

$$\langle \xi | \eta \rangle^* = \langle \eta | \xi \rangle$$

$$\langle \xi | \zeta + \lambda \eta \rangle = \langle \xi | \zeta \rangle + \lambda \langle \xi | \eta \rangle, \quad \xi, \zeta, \eta \in X, \text{ at } \lambda \in \mathbb{C}$$

and

$$\|\xi\|^2 = \|\langle \xi | \xi \rangle\|, \quad \forall \xi \in X.$$

C^*-correspondence: $(X, \mathcal{Q}, \varphi)$

$(X, \mathcal{Q}, \langle \cdot | \cdot \rangle)$ Hilbert C^*-module

$\varphi : \mathcal{Q} \to \mathcal{L}(X)$ $*$-representation into the adjointable operators on X, unital

which allows us to view X as a left $\mathcal{Q}$-module

Example (Concrete C^*-correspondences)

Consider $X \subseteq B(\mathcal{H})$ any closed $\mathcal{Q}$-bimod where $\mathcal{Q} \subseteq B(\mathcal{H})$ is any unital C^*-algebra, satisfying

$$X^* X \subseteq \mathcal{Q}$$

(Here $\langle \xi, \eta \rangle := \xi^* \eta$ and

$\varphi : \mathcal{Q} \to \mathcal{L}(X) : a \to M_a =$ left multiplication by a

Example If $\{V_1, V_2, \dots, V_n\}$ are isometries on $\mathcal{H}$
orthogonal ranges, then

$$X = [\{V_1, V_2, \dots, V_n\}]$$

is a C^*-correspondence with $\mathcal{Q} = \mathbb{C}I$

Example The C^*-correspondence $\mathcal{Q}_\alpha$.

Let $\mathcal{Q}$ be a unital C^*-algebra and $\alpha : \mathcal{Q} \to \mathcal{Q}$ $*$-

Consider

$$\mathcal{Q}_\alpha = \mathcal{Q}$$

as a C^*-correspondence over $\mathcal{Q}$ with

$$\varphi(a)\xi b := \alpha(a)\xi b, \quad \text{and}$$

$$\langle \xi | \eta \rangle := \xi^* \eta, \qquad a, b, \xi, \eta \in \mathcal{Q}.$$

A Toeplitz representation for $(X, \mathcal{Q}, \varphi)$ is a triple $(\pi, t$
where

(i) $\pi : \mathcal{Q} \longrightarrow B$ $\quad$ $*$-representation (non-deg

(ii) $t : X \longrightarrow B$ linear map satisfying

$$t(a \cdot \xi \cdot a') = \pi(a)\, t(\xi)\, \pi(a')$$

and

(iii) $\pi(\langle \xi, \eta \rangle) = t(\xi)^* t(\eta)$, $a, a' \in \mathcal{Q}$, ξ, η.

The Toeplitz C^*-algebra $\mathcal{T}(X, \mathcal{Q}, \varphi)$ or simply is the universal C^*-algebra for all Toeplitz rep of $(X, \mathcal{Q}, \varphi)$.

$$\begin{tikzcd}
(X,\mathcal{Q}) \arrow[r, "{(\bar\pi, \bar t)}", "\text{injective}"'] \arrow[dr, "{(\pi, t)}"'] & \mathcal{T}(X) = C^*(\pi, t) \arrow[d, dashed, "\exists\ \pi \times t"] \\
& B
\end{tikzcd}$$

The tensor algebra $\mathcal{T}^+(X)$ is the norm-closed of $\mathcal{T}(X)$ generated by X and $\mathcal{Q}$.

If $(X, \mathcal{Q})$ is a concrete C^*-correspondence

$$\mathcal{T}(X) \cong C^*(\mathcal{Q} \otimes I,\ X \otimes S) \subseteq B(\mathcal{H} \otimes \ell^2(\mathbb{N}$$

where S is the forward shift on $\ell^2(\mathbb{N})$ (Kats

Let (π, t) be a Toeplitz repn. of $\mathcal{Q}_\alpha$

Then

$$\pi(a)t(1) = t(a\cdot 1) = t(1\cdot\alpha(a))$$

$$= t(1)\pi(\alpha(a))$$

Also

$$t(1)^* t(1) = \pi(\langle 1, 1\rangle) = \pi(1) = I$$

and so the pair $(\pi, t(1))$ forms an isometric covariant representation of $(\mathcal{Q}, \alpha)$.

$$\mathcal{C}(\mathcal{Q}_\alpha) = \mathcal{Q} \rtimes_\alpha^{D.} \mathbb{Z} \qquad \underline{\underline{\mathcal{T}^+(\mathcal{Q}_\alpha) =}}$$

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