Wreath-like product groups and rigidity of their von Neumann algebras

joint work with Ionut Chifan, Denis Osin and Bin Sun

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3. $\mathrm{L}(G)$ is a II_1 factor (∞ dim vNa with a trace and trivial center) $\Leftrightarrow$ G has ∞ conj. classes (icc): $|\{ghg^{-1} \mid g \in G\}| = \infty, \forall h \in G \setminus \{e\}$.

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Lack of rigidity: the vNa forgets algebraic properties of amenable grps.

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- **Popa (2006)** $G \mapsto \mathrm{L}(G)$ is countable-to-1 for icc prop. (T) groups.

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- **I-Popa-Vaes (2010)** examples of W*-superrigid icc groups G.

Remark. These results do not apply to property (T) group II_1 factors. This is because deformation/rigidity applies to II_1 factors which admit deformations, whose presence is typically incompatible with property (T).

Chifan-Das-Houdayer-Khan (2020) examples of icc property (T) groups G such that $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

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Proposition Let $S = \langle [tHt^{-1}, t'Ht'^{-1}] \mid t, t' \in T, t \neq t' \rangle$, for a C-L subgroup $H < G$. Then $S < \langle\langle H\rangle\rangle$, $S \lhd G$ and $G/S \in \mathcal{WR}(H, G/\langle\langle H\rangle\rangle)$.

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Remark. When $G = H * K$, we have $G/\langle\langle H\rangle\rangle = K$ and $G/S = H \wr K$.

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Dahmani-Guirardel-Osin 2011 (group theoretic Dehn filling), Sun 2020

Let $H < G$ with G hyperbolic relative to H.
Then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have that:
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This is surprising since wreath products $A \wr B$ never have prop. (T) !

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

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