Rigidity of Analytic Operator Algebras

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Framework

Main Idea: Encode Structures via Operator Algebras

$$\left\{ \begin{array}{c} S\text{: set of generators} \\ \text{and relations} \end{array} \right\} \quad \rightsquigarrow \quad \left\{ \text{ Subalgebras of } \mathscr{B}(H) \right\}$$

Rigidity

$$\left\{ S_1 \sim S_2 \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{array}{c} \mathrm{C}^*_{S_1} \sim \mathrm{C}^*_{S_2}, \mathscr{A}_{S_1} \sim \mathscr{A}_{S_2} \\ \left\{ \text{ strong Morita equivalence/isomorphisms } \right\} \end{array} \right\}$$

Question: $S_1 \sim S_2 \overset{??}{\iff} \mathrm{C}^*_{S_1} \simeq \mathrm{C}^*_{S_2}$

Example: Let G_1, G_2 be free abelian discrete groups, i.e. $G_i = \mathbb{Z}^{d_i}$. Then:

$$\mathrm{C}^*(G_1) \simeq \mathrm{C}^*(G_2) \text{ iff } C_0(\widehat{G}_1) \simeq C_0(\widehat{G}_2), \text{ iff } \widehat{G}_1 \simeq \widehat{G}_2, \text{ iff } d_1 = d_2, \text{ iff } G_1 \simeq G_2.$$

But in general, we require more elements than just $*$-isomorphism.

1. Hoare-Parry 1960's: C*-crossed products do not classify $\mathbb{Z}$-actions up to conjugacy.
2. Semigroup algebras: $\mathrm{C}^*(S) \simeq \mathrm{C}^*(\mathbb{Z}_+)$ for any positive cone $S \subseteq \mathbb{Z}$.
3. Work on graphs by Eilers-Restorff-Ruiz-Sørensen: moves plus K-theory.

Framework

Positive answers: $S_1 \sim S_2 \iff \mathscr{A}_{S_1} \simeq \mathscr{A}_{S_2}$

It started in 1960's by Arveson. Some examples:

1. Aperiodic C*-correspondences: Muhly-Solel (2000).
2. Graphs: Katsoulis-Kribs (2004), Solel (2004).
3. Dynamical systems: Arveson (1967), Arveson-Josephson (1969), Peters (1984), Hadwin-Hoover (1988), Davidson-Katsoulis (2008^2); Davidson-K. (2012); K.-Katsoulis (2012); Katsoulis-Ramsey (2021).
   Cornelissen-Marcolli use results of Davidson-Katsoulis to settle questions in Number Theory.
4. Topological graphs: Davidson-Roydor (2009).
5. Analytic varieties: Shalit-Solel (2009), Davidson-Ramsey-Shalit (2011), Hartz (2015).
6. Stochastic matrices: Dor-On and Markiewisz (2014).
7. Weighted shifts: Dor-On (2015).
8. Triangular limit algebras: Katsoulis-Ramsey (2015).
9. Subproduct systems: K.-Shalit (2015).
10. Subvarieties of the nc ball: Salomon-Shalit-Shamovich (2018).

Framework

Structural differences

C*-algebras = topological objects vs. Nsa = analytical objects.

Strategy for rigidity of nsa

Let $\mathscr{A}_{S_1}$ and $\mathscr{A}_{S_2}$ be nsa's related to structures S_1 and S_2.

1. Suppose $\mathscr{A}_{S_1}$ and $\mathscr{A}_{S_2}$ are generated by analytic polynomials.
2. Obtain rotations in the automorphism groups.
3. Rotate isomorphisms to vacuum preserving isomorphisms.
4. Apply a Schwarz-type Lemma.
5. Analyze the Fourier co-efficients to get information on the S_1 and S_2.

Rotating to vacuum preserving homomorphisms

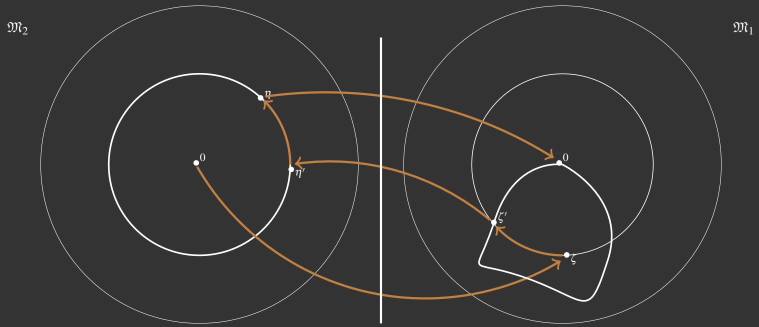
“Theorem” of rotating homomorphisms (Davidson–Ramsey–Shalit 2011)

Let $\rho : \mathscr{A}_1 \to \mathscr{A}_2$ be an isomorphism.

1. Suppose that $\mathrm{Aut}(\mathscr{A}_1)$ and $\mathrm{Aut}(\mathscr{A}_2)$ contain rotations.
2. Suppose that $\rho^* : \mathfrak{M}_2 \to \mathfrak{M}_1$ maps a disc onto a disc.

Then we can rotate ρ to a vacuum preserving isomorphism.

Proof



Part I. Semigroup Operator Algebras

References

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Semigroup algebras

Definition

A *positive cone* S of $\mathbb{Z}_+^d$ is a unital subsemigroup such that: (i) $S \cap (-S) = (0)$; and (ii) for every $g \in \mathbb{Z}^d$ there exist $s, t \in S$ such that $g = s - t$.

Definition

The *Fock representation* $V : S \to \mathscr{B}(\ell^2(S))$ is given by

$$V_s \colon \ell^2(S) \to \ell^2(S) : e_t \mapsto e_{s+t}.$$

We define the *nonselfadjoint semigroup algebra* by $\mathscr{A}(S) := \overline{\operatorname{alg}}\{V_s \mid s \in S\}$.

Example

The prototypical example is the unilateral shift in $\ell^2(\mathbb{Z}_+)$ given by $Ve_n = e_{n+1}$. Recall that the Toeplitz C*-algebra is defined as

$$\mathscr{T} := \mathrm{C}^*(V) = \overline{\operatorname{span}}\{V_n V_m^* \mid n, m \in \mathbb{Z}_+\}.$$

The nonselfadjoint semigroup algebra $\mathscr{A}(\mathbb{Z}_+)$ is a representation of the disc algebra $\mathbb{A}(\mathbb{D})$.

Semigroup C*-algebras

Question

Is it true that $\mathrm{C}^*(S_1) \simeq \mathrm{C}^*(S_2)$ implies $S_1 \simeq S_2$?

Answer: No!

Let $S \subseteq \mathbb{Z}_+$ be a semigroup such that $S \cap (-S) = \{0\}$ and $S - S = \mathbb{Z}$.
Then $\gcd(S) = 1$ and there exists an $N_0 \in S$ such that if $n \geq N_0$ then $n \in S$.
Let N_0 be the minimal.
Let $\gamma \colon \mathbb{Z}_+ \to S$ be a bijective map that respects the total order, i.e., γ is the linear enumeration on S.
It thus satisfies

$$\gamma(k+m) = k + \gamma(m) \text{ for all } k \in \mathbb{Z}_+; \gamma(m) \geq N_0;$$

and so

$$k + m = \gamma^{-1}(k + \gamma(m)) \text{ for all } k \in \mathbb{Z}_+; \gamma(m) \geq N_0.$$

By replacing k with $\gamma(k) \in \mathbb{N}$ we moreover have

$$\gamma^{-1}(\gamma(k) + \gamma(m)) = \gamma(k) + m \text{ for all } k \in \mathbb{Z}_+; m \geq \gamma^{-1}(N_0).$$

Semigroup C*-algebras

Answer cc'ed.

Let $U : \ell^2(\mathbb{Z}_+) \to \ell^2(S)$ by $U e_n = e_{\gamma(n)}$.

Set V be the usual unilateral shift, and T_s be the shift operators that generat $\mathrm{C}^*(S)$.

The properties of γ then imply

$$UT_{\gamma(k)}U^* = V_{\gamma(k)}V_{\gamma^{-1}(N_0)}V^*_{\gamma^{-1}(N_0)} + \sum_{m \notin \gamma^{-1}(N_0)+\mathbb{Z}_+} V_{\gamma^{-1}(\gamma(k)+\gamma(m))}V^*_m p_m,$$

for the projections p_m on e_m.

Likewise we have that

$$U^*V_kU = T_{k+N_0}T^*_{N_0} + \sum_{\gamma(m)\notin N_0+S} T_{\gamma(s+m)}T^*_{\gamma(m)}p_{\gamma(m)}.$$

By construction the set $(N_0+S)^c \cap S$ is finite (thus the sums are finite).

It suffices to show that the p_m projections are in $\mathscr{T}$, and that the $p_{\gamma(m)}$ projections are in $\mathrm{C}^*(S)$, and then ad_U induces the required $*$-isomorphism between $\mathrm{C}^*(S)$ and $\mathscr{T}$.

Semigroup C*-algebras

Answer cc'ed.

On one hand we have

$$p_m := P_{\mathbb{C}e_m} = V_m V_m^* - V_{m+1} V_{m+1}^* \in \mathscr{T}.$$

It remains to show that the projections $p_{\gamma(m)}$ are in $\mathrm{C}^*(S)$. We have that $S \setminus \{0\} = \langle s_1^0, \dots, s_{k_0}^0 \rangle$ is finitely generated and so

$$p_0 = \prod_{j=1}^{k_0} (I - T_{s_j^0} T_{s_j^0}^*) \in \mathrm{C}^*(S).$$

Now consider the first non-zero element $\gamma(1)$ in S. Then $S \setminus \{0, \gamma(1)\}$ is a subsemigroup of $\mathbb{Z}_+$ and contains all natural numbers after a finite step. Hence it is finitely generated, say $S \setminus \{0, \gamma(1)\} = \langle s_1^{\gamma(1)}, \dots, s_{k_1}^{\gamma(1)} \rangle$, and so

$$p_0 + p_{\gamma(1)} = \prod_{j=1}^{k_1} (I - T_{s_j^{\gamma(1)}} T_{s_j^{\gamma(1)}}^*) \in \mathrm{C}^*(S)$$

and thus $p_{\gamma(1)} \in \mathrm{C}^*(S)$. Inductively $p_{\gamma(m)} \in \mathrm{C}^*(S)$ for every m. □

Positive cones

Question

Is it true that $S_1 \simeq S_2$ if and only if $\mathscr{A}(S_1) \simeq \mathscr{A}(S_2)$?

Definition

A positive cone S is a subsemigroup of a discrete abelian group $\mathscr{G}$ such that $S \cap (-S) = \{0\}$ and $S - S = \mathscr{G}$.

Proposition

Let $S \subseteq \mathscr{G}$ be a positive cone. Then there is an isometric map $\mathscr{A}(S) \to \mathrm{C}^*(\mathscr{G}); V_s \mapsto U_s$.

Proof

Since polynomials in $\mathscr{A}(S)$ have a unique expression the map $V_s \mapsto U_s$ admits a unique linear extension. Identify $\ell^2(S)$ with the obvious subspace inside $\ell^2(\mathscr{G})$ and get:

$$\|\sum_{s \in F} \lambda_s V_s\| = \|P_{\ell^2(S)}\left(\sum_{s \in F} \lambda_s U_s\right)|_{\ell^2(S)}\| \leq \|\sum_{s \in S} \lambda_s U_s\|.$$

For the reverse inequality fix $\varepsilon > 0$. Let $\xi = \sum_{i=1}^n k_i e_{g_i}$ in the unit ball of $\ell^2(\mathscr{G})$ such that

$$\|\sum_{s \in F} \lambda_s U_s\| - \varepsilon \leq \|\sum_{s \in F} \lambda_s U_s \xi\|_{\ell^2(\mathscr{G})}.$$

The one-variable case

Proof cont'd.

Since S is a positive cone we have that there are $s_i, t_i \in S$ such that $g_i = s_i - t_i$ for all $i = 1, \ldots, n$. Set $t := \sum_{i=1}^n t_i \in S$ so that $t + g_i \subset S$ for all $i = 1, \ldots, n$. Then the vector

$$\xi' := U_t \xi = \sum_{i=1}^n k_i e_{t+g_i} \in \left(\ell^2(S)\right)_1 .$$

Therefore we obtain

$$\begin{aligned} \| \sum_{s \in F} \lambda_s U_s \| - \varepsilon &\leq \|U_t \sum_{s \in F} \lambda_s U_s \xi \|_{\ell^2(\mathscr{G})} = \| \sum_{s \in F} \lambda_s U_s U_t \xi \|_{\ell^2(\mathscr{G})} \\ &= \| \sum_{s \in F} \lambda_s U_s \xi' \|_{\ell^2(\mathscr{G})} = \| \sum_{s \in F} \lambda_s V_s \xi' \|_{\ell^2(S)} \leq \| \sum_{s \in F} \lambda_s V_s \|. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary we have equality of the norms. □

Corollary

If $S_1 \subseteq S_2 \subseteq \mathscr{G}$ are positive cones then there is an isometric embedding

$$\mathscr{A}(S_1) \hookrightarrow \mathscr{A}(S_2); V_s^{S_1} \mapsto V_s^{S_2}.$$

The one-variable case

Numerical semigroup

1. A positive cone of $\mathbb{Z}$ is called *numerical semigroup* ($S \subseteq \mathbb{Z}_+$ and $S \cap (-S) = \{0\}$).
2. An $S \subseteq \mathbb{Z}_+$ is a numerical semigroup iff there is an $n \in S$ such that $m \in S$ for all $m > n$.

Proposition

Let S be a numerical semigroup. Then the inclusion $\mathscr{A}(S) \hookrightarrow \mathscr{A}(\mathbb{Z}_+)$ induces a homeomorphism

$$\iota^* : \overline{\mathbb{D}} \to \mathfrak{M}_S : \zeta \mapsto \mathrm{ev}_\zeta|_{\mathscr{A}(S)}.$$

Proof

The map ι^* is well defined. For $S \subseteq \mathbb{Z}_+$ there is an $n > 0$ such that $n, n+1 \in S$.

1. Injective: If $\mathrm{ev}_\zeta(V_n) = 0 = \mathrm{ev}_{\zeta'}(V_n)$ then $\zeta^n = (\zeta')^n = 0$. If $\mathrm{ev}_\zeta(V_n) \neq 0$ then

$$\zeta = \frac{\mathrm{ev}_\zeta(V_{n+1})}{\mathrm{ev}_\zeta(V_n)} = \frac{\mathrm{ev}_{\zeta'}(V_{n+1})}{\mathrm{ev}_{\zeta'}(V_n)} = \zeta'.$$

2. Onto: If $\chi(V_n) = 0$ then $\chi = \mathrm{ev}_0$. If $\chi(V_n) \neq 0$ then $\chi = \mathrm{ev}_\zeta|_{\mathscr{A}(S)}$ for $\zeta = \chi(V_{n+1})/\chi(V_n)$.
$\square$

Semigroup algebras

Proposition

Let $S \subset \mathbb{Z}_+^d$ be a positive cone in $\mathbb{Z}^d$. Then any algebraic epimorphism $\rho \colon \mathscr{A} \to \mathscr{A}(S)$ for any Banach algebra $\mathscr{A}$ is automatically continuous.

Proof

By the closed graph theorem ρ is continuous if and only if

$$\mathfrak{S}(\rho) := \{b \in \mathscr{B} \mid \exists (a_n) \subset \mathscr{A} \text{ such that } a_n \to 0 \text{ and } \rho(a_n) \to b\} = (0).$$

Due to a result of Sinclair, for any sequence (b_n) in $\mathscr{A}(S)$ there exists an $N \in \mathbb{N}$ such that

$$\overline{b_1 \cdots b_N \mathfrak{S}(\rho)} = \overline{b_1 \cdots b_n \mathfrak{S}(\rho)} \text{ for all } n \geq N.$$

By applying for all $b_i = V_s$ which is an isometry we get that $\mathfrak{S}(\rho) = \overline{(V_s)^n \mathfrak{S}(\rho)}$ for all $n \in \mathbb{N}$. However the Fourier transform yields $\bigcap_{n \in \mathbb{N}} \overline{(V_s)^n \mathscr{I}} = (0)$ for any ideal $\mathscr{I} \subset \mathscr{A}(S)$. □

Remarks

1. The map $\mathscr{A}(S) \ni V_s \mapsto U_s \in \mathrm{C}^*(\mathbb{Z}^d)$ extends to a ucis repn.
2. $\mathrm{C}^*(\mathbb{Z}^d)$ is the C*-envelope of $\mathscr{A}(S)$.
3. We have that $s \in S$ if and only if there exists an $f \in \mathscr{A}(S)$ such that $f^{(s)}(0) \neq 0$.

The one-variable case

Theorem (K.-Katsoulis-Li 2020)

Let $S_1, S_2 \subset \mathbb{Z}_+$ be numerical semigroups. Then:

$$S_1 = S_2 \text{ if and only if } \mathscr{A}(S_1) \simeq \mathscr{A}(S_2) \text{ by an algebraic isomorphism.}$$

Proof

Let $\rho \colon \mathscr{A}(S_1) \to \mathscr{A}(S_2)$ be an algebraic isomorphism.

1. The algebraic isomorphism ρ is continuous.
2. We have that $s \in S_i$ iff there exists an $f \in \mathscr{A}(S_i)$ such that $f^{(s)}(0) \neq 0$.
3. By disc-trick the homeomoprhism ρ^* is vacuum preserving.
4. By explicit construction it has the form $\rho^*(\zeta) = f(\zeta)/g(\zeta)$ for $f = \rho(V_{n+1})$ and $g = \rho(V_n)$ with $n, n+1 \in S_2$, whenever $g(\zeta) \neq 0$.
5. By Riemann's Theorem and Open Mapping Theorem ρ^* is a biholomorphism of $\mathbb{D}$ fixing zero.
6. By Schwarz Lemma we have $\rho^*(\zeta) = e^{i\theta}\zeta$; wlog $\rho^* = \mathrm{id}$.
7. For $s \in S_1$ and $h = \rho(V_s)$ we get that
$$\zeta^s = \mathrm{ev}_\zeta(V_s) = \rho^*(\mathrm{ev}_\zeta)(V_s) = \mathrm{ev}_\zeta(\rho(V_s)) = h(\zeta) \text{ for all } \zeta \in \mathbb{D}.$$
Thus $V_s = h = \rho(V_s)$.
8. Thus $s \in S_2$, and symmetry finishes the proof. $\square$

The multivariable case

Definition

A positive cone S of a group $\mathscr{G}$ is called a *higher rank numerical semigroup* if

$$S_{\mathrm{sn}} := \{g \in \mathscr{G} \mid ng \in S \text{ eventually for } n \in \mathbb{N}\} \simeq \mathbb{Z}_+^d.$$

Proposition

Let $S \subset \mathbb{Z}_+^d$ be a positive cone of $\mathbb{Z}^d$. Let $\iota^* \colon \overline{\mathbb{D}}^d \to \mathfrak{M}_S$ be the continuous map induced by the embedding $\mathscr{A}(S) \hookrightarrow \mathscr{A}(\mathbb{Z}_+^d)$. Then the following are equivalent:

1. $S_{\mathrm{sn}} = \mathbb{Z}_+^d$;
2. the intersection of S with any axis is a non-trivial positive cone of $\mathbb{Z}$;
3. ι^* is injective.

In particular, ι^* is a homeomorphism when it is injective.

Proof.

[(1) $\Leftrightarrow$ (2)]: “Immediate”.

[(2) $\Rightarrow$ (3)]: As before for each direction independently.

[(3) $\Rightarrow$ (2)]: If $\mathbb{Z}_+ \cdot e_1 \cap S = (0)$, then we would have that $\mathrm{ev}_{(\lambda,0,\dots,0)}|_{\mathscr{A}(S)} = \mathrm{ev}_{(0,0,\dots,0)}|_{\mathscr{A}(S)}$ for any $\lambda \neq 0$, which contradicts injectivity.

– Surjectivity as before from item (2). $\square$

The multivariable case

Theorem (K.-Katsoulis-Li 2020)

Let $S_1 \subset G_1$ and $S_2 \subset G_2$ be higher-rank numerical semigroups. Then $S_1 \simeq S_2$ if and only if $\mathscr{A}(S_1) \simeq \mathscr{A}(S_2)$ by an algebraic isomorphism.

Proof

Wlog assume that $S_1 \subset \mathbb{Z}^{d_1}$ and $S_2 \subset \mathbb{Z}^{d_2}$.

Now move in a similar way by using that:

1. thin sets (thin sets are the zero sets of holomorphic functions, and there is an analogue of Riemann's Theorem for locally bounded functions);
2. $S_1 \simeq S_2$ if and only if $d_1 = d_2$ and $S_1 = S_2$ up to a permutation of the coordinates; and
3. $\mathrm{Aut}(\mathbb{D}^d) \simeq (\times_{i=1}^d \mathrm{Aut}(\mathbb{D})) \rtimes \mathscr{S}_d$.

Corollary

An algebraic isomorphism between higher rank numerical semigroups algebras is vacuum preserving if and only if it is the composition of a permutation of co-ordinates by a rotation.

End of Part I.