An operator system approach to quantum correlations

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Organization

Organization of talk:

- Quantum correlations
- Correlations from abstract structures
- Quantum commuting operator systems
- Matricial AOU spaces
- Correlations from operator systems

Quantum correlations

Throughout, we'll let $[n] = \{1, 2, \ldots, n\}$.

Let $n, m \in \mathbb{N}$. We call a tuple

$$p = \{p(a, b|x, y)\}_{a,b\in[m],x,y\in[n]}$$

a **correlation** if each $p(a, b|x, y) \geq 0$ and for every $x, y \in [n]$

$$\sum_{a,b\in[m]} p(a, b|x, y) = 1.$$

A correlation models a scenario where Alice & Bob each get questions from a set of n questions and must each give answers from a set of m answers. We interpret $p(a, b|x, y)$ to be the probability that (Alice, Bob) return answers (a, b) given that they received questions (x, y).

Nonsignalling correlations

We call a correlation p **nonsignalling** if the **marginal densities** given by

$$p_A(a|x) := \sum_{c\in[m]} p(a,c|x,w) \quad \text{and} \quad p_B(b|y) := \sum_{c\in[m]} p(c,b|z,y)$$

are well-defined.

We call these relations the **nonsignalling conditions** and they ensure Alice and Bob provide answers independently.

We let $C_{ns}(n,m)$ denote the set of all nonsignalling correlations in the n-question m-answer scenario. We have that $C_{ns}(n,m)$ is a convex polytope in $\mathbb{R}^{n^2m^2}$.

Local correlations

A correlation is **deterministic** it is nonsignalling and if every $p_A(a|x), p_B(b|y) \in \{0, 1\}$.

A convex combination of deterministic correlations is called a **local correlation**. We let $C_{loc}(n, m)$ denote the set of all local correlations.

The set $C_{loc}(n, m)$ is a convex polytope by its definition.

Quantum correlations

A correlation p is called a **quantum correlation** if there exist finite dimensional Hilbert spaces H_A, H_B, projections $\{E_{x,a}\} \subseteq B(H_A)$ and $\{F_{y,b}\} \subseteq B(H_B)$ satisfying

- $\sum_a E_{x,a} = I_A$ for all x,
- $\sum_b F_{y,b} = I_B$ for all y,

and a unit vector $\phi \in H_A \otimes H_B$ such that
$p(a, b|x, y) = \langle E_{x,a} \otimes F_{y,b}\phi, \phi\rangle$.

We let $C_q(n, m)$ denote the set of all quantum correlations

If the state ϕ is separable, then p is a local correlation. Hence, correlations in $C_q(n, m) \setminus C_{loc}(n, m)$ arise from entangled states.

Quantum commuting correlations

A correlation p is called a **quantum commuting correlation** if there exists a Hilbert space H, projections $\{E_{x,a}, F_{y,b}\} \subseteq B(H)$ satisfying

- $\sum_a E_{x,a} = I$ for all x,
- $\sum_b F_{y,b} = I$ for all y,
- $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$,

and a unit vector $\phi \in H$ such that $p(a, b|x, y) = \langle E_{x,a}F_{y,b}\phi, \phi\rangle$.

We let $C_{qc}(n, m)$ denote the set of all quantum commuting correlations

The definition of a quantum commuting correlation is based on the Haag-Kastler axioms of quantum mechanics.

Hierarchy of correlation sets

We have the following inclusions:

$$C_{loc}(n,m) \subseteq C_q(n,m) \subseteq C_{qc}(n,m) \subseteq C_{ns}(n,m).$$

Each inclusion has been demonstrated to be proper.

- $C_{loc}(n,m) \subseteq C_q(n,m)$: John Bell, 1960s.
- $C_q(n,m) \subseteq C_{qc}(n,m)$: William Slofstra, 2017.
- $C_{qc}(n,m) \subseteq C_{ns}(n,m)$: Boris Tsirelson, 1980s.

The inclusion $\overline{C_q(n,m)} \subseteq C_{qc}(n,m)$ is proper by work of Ji-Natarajan-Vidick-Wright-Yuen from early 2020. This result solved a 50-year-old open problem in operator algebras, *Connes' embedding problem* (due to work of Fritz and Junge-Navascues-Palazuelos-Perez-Garcia-Scholz-Werner).

Correlations from abstract structures

Operator Theoretic vs. Algebraic definitions

The correlation definition

$$p(a, b|x, y) = \langle E_{x,a} F_{y,b} \phi, \phi \rangle$$

is *physical* or *operator theoretic* in nature:

- Hilbert space $H \simeq$ a physical system.
- State $\phi \simeq$ the state of that system.
- Operators $E_{x,a}$, $F_{y,b} \simeq$ measurements performed in labs.

Is there an essentially *algebraic* framework for generating correlations?

Correlations from C*-algebras

Theorem

A correlation p is quantum commuting if and only if there exists a C-algebra $\mathcal{A}$, projection-valued measures $\{E_{x,a}\}, \{F_{y,b}\} \subseteq \mathcal{A}$ with $[E_{x,a}, F_{y,b}] = 0$ and a state ϕ such that $p(a, b|x, y) = \phi(E_{x,a}F_{y,b})$. Moreover,*

- *$p \in C_q(n, m)$ if and only if the statement holds for a finite-dimensional $\mathcal{A}$.*
- *$p \in C_{loc}(n, m)$ if and only if the statement holds for a commutative $\mathcal{A}$.*

Proof.

$\Longrightarrow$: Consider $\mathcal{A} = C^*(E_{x,a}, F_{y,b})$ and $\phi(x) = \langle xh, h\rangle$.
$\Longleftarrow$: GNS theorem. $\square$

Operator Systems

An **operator system** is a $*$-closed unital subspace of $B(H)$.

An **abstract operator system** consists of a $*$-vector space $\mathcal{V}$, a sequence of cones $C_n \subseteq M_n(\mathcal{V})_h$ satisfying

$$\alpha^* C_n \alpha \subseteq C_m$$

for every $\alpha \in M_{n,m}$, and an element $e \in \mathcal{V}_h$ such that $(M_n(\mathcal{V}), C_n, I_n \otimes e)$ is an AOU space for every $n \in \mathbb{N}$.

Theorem (Choi-Effros)

Let $\mathcal{V}$ be an abstract operator system. Then there exists a Hilbert space H and a unital complete order embedding $\pi : \mathcal{V} \to B(H)$.

Correlations from operator systems?

Assume $p \in C_{qc}(n, m)$ with $p(a, b|x, y) = \langle E_{x,a}F_{y,b}\phi, \phi\rangle$. Then:

$$\mathcal{V} = \text{span}\{E_{x,a}F_{y,b}\}$$

is an operator system and $E_{x,a}F_{y,b} \mapsto \langle E_{x,a}F_{y,b}\phi, \phi\rangle$ is a state on $\mathcal{V}$.

So correlations only require a finite dimensional operator system and a state.

We can abstractly characterize operator systems, but what about operator systems of the form

$$\mathcal{V} = \text{span}\{E_{x,a}F_{y,b}\}?$$

Quantum commuting operator systems

Projections in operator systems

Assume $p \in \mathcal{V} \subseteq B(H)$, and p is a projection, $\mathcal{V}$ an operator system.

If we forget the concrete structure, then p remains a positive contraction in the (abstract) operator system $\mathcal{V}$.

The Choi-Effros Theorem allows us to recover p as an operator on $B(K)$, but does not guarantee that p will be a projection.

Question

Can we detect the presence of a projection p in an abstract operator system $\mathcal{V}$?

Compressions of operator systems

Proposition

Let $p \in \mathcal{V} \subseteq B(H)$, $x \in \mathcal{V}$ with $x = x^*$. Then $pxp \geq 0$ if and only if for every $\epsilon > 0$ there exists a $t > 0$ such that

$$x + \epsilon p + t(I - p) \in \mathcal{V}_+.$$

Thus if $p \in \mathcal{V}$ is a projection, we can detect when $pxp \geq 0$ using only the data of the operator system $(\mathcal{V}, \{C_n\}, e)$.

Characterization of projections

Assume $p \in \mathcal{V} \subseteq B(H)$ and p is a projection. Let $q = I - p$. Then we may decompose each $x \in \mathcal{V} \subseteq B(H) = B(pH \oplus qH)$ as

$$x = \begin{pmatrix} pxp & pxq \\ qxp & qxq \end{pmatrix}.$$

Consider the compression of

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \text{ by } p \oplus q, \text{ i.e. } \begin{pmatrix} \begin{pmatrix} pxp & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & pxq \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ qxp & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & qxq \end{pmatrix} \end{pmatrix}.$$

Observe that $x \geq 0$ if and only if $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ has positive compression by $p \oplus q$.

Characterization of projections

Definition

We call a positive contraction p in an abstract operator system $\mathcal{V}$ an **abstract projection** if the set of $x = x^* \in M_n(\mathcal{V})$ satisfying for every $\epsilon > 0$ there exists $t > 0$ such that

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} + \epsilon I_n \otimes (p \oplus q) + tI_n \otimes (q \oplus p) \geq 0$$

coincides with the positive cone of $M_n(\mathcal{V})$.

Theorem (Araiza, R.)

A positive contraction p in an operator system $(\mathcal{V}, \{C_n\}, e)$ is an abstract projection if and only if there exists a unital complete order embedding $\pi : \mathcal{V} \to B(H)$ such that $\pi(p)$ is a projection.

The set of abstract projections

The theorem allows us to build $\pi : \mathcal{V} \to B(H)$ mapping a single abstract projection p to an honest projection $\pi(p)$. What if there are many abstract projections?

Theorem (Araiza, R.)

Let p be an abstract projection in an operator system $\mathcal{V}$. Then p is a projection in $C_e^*(\mathcal{V})$.

Thus, if $p_1, p_2, \ldots, p_N \in \mathcal{V}$ are all abstract projections, then $p_1, p_2, \ldots, p_N$ are projections in $C_e^*(\mathcal{V})$.

Quantum commuting operator systems

A **quantum commuting operator system** is a finite dimensional operator system with unit e spanned by positive contractions $\{Q(a,b|x,y) : a,b \in [m], x,y \in [n]\}$ such that

- For each $x,y \in [n]$, $\sum_{a,b\in[m]} Q(a,b|x,y) = e$
- For each $x,y \in [n]$ and $a,b \in [m]$, the vectors

  $$E(a|x) := \sum_{c\in[m]} Q(a,c|x,w) \quad \text{and} \quad F(b|y) := \sum_{c\in[m]} Q(c,b|z,y)$$

  are well-defined
- Each generator $Q(a,b|x,y)$ is an abstract projection.

Quantum correlations and operator systems

Theorem (Araiza, R.)

A correlation p is quantum commuting if and only if there exists a quantum commuting operator system $\mathcal{V} = span\{Q(a, b|x, y)\}$ and a state $\phi : \mathcal{V} \to \mathbb{C}$ such that

$$p(a, b|x, y) = \phi(Q(a, b|x, y)).$$

Proof elements:

- The linear relations between $\{Q(a, b|x, y)\}$ ensure p is nonsignalling correlation.
- Each $Q(a, b|x, y)$ is a projection in $C_e^*(\mathcal{V})$.
- The relations $Q(a, b|x, y) = E(a|x)F(y|b) = F(y|b)E(a|x)$ are forced.

Matricial AOU spaces

k-AOU spaces

By a k-AOU space, we mean a $*$-vector space $\mathcal{V}$ with a positive cone $C \subseteq M_k(\mathcal{V})_h$ satisfying $\alpha^* C \alpha \subseteq C$ for all $\alpha \in M_k$, and an element $e \in \mathcal{V}$ such that $(M_k(\mathcal{V}), C, I_k \otimes e)$ is an archimedean order unit space.

Given a k-AOU space $\mathcal{V}$, we can define a canonical operator system $\mathcal{V}_{k\text{-min}}$ by

$$C_n^{k\text{-min}} = \{x \in M_n(\mathcal{V})_h : \alpha^* x \alpha \in C \text{ for all } \alpha \in M_{n,k}\}.$$

If $\varphi : \mathcal{V} \to \mathcal{W}$ is k-positive (k-order embedding), then $\varphi : \mathcal{V}_{k\text{-min}} \to \mathcal{W}_{k\text{-min}}$ is completely positive (complete order embedding).

k-minimal Operator Systems

If an operator system $\mathcal{V}$ is completely order isomoriphic to $\mathcal{V}_{k\text{-min}}$, we call it k-**minimal**.

Theorem (Araiza, R., Tomforde)

An operator system $\mathcal{V}$ is k-minimal if and only if there exists a unital complete order embedding

$$\pi : \mathcal{V} \to \bigoplus_{i \in I} M_{d_i}$$

where each $d_i \leq k$.

Proof idea:
Show that $\mathcal{V}$ satisfies the property that every k-positive map $\phi : \mathcal{W} \to \mathcal{V}$ is completely positive, then apply a theorem of Xhabli.

C*-envelope

Theorem (Araiza-R.-Tomforde)

For a k-AOU space $\mathcal{V}$, $C_e^*(\mathcal{V}_{k\text{-}min})$ is a C-subalgebra of a direct sum $\bigoplus_{i\in\Omega} M_{d_i}$ where $d_i \leq k$.*

Elements of proof:

- We show that injective envelopes exist in the category of k-AOU spaces, and $I(\mathcal{V})_{k\text{-min}} = I(\mathcal{V}_{k\text{-min}})$. In particular, $I(\mathcal{V}_{k\text{-min}})$ is k-minimal.
- By a result of Hamana, $C_e^*(\mathcal{V}_{k\text{-min}}) \subseteq I(\mathcal{V}_{k\text{-min}})$.
- We argue that the irreducible representations of any k-minimal C*-algebra have the form $\pi : \mathcal{A} \to M_{d_i}$ with $d_i \leq k$.

Projections in k-AOU spaces

Given a k-AOU space $\mathcal{V}$ with positive cone $C \subseteq M_k(\mathcal{V})$, we can characterize the abstract projections of $\mathcal{V}_{k\text{-min}}$. An element $p \in \mathcal{V}$, is called an **abstract projection** in $\mathcal{V}$ if the set of $x \in M_k(\mathcal{V})_h$ such that for every $\epsilon > 0$ there exists $t > 0$ such that

$$(\alpha+\beta)^* x(\alpha+\beta) + (\epsilon\alpha^*\alpha + t\beta^*\beta) \otimes p + (\epsilon\beta^*\beta + t\alpha^*\alpha) \otimes q \in C$$

for every $\alpha, \beta \in M_k$ coincides with the cone $C \subseteq M_k(V)$.

Theorem (Araiza-R.-Tomforde)

For a k-AOU space $\mathcal{V}$, the following statements are equivalent:

- *p is an abstract projection in $\mathcal{V}$.*
- *p is an abstract projection in $\mathcal{V}_{k\text{-min}}$.*
- *p is a projection in $C_e^*(\mathcal{V}_{K\text{-min}})$.*

When p satisfies any (hence all) of these statements, we call p an abstract projection in $\mathcal{V}$.

Quantum k-AOU space

A **quantum k-AOU space** is a finite dimensional k-AOU space with unit e spanned by positive contractions $\{Q(a,b|x,y) : a,b \in [m], x,y \in [n]\}$ such that

- For each $x,y \in [n]$, $\sum_{a,b\in[m]} Q(a,b|x,y) = e$
- For each $x,y \in [n]$ and $a,b \in [m]$, the vectors

  $$E(a|x) := \sum_{c\in[m]} Q(a,c|x,w) \quad \text{and} \quad F(b|y) := \sum_{c\in[m]} Q(c,b|z,y)$$

  are well-defined
- Each generator $Q(a,b|x,y)$ is an abstract projection.

Correlations from k-AOU spaces

Theorem (Araiza-R.-Tomforde)

A correlation p is quantum if and only if there exists a quantum k-AOU space $\mathcal{V} = span\{Q(a,b|x,y)\}$ and a state $\phi : \mathcal{V} \to \mathbb{C}$ such that

$$p(a,b|x,y) = \phi(Q(a,b|x,y)).$$

Proof elements:

- The definitions ensure $\mathcal{V}_{k\text{-min}}$ is a quantum commuting operator system, so p is quantum commuting.
- Each $Q(a,b|x,y)$ is a projection in $C_e^*(\mathcal{V}_{k\text{-min}})$, which is a C*-subalgebra of $\bigoplus M_{d_i}$ with each $d_i \leq k$.
- The resulting correlation is a convex combination of quantum correlations. Apply Caratheodory's theorem.

Correlations from Operator Systems

The paper $MIP^* = RE$ implies that $C_{qc}(n,m) \setminus \overline{C_q(n,m)}$ is non-empty for some $n, m \in \mathbb{N}$.

Theorem (Araiza-R.-Tomforde)

A correlation $p \in C_{qc}(n,m) \setminus \overline{C_q(n,m)}$ if and only if there exists a qc operator system $\mathcal{V}$ with generators $Q(a,b|x,y)$, a state φ on $\mathcal{V}$, and an $\epsilon > 0$ such that whenever $\mathcal{W}$ is a q k-AOU space with generators $R(a,b|x,y)$ and ψ is a state on $\mathcal{W}$ we have

$$|\varphi(Q(a',b'|x',y')) - \psi(R(a',b'|x',y'))| > \epsilon$$

for some $a', b' \in [m]$ and $x', y' \in [n]$.

An equivalent statement is something like the following...

Theorem

There exists a quantum commuting operator system $\mathcal{V}$ which cannot be approximated by any quantum k-AOU space at its first matrix level.

Question

What obstructions prevent the cone C_1 in a QC operator system from being approximated by $C_1^{k\text{-}min}$ of a quantum k-AOU space?

Thanks!

References (on arXiv):

- "An abstract characterization for projections in operator systems", Roy Araiza and Travis Russell *To appear, Journal of Operator Theory*.
- "A universal representation for quantum commuting correlations", Roy Araiza, Travis Russell and Mark Tomforde *Preprint*.
- "Matricial Archimedean order unit spaces and quantum correlations" Roy Araiza, Travis Russell and Mark Tomforde *To appear, Indiana University Journal of Mathematics*.