

Part I: Fell bundles + topological gradings + coactions

Defn: $\mathcal{B} = \{\mathcal{B}_g\}_{g \in G}$ is a Fell bundle means:

(1) Every \mathcal{B}_g is a Banach space

(2) \exists multiplication

(3) \exists involution

$$(4) b_g^* = b_g$$

$$(5) (b_g b_h)^* = b_h^* b_g^*$$

$$(6) \|b_g b_h\|_{\mathcal{B}_h} \leq \|b_g\|_{\mathcal{B}_g} \cdot \|b_h\|_{\mathcal{B}_h}$$

$$(7) \|b_g^* b_g\| = \|b_g\|^2$$

$$(8) \forall b_g \in \mathcal{B}_g \exists a \in \mathcal{B}_e \text{ s.t. } b_g^* b_g = a^* a$$

$$\begin{aligned} \cdot & \mathcal{B}_g \times \mathcal{B}_h \longrightarrow \mathcal{B}_{gh} && \text{bilinear + associative} \\ * & : \mathcal{B}_g \longrightarrow \mathcal{B}_g^* && \text{cong-linear + idempotent} \end{aligned}$$

Rem: • \mathcal{B}_g is called the g -fiber of \mathcal{B}

• $\mathcal{B}_g \cdot \mathcal{B}_g^*$ ideal of \mathcal{B}_e .

Defn. Set $C_c(\mathcal{B}) = \bigoplus_{g \in G} \mathcal{B}_g$ and define:

- (1) $*: C_c(\mathcal{B}) \times C_c(\mathcal{B}) \rightarrow C_c(\mathcal{B})$ s.t. $(\beta * \eta)(h) = \sum_g \beta(g) \eta(g^{-1}h)$
- (2) ${}^*: C_c(\mathcal{B}) \rightarrow C_c(\mathcal{B})$ s.t. $\beta^*(h) = \beta(h^{-1})^*$.

Rem.: (1) $\mathcal{B}_g \hookrightarrow C_c(\mathcal{B})$ s.t. $b_g \mapsto \delta_{bg}$.

(2) If $\|\cdot\|_1$ on $C_c(\mathcal{B})$ is given by $\|\beta\|_1 := \sum_{g \in G} \|\beta(g)\|_{\mathcal{B}_g}$ then every $*$ -homomorphism $C_c(\mathcal{B}) \rightarrow \mathcal{B}(H)$ is $\|\cdot\|_1$ -continuous.

(3) $*$ -homomorphisms of $C_c(\mathcal{B})$ are in bijection with collections $\pi = \{\pi_g\}_{g \in G}$ of linear maps $\pi_g: \mathcal{B}_g \rightarrow \mathcal{B}(H)$ s.t.

$$\pi_g(b_g) \pi_h(b_h) = \pi_{gh}(b_{gh})$$

$$\pi_g(b_g)^* = \pi_{g^{-1}}(b_{g^{-1}}).$$

Thm.: A usual argument produces the universal $C^*(\mathcal{B})$, i.e., the C^* -algebra s.t. every π of \mathcal{B} gives a $*$ -epimorphism

$$C^*(\mathcal{B}) \xrightarrow{\cong} C^*(\pi) := C^*(\pi_g(\mathcal{B}_g) : g \in G).$$

Defn: Consider \mathcal{B}_g as a right Hilbert module over $\mathcal{B}e$.

- Set $\ell^2(\mathcal{B}) = \ell^2 - \bigoplus_{g \in G} \mathcal{B}_g$ as a Hilbert module over $\mathcal{B}e$.
- Define $\mathcal{I}: \mathcal{B} \rightarrow L(\ell^2(\mathcal{B}))$ s.t. $\mathcal{I}(b_g) g_n = b_g \delta_{gn}$
- Define $C^*_r(\mathcal{B}) = C^*(\mathcal{I}) := C^*(\mathcal{I}(\mathcal{B}_g) : g \in G)$.

This is called the reduced crossed sectional C^* -algebra.

Rem: If $\mathcal{B}_g = \mathbb{C}$ then $C^*(\mathcal{B}) = C(G)$ and $G(\mathcal{B}) = \mathcal{G}(G)$.

Thm: A conditional expectation $E_g: G(\mathcal{B}) \rightarrow \mathcal{B}e$ given by compressing to $\mathcal{B}e \subset \ell^2 - \bigoplus_{g \in G} \mathcal{B}_g$.

[Cond. exp over $\mathcal{B}e$ means: contractive, idempotent map, $\mathcal{B}e$ in multiplicative domain i.e. $E_g(x \cdot b) = E_g(x)b$ if $b \in \mathcal{B}e$].

A contractive $E_g: G(\mathcal{B}) \rightarrow \mathcal{B}_g$ s.t. $E_g(\mathcal{I}(b_n)) = \delta_{g,n} \mathcal{I}(b_n)$.

Defn.: Now let C be a C^* -algebra. A collection $\{C_g\}_{g \in G}$ is called a grading for C if: (1) $C_g = C_g^*$, (2) $C_g C_h \subseteq C_{gh}$, (3) $\{\langle g \rangle\}_{g \in G}$ is linearly independent and $\bigoplus_{g \in G} C_g$ is dense in C .

- It is called a topological grading if there exists a conditional expectation $E : C \rightarrow C_E$.

Thm: Let $\{C_g\}_{g \in G}$ be a top. grading for C . Then $\{C_g\}_{g \in G}$ is a Fell bundle, and $\exists \Phi : C \rightarrow S_C(c) : c_g \mapsto \Phi(c_g)$.

- Pf:
- (1) Let C "become" a C_E -module with $\langle x, y \rangle = E(x^*y)$.
 - (2) For $c \in C$: $\langle c x, c x \rangle = E(x^* c^* c x) \leq \|c\|^2 E(x^* x) = \|c\|^2 \langle x, x \rangle$.
 - (3) \exists a π -homomorphism $C \ni c \mapsto h_c \in L(C_E) \quad h_c(x) = cx$.
 - (4) The map $U : C_E \rightarrow l^2(C)$ s.t. $U(\sum c_g) := \sum c_g$ is an isometry and onto: $\langle U c_g, U c_h \rangle = \langle c_g, c_h \rangle = \delta_{gh} \langle c_g, c_h \rangle = E(c_g^* c_h)$.
 - (5) $C \ni c \mapsto L_c := U h_c U^* = \Phi(c_g)$ gives the required. ◻

Recall the defn of a coaction $\delta: C \rightarrow C \otimes C^{\circ}(G)$
 i.e. that satisfies $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta$.

Then the fibers $C_g := \{c \in C : \delta(c) = c \otimes c^g\}$ define a topological grading and thus a Fell bundle.

Thm: Sps $\bar{\Phi}: C \rightarrow B(H)$ admits a co-action on (C, δ) .
 Then \exists *-epimorphism: $C^*(\{C_g\}_{g \in G}) \rightarrow C^*(\{\Phi(C_g)\}_{g \in G})$.

Pf.

$$\begin{array}{ccc} C^*(\{C_g\}_{g \in G}) & \xrightarrow{\quad \text{Id} \quad} & C^*(\Phi) \\ \downarrow \varphi' & \nearrow \bar{\Phi} & \downarrow \text{Ep} \\ C^*(\{C_g\}_{g \in G}) & & \end{array}$$

Hts: $\ker \varphi' \supseteq \ker \lambda$.

$$\text{Sps f-ker} \lambda \Rightarrow \lambda E(f^*) = E, \lambda(f^*) = 0 \xrightarrow{E(f^*) \in C^*} E(f^*) = 0$$

$$\Rightarrow E \Phi(f^*) = \Phi E(f^*) = 0 \xrightarrow{\Phi \text{ faithful}} \bar{\Phi}(f^*) = 0 \Rightarrow \text{f-ker } \bar{\Phi}. \quad \square$$

Thm: Let $\tilde{\mathcal{I}} \trianglelefteq C$ and $C = \overline{\bigoplus_{g \in G} C_g}$ be a topological grading.

Set: (i) $\tilde{\mathcal{I}}_1 = \langle \tilde{\mathcal{I}} \cap C \rangle$

(ii) $\tilde{\mathcal{I}}_2 = \{c \in C : E_g(c) \in \tilde{\mathcal{I}} \text{ for } g \in G\}$

(iii) $\tilde{\mathcal{I}}_3 = \{c \in C : E(c^*c) \in \tilde{\mathcal{I}}\}$.

Then: $\tilde{\mathcal{I}}_3 \subseteq \tilde{\mathcal{I}}_2 = \tilde{\mathcal{I}}_1$.

Defn: If $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_1$ then $\tilde{\mathcal{I}}$ is called induced.

If $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_3$ then $\tilde{\mathcal{I}}$ is called Fourier.

Thm: If $\delta: C \rightarrow C \otimes C^*(G)$ is a coaction and $\tilde{\mathcal{I}}$ is induced on C

then δ descends to $\tilde{\delta}: C/\tilde{\mathcal{I}} \rightarrow C/\tilde{\mathcal{I}} \otimes C^*(G)$.

If δ is normal then so is $\tilde{\delta}$.

Rmk: Every induced is a Fourier. If G is exact and E faithful we get that induced \cong Fourier.

Point I. Product systems.

Def. A p.s. over P is a family $\{X_p\}_{p \in P}$ of C^* -correspondences over $A = X_0$ together with unitary maps $M_{P,Q}: X_P \otimes_A X_Q \xrightarrow{\sim} X_{PQ}$

s.t.

$$\begin{array}{ccc} X_P \otimes_A X_Q \otimes_A X_R & \xrightarrow{M_{P,Q} \otimes \text{id}} & X_{PR} \otimes_A X_R \\ \downarrow M_{P,Q} \otimes M_{Q,R} & & \downarrow M_{PQ,R} \\ X_P \otimes_A X_{QR} & \xrightarrow{M_{P,QR}} & X_{PQR} \end{array}$$

and $M_{A,P}: A \otimes_A X_P \rightarrow \overline{A \cdot X_P}$. $M_{P,A}: X_P \otimes_A A \rightarrow \overline{X_P \circ A}$

Rem: We assume here that $\overline{A \cdot X_P} = X_P$.

Rem: We will write $\beta_P \beta_Q = M_{P,Q}(\beta_P \otimes \beta_Q)$.

Rem: We get $l_P^{**}: L X_P \rightarrow L X_Q$ s.t. $l_P^{**}(s)(\beta_P \beta_Q) = S(\beta_P) \beta_Q$

Rem: If \star is invertible in P then $l_P^{**}: L X_P \rightarrow L X_Q$ is a $*$ -isomorphism with inverse $l_{PQ}^{**}: L X_Q \rightarrow L X_P$.

Defn.: A Toeplitz repn $t = \{t_p\}_{p \in P}$ of a p.s. $X = \{x_p\}_{p \in P}$ is a family of maps s.t. (1) (t_α, t_β) is a repn of x_p over A .

$$(2) \quad t_p(\{z_p\}) t_q(\{z_q\}) = t_{pq}(\{z_p z_q\}).$$

The Toeplitz algebra $\tilde{\Gamma}(X)$ is the universal C^* -algebra w.r.t. the Toeplitz repns of X .

For a Toeplitz repn t of X we write $t_X : \tilde{\Gamma}(X) \rightarrow C^*(t)$ for the induced *-repn.

Rem. (Fock repn). Set $FX := \ell^2 - \sum_{p \in P} X_p$. For $\{z_p\}_{p \in P}$ define

$$\bar{t}_p(z_p) v_r := z_p v_r.$$

We get that $\bar{t}_p(z_p) v_r = 0$ if $r \notin P$ and $\bar{t}_p(z_p) v_p v_{p'} = \langle z_p, v_p \rangle v_{p'}$.
Then $\bar{t} = \{\bar{t}_p\}_{p \in P}$ is a Toeplitz repn.

We call $\Gamma(X) := C^*(\bar{t})$ the fock C^* -algebra.

Prop. Let t be a Toeplitz repn of \mathcal{X} .

(1) If $r \in P \cap P^*$, then $\text{tr}(x_r)^* = t_{r^*}(x_{r^*})$.

(2) If $w \in P$ and $r \in P \cap P^*$, then $\text{tr}^w(k_w) \in K \times_{wr}$ and $t(\text{tr}^w(k_w)) = t(k_w)$ for all $k_w \in K \times_w$.

Pf. (1) $A = \mathcal{X}_e \cong \mathcal{X}_r \otimes_A \mathcal{X}_{r^*} \Rightarrow t_e(A) = [\text{tr}(x_r) \text{tr}^*(x_{r^*})]$.
 $x_{r^*} \cong A \otimes_A \mathcal{X}_{r^*} \Rightarrow [t_e(A) \text{tr}^*(x_{r^*})] = \text{tr}^*(x_{r^*})$

$$\Rightarrow \text{tr}(x_r)^* \subseteq [\text{tr}(x_r)^* \text{tr}(x_r) \text{tr}(x_{r^*})] \subseteq [\text{tr}(A) \text{tr}(x_{r^*})] \subseteq \text{tr}^*(x_{r^*})$$

Adjoints + apply for r^* gives the reverse inclusion.

(2) tr^w is a *-isomorphism and thus preserves compacts.

• We can see that $t(\text{tr}^w(k_w))$ and $t(k_w)$ coincide on $[t_w(K \times_{wr}) H]$.

• By (1): $\text{tr}^w(K \times_{wr}) = [\text{tr}_w(x_w) \text{tr}(x_r) \underbrace{\text{tr}(x_r)^*}_{\in A} \text{tr}_w(x_w)] = t(K \times_w)$

• $t(k_w)$ is completely identified by $[t(K \times_w) H] = [t(K \times_{wr}) H]$

$$\Rightarrow t(\text{tr}^w(k_w)) = t(k_w).$$



Sps: $\tau(x) = C^*(\tilde{x})$ and $\tau_x(x) = C^*(\tilde{x}) \quad P \in G.$ - 4 -

Prop: (1) $\tau(x)$ admits a co-action $\tilde{s}: \tau(x) \rightarrow \tau(x) \otimes C^*(G)$
 (2) $\tau_x(x)$ admits a normal co-action $\bar{s}: \tau_x(x) \rightarrow \tau_x(x) \otimes C^*(G).$

Pf: (1) The family $\{\tilde{t}_p \otimes u_p\}$ with $(\tilde{t}_p \otimes u_p)(z_p) = \tilde{t}_p(z_p) \otimes u_p$ defines
 a repn of $\tau(x)$, so:

$$\tau(x) \longrightarrow \tau(x) \otimes C^*(G) \xrightarrow{id \otimes x} \tau(x).$$

(2) let $U: Fx \otimes l^2(G) \rightarrow l^2(G)$ with $U(z_r \otimes e_h) = z_r \otimes e_{rh}$ $\forall P, h \in G$
 Then U is a unitary and $U(\tilde{t}_p(z_p) \otimes I)U^* = \tilde{t}_p(z_p) \otimes J_p$

So: $\tau_x(x) \xrightarrow{\cong} \tau_x(x) \otimes I \xrightarrow{ad U} C^*(\tilde{t}_p(z_p) \otimes J_p : \dots)$

This gives a reduced coaction: $\tilde{\tau}_x(x) \rightarrow \tilde{\tau}_x(x) \otimes C^*_r(G)$
 that promotes to a normal coaction

Note: $\tilde{\tau}_x(x) \otimes C^*_r(G) \subseteq L(Fx) \otimes B(l^2(G)) \subseteq L(Fx \otimes l^2(G)).$ 22

Rem. (fibers). The coaction induce Fell bundle structures:

- $\Gamma(x)_g = \overline{\text{span}} \{ \hat{t}_{p_1}(x_{p_1}) \hat{t}_{p_2}(x_{p_2})^* \dots \hat{t}_{p_n}(x_{p_n})^* : p_1 p_2 \dots p_n = g \}$
- $\tilde{\Gamma}_\lambda(x)_g = \overline{\text{span}} \{ \tilde{t}_{p_1}(x_{p_1}) \tilde{t}_{p_2}(x_{p_2})^* \dots \tilde{t}_{p_n}(x_{p_n})^* : p_1 p_2 \dots p_n = g \}$.

So we get two Fell bundles that may not be isomorphic.
The $\tilde{\Gamma}_\lambda(x)$ -bundle is more interesting.

Example. P semigroup $x_p = e$. Then:

$$\Gamma(x) = C_{iso}(P) \quad \text{and} \quad \tilde{\Gamma}_\lambda(x) = G_\lambda^*(P).$$

(Murphy) $C_{iso}(N^2)$ is not nuclear.

(...) $G_\lambda^*(N^2)$ is nuclear.

G_λ^* has more relations and an interesting structure of projections.