

Part II. Product systems over right LCM-semigroups

Defn. A semigroup P is called right LCM if:

(1) it is left cancellative: $pr = ps \Rightarrow r = s$.

(2) satisfies Clifford condition: $p^P \cap q^P = \emptyset$ or $p^P \cap q^P = w^P$.

Rem? If $p^P \cap q^P = w^P$ then w is called (the) right Least Common Multiple. wx is so for any $x \in P \cap P^{-1}$.

(2) if $P \cap P^{-1} = \{e\}$ then right LCM means quasi-lattice.

(3) If $P \hookrightarrow G$ then we do not assume an LCM structure in G .

Examples: (1) Artin monoids



(2) $B(m,n)^* = \langle a, b \mid ab^m = b^n \rangle$ Brunsbag-Solitar

(3) $R \rtimes R^\times$ over an integral domain R iff R satisfies GCD.

(4) We don't have a partial order in a right LCM but ...

Def. let $P \in G$ right LCM. A finite set F of P is said to be $\overset{-2-}{v\text{-closed}}$ if: $\forall p, q \in P$ with $pP \cap qP \neq \emptyset$ \exists unique $w \in F$ s.t.
 $pP \cap qP = wP$.

Rem.(1) The right LCM defines a partial order on $v\text{-closed}$ finite sets.

(2) for every finite F we write F^v for ~~α~~ $v\text{-closure}$ by considering a minimal set of LCM's.

$$\begin{aligned} F = \{p_1, \dots, p_n\} &\rightsquigarrow \{p_1P, \dots, p_nP\} \rightsquigarrow \\ &\rightsquigarrow \langle \{p_1P, \dots, p_nP\} \rangle^v = \{q_1P, \dots, q_sP\} \\ &\rightsquigarrow F^v = \{q_1, \dots, q_s\}. \end{aligned}$$

~~Def.~~ let X p.s. over right LCM P . It is called compactly aligned if: $\forall p, q \in P$ with $pP \cap qP = wP$ we have:

$$L_p^w(k_p) L_q^w(k_q) \in kX_w \quad \forall k_p \in kX_p, k_q \in kX_q.$$

Prop: let P be right LCM and \times p.s. over P . TFAE:

(1) \times is compactly aligned

(2) $\bar{t}(k_p) \bar{t}(k_q) \subseteq \bar{t}(k_{pq})$ whenever $pP \cap qP = wP$, otherwise \emptyset .

(3) (2) for any injective repn t .

Def. let P be right LCM, \times compactly aligned over P . A repn t of \times is said to satisfy the Nica-covariant condition

$$\text{if: } t(k_p) t(k_q) = \begin{cases} t(l_p^w(k_p) l_q^w(k_q)) & , pP \cap qP \neq wP \\ \emptyset & , pP \cap qP = \emptyset . \end{cases}$$

Rem. (1) Considered by Fwonsiewski-Larsen, Brownlowe-Larsen-Stammelie.

(2) Note that Nica-covariance is independent of the choice of w .

Indeed: if $pP \cap qP = w'P$ then $w=w'$ for $x \in P \cap P'$ and we have shown that $t(k_w) = t(l_w^w(k_w))$.

Facts: let t be Nica-covariant repn's. Then:

- ① $C^*(t) = \overline{\text{span}} \{ t_p(x_p) t_p(x_p)^*: p \in P \}$.
- ② If f : finite V -closed then $B_{f,t} := \text{span} \{ t_p(x_p) t_p(x_p)^*: p \in P \}$ is a C^* -subalgebra.
- ③ If t admits a coaction by G $t_p(\beta_p) \mapsto t_p(\beta_p) \otimes \iota_p$ then its fixed point algebra is described:

$$\overline{B_{P,t}} := \overline{\text{span}} \{ t_p(x_p) t_p(x_p)^*: p \in P \} = \overline{\bigcup_{\substack{f \text{ finite} \\ V\text{-closed}}} B_{f,t}}$$

Rew: For any finite $F \sim F'$ and note that if $q = px$ for $x \in P \cap P'$ then $t(k \gamma_q) = [t_q(x_q) t_q(x_q)^*] =$

$$= [t_p(x_p) t_x(x_x) t_x(x_x)^* t_p(x_p)^*] =$$

$$= [t_p(x_p) t_p(x_p)^*] = t(kx_p).$$

Defn. Write $\mathcal{NT}(x)$ for the universal C^* -algebra w.r.t Maa.-co-^{-5-}
variant repr's of a c.e. ~~not~~ p.s. x over right LCM $P \subseteq G$.

- Prop:
- ① $\mathcal{NT}(x)$ admits a coaction by G .
 - ② $\mathcal{NT}(x) \cong C_{\max}(\alpha x)$ for the full bundle $\{\alpha x = \{[\mathcal{NT}(x)]_g\}_{g \in G}\}$
 - ③ $\gamma_x(x) \cong C^*(\alpha x)$.

Pl. ① by universality. ② all relations are graded.

③ $\Rightarrow [\mathcal{NT}(x)]_e \cong [\tilde{T}_e(x)]_e$. Sufficient for B_f -cores.

Let $\mathcal{NT}(x) = C(\hat{x})$ and take $f = \sum_{p \in F} b_p (t_p) \otimes \text{ker } \Phi$

let t_0 minimal in F s.t. $\hat{x}(t_0) \neq 0$. Then $b_0 \neq 0$. But then:

$$b_0 = Q_{p_0} \left(\sum_{p \in F} \hat{x}(t_p) \right) Q_{p_0} = Q_{p_0} \Phi(f) Q_{p_0} = 0$$

which is a contradiction. (Here $Q_{p_0}: Fx \rightarrow x_{p_0}$.)



Part IV. Co-universality.

Let X be a compactly aligned product system over P right LCN-semigroup and $P \subseteq G$.

Goal 1. Show that $\exists t'$ repn of X s.t.

(1) t' is Nice-covariant + injective on A

(2) $\forall t$ Nice-covariant + injective on A + coaction of G

$\exists \bar{\Phi}: C^*(t) \rightarrow C^*(t')$ s.t. $b_P(\gamma_P) \mapsto b'_P(\gamma_P)$.

Goal 2. Identify the α -algebraic C^* -structure of that co-universal C^* -algebra.

Recall: $\mathbb{G}^*(\rho) = C(V_\rho; \rho \in P)$ $V_\rho: l^2(\rho) \rightarrow l^2(\rho) : e_i \mapsto e_{\rho i}$.

Prop. Let $t = \{t_p\}_{p \in \mathbb{P}}$ ^{n.f.} repn of X . Then \exists faithful $*\text{-hom}$

$$\tilde{\tau}_x(x) \rightarrow C^*(t) \otimes G^*(\mathbb{P}) : \tilde{\tau}_p(z_p) \mapsto t_p(z_p) \otimes v_p.$$

Pf. Step 1. $t \otimes V : NT(X) \rightarrow C^*(t) \otimes G^*(\mathbb{P})$.

Then $(t \otimes V)_*$ is faithful on $[NT(X)]_*$: Sps $NT(X) = C^*(\hat{t})$ and take $f = \sum_{p \in F} \hat{t}(z_p) \in \ker(t \otimes V)_*$, let p_0 of minimal s.t. $\hat{t}(z_{p_0}) \neq 0 \Rightarrow z_{p_0} \neq 0 \Rightarrow t(z_{p_0}) \neq 0$. But then:

$$\begin{aligned} t(z_{p_0}) &= (I \otimes P_{p_0}) \left(\sum_{p \in F} t(z_p) \otimes v_p v_p^* \right) (I \otimes P_{p_0}) = \\ &\Rightarrow (I \otimes P_{p_0}) (t \otimes V)_* \left(\sum_{p \in F} \hat{t}(z_p) \right) (I \otimes P_{p_0}) = 0. \end{aligned}$$

Step 2. $t \otimes V$ admits a coaction as $G^*(\mathbb{P})$ admits one. This is normal as so is the one on $G^*(\mathbb{P})$. Hence:

$$C^*((t \otimes V)_*) \simeq G^*(NX) \simeq \tilde{\tau}_x(x).$$

Defn. In $\mathfrak{T}_\lambda(x)$ define $\mathfrak{T}(x) = \overline{\text{alg}}\{X_\rho : \rho \in \mathcal{P}\}$ the fact tensor algebra.
 The normal coaction on $\mathfrak{T}_\lambda(x)$ restricts to a normal coaction
 on $\mathfrak{T}(x)^*$.

Prop. Let $\Phi: \mathfrak{T}_\lambda(x) \rightarrow B(H)$ be a $*$ -repm with a coaction s.t.
 $\Phi|_A$ is injective. Then $\alpha(\Phi)$ defines a C^* -cover for $(\mathfrak{T}_\lambda(x)^*, G, \bar{s})$

Pf.

$$\begin{array}{ccc} \mathfrak{T}_\lambda(x)^* & \xrightarrow{\quad \cong \quad} & C^*(\Phi) \otimes G^*(\rho) \\ \Phi \downarrow & \cong & \uparrow d \otimes \psi \\ C^*(\Phi) & \longrightarrow & C^*(\Phi) \otimes C(G) \end{array}$$

where $\psi: C^*(G) \longrightarrow B(\ell^2(G)) \longrightarrow B(\ell^2(\rho))$.

$$u_g \longmapsto U_g \longrightarrow P_{\ell^2(\rho)} u_g |_{\ell^2(\rho)}$$

$$v_{\mu\rho} \longmapsto U_\rho \longmapsto V_\rho .$$



Thm DK12: If t' nice-coverent, injective, equivariant repn of X with the comiversal property:

Pf: let t be nice-coverent, injective, equivariant and set
 $\mathcal{F} = \{C^*(t)_g\}_{g \in G}$ be the Fell bundle. Then:

$$\begin{array}{ccc} \mathcal{F}(x) = C^*_{max}(x) & \rightarrow & C^*(t) \rightarrow \mathcal{G}(\mathcal{F}) \\ & \searrow & \nearrow \\ & \tilde{\tau}_x(x) = C^*_r(x) & \longrightarrow C^*_e(\tilde{\tau}_x(x), G, \bar{s}) \end{array}$$

□

... and this comiversal C^* -algebra is
 $C^*_e(\tilde{\tau}_x(x), G, \bar{s})$.

Strong covariant relations:

① Set $I_{(r,g)} = \begin{cases} \cap_{t \in r \cap g^P} \text{ker } \varphi_{r,t}, & \text{if } r \cap g^P \neq \emptyset, \\ A & \text{otherwise} \end{cases}$

Set $I_{r^*(r \vee f)} = \bigcap_{g \in f} I_{(r,g)}$ for $f \subseteq G$ finite.

② Set $X_F = \sum_{r \in P} X_r I_{r^*(r \vee F)}$ $X_{F^+} = \sum_{g \in G} X_g F$

③ $[\gamma(x)]_e \rightarrow [\tilde{\gamma}_\lambda(x)]_e \xrightarrow{③ \bar{\Phi}_F} \prod_{\substack{F \subseteq G \\ \text{fin}}} B(X_F) \rightarrow \frac{\prod B(X_F)}{G(B(X_F))}$

let \mathcal{I}_e be the kernel of this repn.

let $\mathcal{L}_\infty = \langle \mathcal{I}_e \rangle \times \gamma(x)$

Def.: $A \times_P = \gamma(x)/\mathcal{L}_\infty$.

Properties : ① independent of G

② $A \hookrightarrow A \rtimes_{\alpha} P$

③ $A \rtimes_{\alpha} P$ satisfies Nica-covariant repn.

④ $A \rtimes_{\alpha} P$ admits coaction; write scx for the Fell bundle

⑤ $\Phi: A \rtimes_{\alpha} P \rightarrow B(H)$ is faithful on A . If faithful on $[A \rtimes_{\alpha} P]_e$.

⑥ α defines repn of $A \rtimes_{\alpha} P$ iff Nica-covariant and satisfies

$$\sum_{P \in F} \alpha_f(k_P) = 0 \Leftrightarrow \sum_{P \in F} b_P(k_P) = 0 \quad \text{if } f \subseteq G \text{ finite.}$$

Thm DKL20 : ① $C_c^*(\tilde{\alpha}(x)^*; G, S) \cong \tilde{G}^*(\text{scx})$

② If G is exact ~~and~~ then $\cong \tilde{T}_x(\tilde{x}) / \tilde{\alpha}(x_a)$.

Cor DKL20 : Sp $\alpha: G \rightarrow \text{Aut}(\tilde{\alpha}(x))$. Then :

$$C_c^*(\text{sc}(x \rtimes_{\alpha, \beta} G)) \cong \tilde{G}^*(\text{scx}) \rtimes_{\alpha, \beta} G.$$

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