Part II. Subproduct systems

References

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Subproduct Systems

Definition

Let $d < \infty$. A subproduct system over $\mathbb{C}$ consists of a family of subspaces:

$$X(n) \subseteq (\mathbb{C}^d)^{\otimes n} \text{ s.t. } X(n) \subseteq X(k) \otimes X(m) \text{ for all } k, m \text{ with } n = k + m.$$

We write p_n for the projection onto $X(n)$. We write $\mathscr{F}_d = \sum_n^{\oplus} (\mathbb{C}^d)^{\otimes n}$ and $\mathscr{F}_X = p\mathscr{F}_d$ for $p = \sum_n p_n$.

Notation

Let $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ be the polynomial ring in d noncommuting variables. Let $\{e_1, \ldots, e_d\}$ be the o.n. basis of $\mathbb{C}^d$. We write

$$f(\underline{e}) = \sum_{w \in \mathbb{F}_+^d} \lambda_w e_{w_1} \otimes \cdots \otimes e_{w_{|w|}} \text{ when } f(\underline{x}) = \sum_{w \in \mathbb{F}_+^d} \lambda_w \underline{x}^w \in \mathbb{C}\langle x_1, \ldots, x_d \rangle.$$

Theorem (Shalit-Solel 2009)

There is a bijection between the s.p.s. $X = (X(n))$ on $d < \infty$ variables, and the homogeneous ideals $I = \cup_n I(n) \lhd \mathbb{C}\langle x_1, \ldots, x_d \rangle$ in the sense

$$\{f(\underline{e}) \mid f \in I(n)\} =: X(n)^{\perp} \longleftrightarrow I(n) := \operatorname{span}\{f \mid f(\underline{e}) \in X(n)^{\perp}\}$$

Subproduct Systems

Quantization

- Let $\mathscr{F}_d = \sum_n^{\oplus} (\mathbb{C}^d)^{\otimes n}$ and let $L_i \colon \mathscr{F}_d \to \mathscr{F}_d$ be the canonical shift operators. We write $\mathfrak{A}_d = \overline{\mathrm{alg}}\{L_i \mid i = 1, \ldots, d\}$ for Popescu's non-commutative disc algebra.
- If $X = (X(n), p_n)$ is a s.p.s. then let $S_i = pL_i p = pL_i$ and write $\mathscr{F}_X = p\mathscr{F}_d$. We write $\mathscr{A}_X := \overline{\mathrm{alg}}\{S_i \mid i = 1, \ldots, d\} = \overline{p \cdot \mathfrak{A}_d}$.

Theorem (Shalit-Solel 2009)

$\mathscr{A}_X$ has the following property: if $\underline{T} = [T_1, \ldots, T_d]$ is a row contraction that satisfies $f(T_1, \ldots, T_d) = 0$ for all $f \in I_X$ then there exists a completely conctractive map s.t. $S_i \mapsto T_i$ for all $i = 1, \ldots, d$.

Proof.

(Popescu's Poisson Transform). For $0 \leq r < 1$ set $K_r(\underline{T}) \colon H \to \mathscr{F}_d \otimes H$ by

$$K_r(\underline{T})h = \sum_{w \in \mathbb{F}_d} e_w \otimes (r^{|w|} \Delta(r\underline{T})^{1/2} (\underline{T}^w)^* h) \text{ where } \Delta(\underline{T}) = I - \sum_i T_i T_i^*.$$

Then the required map $\Phi \colon \mathscr{A}_X \to \mathscr{B}(H)$ is given by

$$\Phi(a) = \lim_{r \uparrow 1} K_r(\underline{T})^* (a \otimes I) K_r(\underline{T}).$$

$\square$

Example

The Symmetric Fock space

For every permutation σ on n elements define the unitary U_σ on $(\mathbb{C}^d)^{\otimes n}$ by

$$U_\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

The *n-th fold symmetric tensor product of* $\mathbb{C}^d$, denoted by E^n, is the subspace consisting of the vectors fixed by U_σ for all σ. The *symmetric Fock space* is given by $\mathscr{F}_+(\mathbb{C}^d) = \sum_n^\oplus E^n$. Writing $p_n \colon (\mathbb{C}^d)^{\otimes n} \to E^n$ we get that the resulting algebra $\overline{\mathrm{alg}}\{pL_i \mid i = 1, \dots, d\}$ corresponds to $\mathscr{A}_{X_I}$ for the (commutator) ideal (as) generated by $x_i x_j = x_j x_i$, denoted by $\mathscr{A}_d$.

Alternatively, consider $\mathbb{C}[x_1, \dots, x_d]$ the ring of polynomials on commuting variables, with i.p.:

$$\langle x^\alpha, x^\beta \rangle = \delta_{\alpha,\beta}\, \alpha!/|\alpha|!, \text{ where } (\alpha_1, \dots, \alpha_n)! = \alpha_1! \cdots \alpha_n! \text{ and } |\alpha| = \alpha_1 + \cdots + \alpha_d,$$

and take H_d^2 the completion. Then H_d^2 is identified with the space of holomorphic functions $f \colon \mathbb{B}_d \to \mathbb{C}$ with power series $f = \sum_\alpha c_\alpha x^\alpha$ such that

$$\|f\|_{H_d^2}^2 = \sum_\alpha c_\alpha (\alpha!/|\alpha|!) < \infty.$$

Then the above $\mathscr{A}_d$ is unitarily equivalent to the norm closure of polynomials inside

$$\mathrm{Mult}(H_d^2) := \{f \colon \mathbb{B}_d \to \mathbb{C} \mid fh \in H_d^2 \text{ for all } h \in H_d^2\} = \overline{\mathrm{alg}}^{\mathrm{w}*}\{M_{x^i} \mid i = 1, \dots, d\}.$$

The identification is given by the unitary $V : x^\alpha \mapsto p\underline{x}^\alpha$. Hence the resulting tensor algebra is an algebra of homolomorphic functions.

Rigidity of the tensor algebras of subproduct systems

Definition

(i) $X = (X(n), p_n)$ is *similar* to $Y = (Y(n), q_n)$ if there are invertible $V_n \colon X(n) \to Y(n)$ s.t.

$$\underbrace{V_1 \otimes \cdots \otimes V_1}_{n-\text{times}} = \begin{bmatrix} V_n & 0 \\ * & * \end{bmatrix} \in \mathscr{B}(X(n) \oplus X(n)^\perp, Y(n) \oplus Y(n)^\perp).$$

such that $\sup_n \{\|V_n\|, \|V_n^{-1}\|\} < \infty$.
(ii) $X = (X(n), p_n)$ is *isomorphic* to $Y = (Y(n), q_n)$ if they are similar by unitaries $(U(n))$, i.e.

$$\underbrace{U_1 \otimes \cdots \otimes U_1}_{n-\text{times}} = \begin{bmatrix} U_n & 0 \\ 0 & * \end{bmatrix} \in \mathscr{B}(X(n) \oplus X(n)^\perp, Y(n) \oplus Y(n)^\perp).$$

Theorem

Let X and Y be subproduct systems. Then:

(i) $\mathscr{A}_X$ and $\mathscr{A}_Y$ are cbis iff $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isomorphic as top. alg. iff X and Y are similar.
(ii) $\mathscr{A}_X$ and $\mathscr{A}_Y$ are cisis iff $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isis iff X and Y are isomorphic;

Remark

Combines results from Shalit–Solel (2009), Davidson–Ramsey–Shalit (2010), Dor-On–Markiewisz (2013), K.–Shalit (2015).

Subproduct Systems

Proposition

Let $X = (X(n), p_n)$ be a s.p.s.. If $\bar{I}$ denotes the closure of $I \equiv \{f(\underline{L}) \mid f \in I_X\}$ in $\mathfrak{A}_d$, then $\mathscr{A}_X$ is completely isometrically isomorphic to $\mathfrak{A}_d/\bar{I}$. In particular, the isomorphism is given by $\psi(x+\bar{I}) = px$, and $\mathscr{A}_X = p \cdot \mathfrak{A}_d$.

Proof.

By construction there is a unital c.c. homomorphism $\mathfrak{A}_d \to \mathscr{A}_X : L_i \mapsto S_i$. Its kernel contains I and consequently it contains $\bar{I}$. Thus we obtain a c.c. homomorphism

$$\psi \colon \mathfrak{A}_d/\bar{I} \to \mathscr{A}_X; L_i + \bar{I} \mapsto T_i.$$

On the other hand the row contraction $\widehat{\underline{L}} = [L_1 + \bar{I}, \ldots, L_d + \bar{I}]$ satisfies $f(\widehat{\underline{L}}) = 0$ for all $f \in I_X$, and thus gives a c.c. homomorphism

$$\sigma \colon \mathscr{A}_X \to \mathfrak{A}_d/\bar{I}; S_i \mapsto L_i + \bar{I}$$

which is the inverse of ψ (thus ψ and σ are c.is.). $\square$

Subproduct Systems

Definition

A character of an operator algebra is an algebraic homomorphism in $\mathbb{C}$ (and automatically c.c.).

The character space

For a s.p.s. X we write

$$Z(I_X) = \{z \in \overline{\mathbb{B}}_d \mid f(z) = 0 \text{ for all } f \in I_X\}.$$

Then there is a bijection

$$\mathscr{M}_{\mathscr{A}_X} \ni \rho \longleftrightarrow (\rho(S_1), \dots, \rho(S_d)) \in Z(I_X).$$

Indeed:

1. A character ρ of $\mathscr{A}_X$ defines a character, say ρ' on $\mathfrak{A}_d$ which must annihilate $I \equiv \{f(\underline{L}) \mid f \in I_X\}$. For $f \in I_X$ we have
$$f(\rho(\underline{S})) = \rho'(f(\underline{L})) = 0 \Rightarrow \rho(\underline{S}) \in Z(I_X).$$
2. Conversely, for $\underline{z} \in Z(I_X)$ we have that $f(\underline{z}) = 0$ and $\underline{z} = (z_1, \dots, z_d)$ is a row contraction. Thus $\rho_{\underline{z}}$ lifts to a c.c. homomorphism of $\mathscr{A}_X$ (due to the property of $\mathscr{A}_X$).

Subproduct Systems

Lemma

Let X be a s.p.s. and let $\underline{z} \in Z(I_X)$. Then we have that the following diagram

$$\begin{tikzcd} \mathfrak{A}_d \arrow[rr, "\rho_{\underline{z}}"] \arrow[dr] & & \mathbb{C} \\ & \mathscr{A}_X \arrow[ur, "\rho_{\underline{z}}"'] & \end{tikzcd}$$

is commutative. In particular, for every $x \in \mathfrak{A}_d$ the map $\widehat{x}\colon \overline{\mathbb{B}}_d \to \mathbb{C}$ given by $\lambda \mapsto \rho_\lambda(x)$ is continuous on $\overline{\mathbb{B}}_d$ and holomorphic in $\mathbb{B}_d$.

Proof.

We may apply in particular for the commutator ideal I_d to obtain that the function

$$\lambda \mapsto \rho_\lambda(x) = \rho_\lambda(x + \bar{I}_d)$$

is in $\mathscr{A}_d$. The latter is the norm closure of polynomials in $\mathrm{Mult}(H^2_d)$. $\square$

Subproduct Systems

Lemma (K.-Shalit 2015)

Let $X = (X(n), p_n)$ and $Y = (Y(n), q_n)$ be s.p.s. on d and d' variables, respectively. If $\phi \colon \mathscr{A}_X \to \mathscr{A}_Y$ is an algebraic (resp. bounded, isometric) isomorphism, then there exists a continuous map $\widetilde{F} \colon \overline{\mathbb{B}}_{d'} \to \mathbb{C}^d$ that is holomorphic on $\mathbb{B}_{d'}$ and extends $\phi^* \colon \mathscr{M}_{\mathscr{A}_Y} \to \mathscr{M}_{\mathscr{A}_X}$.

Proof.

For every $i = 1, \dots, d$, let $W_i \in \mathfrak{A}_{d'}$ so that

$$\phi(S_i) = qW_i \in \mathscr{A}_Y.$$

Let the (continuous) map

$$\phi^* \colon Z(I_Y) \to Z(I_X); \underline{z} \mapsto (\rho_{\underline{z}}\phi(S_1), \dots, \rho_{\underline{z}}\phi(S_d)).$$

We then see that the map $\lambda \mapsto \rho_\lambda(\phi(S_i))$ extends to the continuous map $\widehat{W_i} \colon \overline{\mathbb{B}}_{d'} \mapsto \mathbb{C}$, which is holomorphic on $\mathbb{B}_{d'}$ by Lemma 4. The required map $\widetilde{F} \colon \overline{\mathbb{B}}_{d'} \to \mathbb{C}^d$ is then defined by

$$\widetilde{F}(\underline{z}) = (\widehat{W_1}(\underline{z}), \dots, \widehat{W_d}(\underline{z})).$$

$\square$

Subproduct Systems

Lemma (Davidson-Ramsey-Shalit 2010)

Let $X = (X(n), p_n)$ and $Y = (Y(n), q_n)$ be s.p.s. on d and d' variables, respectively. If $\phi \colon \mathscr{A}_X \to \mathscr{A}_Y$ is an algebraic (resp. bounded, isometric) isomorphism then there exists a vacuum preserving algebraic (resp. bounded, isometric) isomorphism $\phi' \colon \mathscr{A}_X \to \mathscr{A}_Y$.

Proof.

Step 1. If $V = V(I)$ is a variety in $\mathbb{C}^d$ for a homogeneous ideal I then either it has singular points or it is a linear subspace. We denote by $\mathrm{Sing}(V)$ the singular points of V and define

$$N(V) = \mathrm{Sing}(\cdots(\mathrm{Sing}(V))\cdots) \text{ called } \textit{the singular nucleus}.$$

Step 2. For X and Y we see that ϕ^* maps $\mathbb{B}_{d'} \cap N(V(I_Y))$ onto $\mathbb{B}_d \cap N(V(I_X))$. These are the same and thus a ball of dimension say n (up to permutation of the coordinates).

Step 3. Let $\psi \in \mathrm{Aut}(\mathbb{B}_n)$. Then we can write $\phi^* = U \circ \varphi_v$ for

1. some unique unitary U,
2. $\varphi_v(\underline{z}) = (v - P_v\underline{z} - (1-|v|^2)^{1/2} Q_v\underline{z})(1 - \langle \underline{z}, v\rangle)^{-1}$, where P_v is the projection onto $\mathbb{C}v$ and $Q_v = I - P_v$, for $v \in \mathbb{B}_n$.

If $v = 0$ take $D_1 = \mathbb{B}_n \cap L$ and $D_2 = UD_1$ for any one-dimensional space L in $\mathbb{C}^n$.
Otherwise, take $D_1 = \mathbb{C}v \cap \mathbb{B}_n$ and $D_2 = U(D_1)$. Note that $\varphi_v|_{D_1} = D_1$.
In either case we have two "discs" so that $\phi^*(D_1) = D_2$.

Step 4. Apply the disc-trick. $\square$

Subproduct Systems

Theorem (Dor-On–Markiewisz 2014)

Let X and Y be s.p.s. . Then $\mathscr{A}_X$ and $\mathscr{A}_Y$ are cbis iff $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isomorphic as top. alg. iff X and Y are similar.

Proof.

Wlog let $\phi : \mathscr{A}_X \to \mathscr{A}_Y$ be vaccum preserving.

In this case ϕ is semi-graded in the sense that:

if the minimal degree in $f(S^X)$ is n then so it is for $\phi(f(S^X))$.

Indeed it suffices to show that the minimal degree of $\phi(S_1^X)$ is 1. If $\phi(S_1^X) = \lambda I^Y + T$ then

$$\lambda = \rho_{\underline{0}}(\phi(S_1^X)) = \phi^* \circ \rho_{\underline{0}}(S_1^X) = \rho_{\underline{0}}(S_1^X) = 0.$$

So ϕ does not drop degree and thus the minimal degree of $\phi(S_1^X)$ is at least 1. By symmetry on ϕ^{-1} we get that it is exactly 1.

The required map then is given by

$$V_n^\phi : X(n) \to Y(n); f(S^X) \mapsto F_n\phi(f(S^X)) \text{ for the } n\text{-th Fourier co-efficient } F_n.$$

The key observation here is that $F_n\phi = F_n\phi F_n$ (and likewise for ϕ^{-1}), and so for $f(S_X) = F_n(f(S^X))$ we get that

$$f(S^X) = F_n(f(S^X)) = F_n\phi^{-1}\phi(f(S^X)) = F_n\phi^{-1}F_n\phi(f(S^X)).$$

Subproduct Systems

Theorem (Shalit-Solel 2009)

Let X and Y be s.p.s. . Then $\mathscr{A}_X$ and $\mathscr{A}_Y$ are cisis iff $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isis iff X and Y are isomorphic.

Proof.

Wlog let $\phi : \mathscr{A}_X \to \mathscr{A}_Y$ be vaccum preserving.

By writing $e_\emptyset$ for the vaccum vector we have that $\|f(S)e_\emptyset\| = \|f(S)\|$ for all $f \in X(n)$.

In this case ϕ preserves the grading. Indeed it suffices to show that if $\phi(S_1^X) = \sum_i \lambda_i S_i^Y + T$ then $T = 0$. If $T \neq 0$ then $\phi^{-1}(T) \neq 0$ (with minimal degree greater than 1) and so

$$1 = \|S_1^X\| = \|S_1^X e_\emptyset\| < \|(S_1^X - \phi^{-1}(T))e_\emptyset\| \leq \|S_1^X - \phi^{-1}(T)\| = \|\sum_i \lambda_i S_i^Y\|$$

while

$$\|\sum_i \lambda_i S_i^Y\| = \|\sum_i \lambda_i S_i^Y e_\emptyset\| \leq \|(\sum_i \lambda_i S_i^Y + T)e_\emptyset\| \leq \|\phi(S_1^X)\| = 1,$$

which is a contradiction.

Thus the map

$$V_n^\phi : X(n) \to Y(n); f(S^X) \mapsto \phi(f(S^X)$$

is a unitary at every level, that sends $X(n)$ into $Y(n)$. $\square$

Applications to monomial ideals

Definition

An ideal I of $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ is called *monomial* if it is generated by monomials. Here we get:

$$T_i e_\mu = \begin{cases} e_{i\mu} & \text{if } i\mu \notin I \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (K.-Shalit 2015)

Let X and Y be subproduct systems associated with the monomial ideals $I \lhd \mathbb{C}\langle x_1,\ldots,x_d\rangle$ and $J \lhd \mathbb{C}\langle y_1,\ldots,y_{d'}\rangle$. Wlog suppose that $x_i \notin I$ and $y_j \notin J$ for all i, j. TFAE:

(i) $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isometrically isomorphic;

(ii) $\mathscr{A}_X$ and $\mathscr{A}_Y$ are algebraically isomorphic;

(iii) $\mathbb{C}\langle x_1,\ldots,x_d\rangle/I$ and $\mathbb{C}\langle y_1,\ldots,y_{d'}\rangle/J$ are isomorphic by a graded isomorphism;

(iv) X and Y are similar;

(v) X and Y are isomorphic;

(vi) $d = d'$ and $I = J$ up to a permutation of the variables.

Proof

H.t.s. that $[(v) \Rightarrow (vi)]$.

Applications to monomial ideals

Proof: H.t.s. that $[(v) \Rightarrow (vi)]$.

Note that the graded isomorphism ϕ is given by invertible matrices V_n.

For $n = 1$ we get an invertible V_1, thus $d = d'$.

The groups of graded automorphisms of the quotients are linear algebraic groups.

Then $T_J = \{\rho \mid \rho(y_i) = a_i y_i, \underline{a} \in \mathbb{C}^d\}$ forms a maximal torus.

By Borel's Theorem the tori T_J and the $\phi T_I \phi^{-1}$ are conjugate.

Wlog we get a graded isomorphism s.t. $\phi T_I = T_J \phi$.

If $V_1 = [a_{ij}]$ then we obtain that:

$$\forall \text{ diagonal } D_1 \ \exists \text{ diagonal } D_2 \text{ s.t. } [a_{ij}]D_1 = D_2[a_{ij}].$$

By Linear Algebra then V_1 is diagonal up to a permutation, say π.

Thus we get $\phi(x_i + I) = a_{\pi(i)i} y_{\pi(i)} + J$ with $a_{\pi(i)i} \neq 0$. $\square$

End of Part II.

Thank you for your attention!

Stay safe, and physically and mentally healthy.