# Harmonic Operators and Crossed products

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# Abstract

*We study the space of harmonic operators for a probability measure $\mu$ (or a family of measures) on a locally compact group G, as a 'quantization' of $\mu$-harmonic (or jointly harmonic) functions on G.*
*This leads to two different notions of crossed product of operator spaces by actions of G which coincide when G satisfies a certain approximation property.*
*The corresponding (dual) notions of crossed products of (co-) actions by the von Neumann algebra of G always coincide.*

*This is a survey of joint work with M. Anoussis and I.G. Todorov, and of recent work by D. Andreou.*

# Harmonic functions, Harmonic operators

Let $\mu \in M(G)$ be a probability measure on a loc. compact group $G$.

• Say a function $\phi : G \to \mathbb{C}$ is a $\mu$-harmonic function if

$$\int_G \phi(st)d\mu(t) = \phi(s)\,. \quad \text{Say } \ \phi \in \mathcal{H}(\mu)\,.$$

So, $\phi$ is a fixed point of the map $P_\mu$ given by

$$(P_\mu\phi)(s) = \int_G \phi(st)d\mu(t)$$

which is (compl.) positive, unital, w*-continuous on $L^\infty(G)$.

• Quantisation: Say an operator $T \in \mathcal{B}(L^2(G))$ is a $\mu$-harmonic operator if

$$\int_G \rho_t T \rho_t^{-1} d\mu(t) = T)\,. \quad \text{Say } \ T \in \widetilde{\mathcal{H}}(\mu)\,.$$

(where $\rho$ is the right regular rep. $G \curvearrowright L^2(G)$.)

# Harmonic functions, Harmonic operators

So $\mu$-harmonic operators are fixed points of the map

$$\Theta_\mu : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G)) : T \to \int_G \rho_t T \rho_t^{-1} d\mu(t)$$

which is weak-* continuous (unital completely positive) and extends $P_\mu$.

The map $\Theta_\mu$ commutes with left or right multiplication by all left translation operators $\lambda_s$ on $L^2(G)$, hence $\Theta_\mu(ATB) = A\Theta_\mu(T)B$ for all $A, B$ in the w*-closed lin. span of $\{\lambda_s : s \in G\}$
(is a bimodule map over $\mathrm{vN}(G)$- the von Neumann algebra of $G$).

[ Recall $\mathcal{B}(L^2 G)$ is the dual of the trace class. So it has a w*-topology. Recall $\mathrm{vN}(G) \subseteq \mathcal{B}(L^2 G)$ is w*-closed, so it has a predual. ]

# Left Ideals of $L^1(G)$ and $\mathrm{vN}(G)$ bimodules

Let $J \subseteq L^1(G)$ be a left ideal. (So $J^\perp \subseteq L^\infty(G)$ is left translation invariant.) Consider

$$\begin{aligned}\ker \Theta(J) &:= \left\{ T \in \mathcal{B}(L^2(G)) : \int_G \rho_t T \rho_t^{-1} f(t)dt = 0 \ \text{ for all } \ f \in J \right\} \\ &= \bigcap_{f \in J} \ker \Theta_f\end{aligned}$$

If $A, B \in \mathrm{vN}(G)$ and $\phi \in J^\perp \subseteq L^\infty(G) \subseteq \mathcal{B}(L^2(G))$ then $A\phi B \in \ker \Theta(J)$ (since each $\Theta_f$ is a $\mathrm{vN}(G)$ bimodule map); Thus,

$$\mathrm{Bim}_{vn}(J^\perp) \subseteq \ker \Theta(J) .$$

where $\mathrm{Bim}_{vn}(J^\perp)$ is the weak-* closed linear span of all products $A\phi B$ as above. [1]

[1] identifying $\phi$ with multiplication by $\phi$ on $L^2(G)$

# Left Ideals of $L^1(G)$ and $\mathrm{vN}(G)$ bimodules

For every closed left ideal $J \subseteq L^1(G)$,

$$\mathrm{Bim}_{vn}(J^\perp) \subseteq \ker \Theta(J)\,.$$

**Theorem**

*If G has the Approximation Property of Haagerup-Kraus then the equality*

$$\mathrm{Bim}_{vn}(J^\perp) = \ker \Theta(J)\,.$$

*holds for every left ideal $J \subseteq L^1(G)$.*

First proved for $G$ abelian, or compact, or weakly amenable discrete with M. Anoussis and I.G. Todorov [AKT]. Then generalised as above by [Crann – Neufang].

# The approximation property AP

Roughly: The predual $A(G)$ of $\mathrm{vN}(G)$ contains an (unbounded) approximate identity of a weak form:

[Haagerup-Kraus, 1993]
A locally compact group $G$ has the AP if and only if there exists a net $\{u_i\}_{i\in I}$ in $A(G)$, (i.e. $u_i(s) = (\lambda_s f_i, g_i), \ \ f_i, g_i \in L^2(G)$) such that

$$(\mathrm{Id} \otimes M_{u_i})(T) \xrightarrow{w^*} T$$

for all $T \in \mathcal{B}(\ell^2)\bar{\otimes}\,\mathrm{vN}(G)$, where for $A \in \mathcal{B}(\ell^2)$ and $s \in G$, $(\mathrm{Id} \otimes M_u)(A \otimes \lambda_s) = A \otimes u(s)\lambda_s$.

Examples: Groups with AP: Abelian, compact, $\mathbb{F}_n$.
Groups without AP: $SL(3,\mathbb{Z}),\ SL(3,\mathbb{R})$.

# Application to jointly harmonic operators

Apply the Theorem to ideals $J \subseteq L^1(G)$ of the form $J^\perp = \mathcal{H}(\mu)$ or $J^\perp = \mathcal{H}(\Lambda) := \bigcap_{\mu \in \Lambda} \mathcal{H}(\mu)$ for a family $\Lambda \subseteq M(G)$. If we define

$$\begin{aligned} \widetilde{\mathcal{H}}(\Lambda) &:= \{T \in \mathcal{B}(L^2(G)) : \mu\text{-harmonic for all } \ \mu \in \Lambda\} \\ &= \{T \in \mathcal{B}(L^2(G)) : \Theta(\mu)(T) = T \text{ for all } \ \mu \in \Lambda\} . \end{aligned}$$

then $\widetilde{\mathcal{H}}(\Lambda) \supseteq \mathrm{Bim}_{vn}(\mathcal{H}(\Lambda))$.

## Theorem

*Suppose G has the Approximation Property.*
*For any $\Lambda \subseteq M(G)$,*

$$\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{vn}(\mathcal{H}(\Lambda)) .$$

Remark For $\Lambda = \{\mu\}$ with a prob. measure $\mu$, the result holds for all groups (Izumi, Jaworski-Neufang). Reason: $\mathcal{H}(\mu)$ is linearly and covariantly completely isometrically isomorphic to a Von Neumann algebra.

# Change of perspective: The crossed product

Let $\mathcal{V} \subseteq \mathcal{B}(H)$ be a w*-closed operator space, and let $s \to \alpha_s$ be an action of $G$ on $\mathcal{V}$ by weak-* continuous unital complete isometries.
To represent both $G$ and $\mathcal{V}$ simultaneously and covariantly:
Define, for each $v \in \mathcal{V}$, an element $\tilde{\alpha}(v) \in \mathcal{V}\bar{\otimes}L^\infty(G)$ by duality:

$$\langle \tilde{\alpha}(v), \omega \otimes h \rangle := \int_G \langle \alpha_s^{-1}(v), \omega \rangle h(s) ds, \qquad \omega \in \mathcal{V}_*, h \in L^1(G) \, .$$

Since $L^\infty(G) \subseteq \mathcal{B} := \mathcal{B}(L^2(G))$ we have a map

$$\tilde{\alpha} : \mathcal{V} \to \mathcal{V}\bar{\otimes}L^\infty(G) \subseteq \mathcal{V}\bar{\otimes}\mathcal{B} \, .$$

Also $\tilde{\lambda} : G \to \mathcal{V}\bar{\otimes}\mathcal{B} : s \to \tilde{\lambda}_s := \mathrm{Id}_H \otimes \lambda_s \, .$

Covariance:

$$\tilde{\alpha}(\alpha_s(v)) = \tilde{\lambda}_s \tilde{\alpha}(v) \tilde{\lambda}_s^{-1} \, .$$

# The crossed products

$$\tilde{\alpha} : \mathcal{V} \to \mathcal{V} \bar{\otimes} L^\infty(G) \subseteq \mathcal{V} \bar{\otimes} \mathcal{B} .$$
$$\tilde{\lambda} : G \to \mathcal{V} \bar{\otimes} \mathcal{B} : s \to \tilde{\lambda}_s := \mathrm{Id}_H \otimes \lambda_s .$$

The *spatial crossed product* $\mathcal{V} \rtimes_\alpha G$ is defined to be the subspace of $\mathcal{V} \bar{\otimes} \mathcal{B}$ generated by $\tilde{\alpha}(\mathcal{V}) \cdot \tilde{\lambda}(G)$: it is the weak* closed space

$$\mathcal{V} \rtimes_\alpha G := \overline{\operatorname{span}\{\tilde{\alpha}(v)\tilde{\lambda}_s,\ v \in \mathcal{V}, s \in G\}}^{w*} \subseteq \mathcal{V} \bar{\otimes} \mathcal{B}.$$

The *Fubini crossed product* $\mathcal{V} \rtimes_\alpha^F G$ is defined to be the fixed point subspace $(\mathcal{V} \otimes \mathcal{B})^{\tilde{\alpha}}$ of $\mathcal{V} \bar{\otimes} \mathcal{B}$ for a certain map $\tilde{\alpha}$ which essentially comes from the action $\alpha$ on $\mathcal{V}$ and the right regular representation of $G$ on $\mathcal{B}$.

$$\mathcal{V} \rtimes_\alpha G \subseteq \mathcal{V} \rtimes_\alpha^F G \subseteq \mathcal{V} \otimes \mathcal{B} .$$

# Bimodules and Crossed products

Specialize to the case of the action $G \overset{\alpha}{\curvearrowright} L^\infty(G)$ by left translation.

## Proposition (D. Andreou)

*For any left closed ideal $J$ of $L^1(G)$, we have*

$$\mathrm{Bim}_{vn}(J^\perp) \overset{\Psi}{\simeq} J^\perp \rtimes_\alpha G \quad \textit{and} \quad \ker\Theta(J) \overset{\Psi}{\simeq} J^\perp \rtimes_\alpha^F G.$$

Here $\Psi : \mathcal{B} \to \mathcal{B}\bar{\otimes}\mathcal{B}$ is an isometric normal *-morphism such that $\lambda_s \to 1 \otimes \lambda_S$ and $f \to \tilde{\alpha}(f)$ where $\tilde{\alpha}(f)(s,t) = f(ts)$.

## Theorem (D. Andreou)

*$G$ has the AP $\iff$ $\mathcal{V} \rtimes_\alpha G = \mathcal{V} \rtimes_\alpha^F G$ for all dual operator spaces.*

$(\Rightarrow)$ proved also by Crann - Neufang.

# Bimodules and Crossed products

So we know that, if a group $G$ has the AP, then $J^{\perp} \rtimes_{\alpha} G = J^{\perp} \rtimes_{\alpha}^{F} G$ for all closed left ideals $J$ of $L^1(G)$.

Question: Is the AP necessary for this?

Or some weaker approximation property?

Or is it true for all $G$?

# Part 2: Ideals of $A(G)$ and $L^\infty(G)$ bimodules

In Part 1 we studied ideals of $L^1(G)$ and $\mathrm{vN}(G)$ bimodules.

Replace the dual pair $\left\langle L^1(G), L^\infty(G)\right\rangle$ by $\langle A(G), \mathrm{vN}(G)\rangle$.

*We now study w*-closed subspaces of $\mathcal{B}$ which are bimodules over $L^\infty(G) \curvearrowright L^2(G)$.*

Recall: the predual $A(G)$ of $\mathrm{vN}(G)$ consists of all functions $u : G \to \mathbb{C}$ of the form $u(s) = (\lambda_s f, g)$ where $f, g \in L^2(G)$. It is an abelian Banach algebra.

# Action of $A(G)$ on $\mathcal{B} := \mathcal{B}(L^2(G))$

Given $u \in A(G)$ define

$$\widehat{\Theta}_u(f\lambda_s) := u(s)\, f\lambda_s \quad f \in L^\infty(G),\ s \in G\,.$$

[This coincides with $S_{Nu}(f\lambda_s)$, where $S_{Nu}$ is the Herz-Schur multiplier.]

(Recall $f$ is identified with the multiplication operator and $(\lambda_s g)(t) := g(s^{-1}t)$ on $L^2(G)$).

The map $\widehat{\Theta}_u$ extends to a w*-continuous $L^\infty(G)$-bimodule map on $\mathcal{B} = \overline{\mathrm{span}}^{\mathrm{w}^*}(L^\infty(G) \cdot \mathrm{vN}(G)) \simeq L^\infty(G) \rtimes_\alpha G$.

# From $A(G)$-ideals to $L^\infty(G)$-bimodules

Given (closed) ideal $J \subseteq A(G)$ first consider the annihilator $J^\perp \subseteq \mathrm{vN}(G) \subseteq \mathcal{B}$ and then the weak* closed $L^\infty(G)$-bimodule $\mathrm{Bim}_m(J^\perp)$ generated by $J^\perp$:

$$\mathrm{Bim}_m(J^\perp) = \overline{\mathrm{span}\{fTg : T \in J^\perp, f, g \in L^\infty(G)\}}^{w*}$$

$$\begin{array}{ccccc}
A(G) & & \mathrm{vN}(G) & \subseteq & \mathcal{B} \\
\cup & & \cup & & \cup \\
J & \longrightarrow & J^\perp & \longrightarrow & \mathrm{Bim}(J^\perp)
\end{array}$$

Another way: Consider

$$\ker \widehat{\Theta}(J) = \{T \in \mathcal{B} : \widehat{\Theta}_u(T) = 0 \ \forall u \in J\}$$

Also a weak* closed $L^\infty(G)$-bimodule. Easy to see

$$\mathrm{Bim}_m(J^\perp) \subseteq \ker \widehat{\Theta}(J) \, .$$

# From $A(G)$-ideals to $L^\infty(G)$-bimodules

**Theorem (AKT)**

*For every (closed) ideal $J \subseteq A(G)$, we have* $\mathrm{Bim}_m(J^\perp) = \ker \widehat{\Theta}(J)$ .

Application. Proof (under $D_\infty$) that a closed $E \subseteq G$ is a set of spectral synthesis iff the associated $E^* \subseteq G \times G$ is operator synthetic [Ludwig-Turowska].

$D_\infty : \ \forall u \in A(G), \ u \in \overline{uA(G)}$.

# Crossed products by co-actions

Instead of the action

$$\tilde{\alpha} : \mathcal{B} \to \mathcal{B}\bar{\otimes}L^\infty(G) \subseteq \mathcal{B}\bar{\otimes}\mathcal{B}$$

of $L^\infty(G)$ on $\mathcal{B}$ induced by translation, consider a "dual" action

$$\delta : \mathcal{B} \to \mathcal{B}\bar{\otimes}\mathrm{vN}(G) \subseteq \mathcal{B}\bar{\otimes}\mathcal{B}$$

of $\mathrm{vN}(G)$ given by $\delta(f) = f \otimes 1$ for $f \in L^\infty(G)$ and $\delta(\lambda_s) = \lambda_s \otimes \lambda_s$ for $s \in G$.

If $\mathcal{V} \subseteq \mathcal{B}$ is invariant under $\delta$, i.e. $\delta(\mathcal{V}) \subseteq \mathcal{V} \otimes \mathcal{B}$, then:

The *spatial dual crossed product* of $\mathcal{V}$ by $\delta$ is defined to be the subspace of $\mathcal{V}\bar{\otimes}\mathcal{B}$

$$\mathcal{V} \ltimes_\delta G = \overline{\mathrm{span}}^{w^*} \left\{(1 \otimes f)\,\delta(v) : \; f \in L^\infty(G), \; v \in \mathcal{V}\right\} .$$

The Fubini dual crossed product of $\mathcal{V}$ by $\delta$ is defined to be

$$\mathcal{V} \ltimes_\delta^{\mathcal{F}} G = (\mathcal{V}\overline{\otimes}\mathcal{B})^{\widetilde{\delta}} .$$

where $\widetilde{\delta}$ is a certain map induced by an action of $\mathrm{vN}(G)$.

# Dual crossed products and bimodules

**Proposition (D. Andreou)**

*For any closed ideal $J$ of $A(G)$, we have*

$$\mathrm{Bim}_m(J^\perp) \overset{\Phi}{\simeq} J^\perp \ltimes_\delta G \quad \textit{and} \quad \ker\Theta(J) \overset{\Phi}{\simeq} J^\perp \ltimes_\delta^F G.$$

Here $\Phi : \mathcal{B} \to \mathcal{B}\bar{\otimes}\mathcal{B}$ is an isometric normal *-morphism such that $\lambda_s \to \lambda_s \otimes \lambda_s$ and $f \to 1 \otimes f$.

**Theorem (D. Andreou)**

$\mathcal{V} \ltimes_\delta G = \mathcal{V} \ltimes_\delta^F G$ *for all dual operator spaces* $\mathcal{V}$.

Yields new approach to proof of $\mathrm{Bim}_m(J^\perp) = \ker\widetilde{\Theta}(J)$.

# References


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