.

The Hilbert matrix,

discrete and continuous

Aristomenis G. Siskakis
Univ. of Thessaloniki

Functional Analysis and Operator Algebras
Athens 2021

1. The Hilbert matrix, brief history.

The (one sided) Hilbert matrix is

$$H = \left(\frac{1}{i+j+1}\right)_{i,j=0}^{\infty} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix}$$

Its (i,j) entry is

$$\frac{1}{i+j+1} = \mu_{i+j}$$

where (μ_n) is the moment sequence

$$\mu_n = \int_0^1 x^n \, dx, \quad n = 0, 1, \cdots$$

of the Lebesgue measure on $[0,1]$.

We will present some known results for the induced operator on spaces of analytic functions, and then discuss the continuous version of the operator

Brief history

Hilbert's double series theorem (1900):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_n a_k}{n+k+1} < \pi\sum_{n=0}^{\infty}a_n^2, \quad a_n \geq 0.$$

Proof published in H. Weyl's dissertation (1908), and the best constant π was found by I. Schur.

Generalized by Hardy and M. Riesz (1925) on the l^p spaces:

If $(a_n) \in l^p,\ (b_n) \in l^q$, $1/p + 1/q = 1$, then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{|a_n||b_k|}{n+k+1} \leq \frac{\pi}{\sin(\frac{\pi}{p})}\|a_n\|_p\|b_n\|_q.$$

with the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ best possible.

By duality, if $1 < p < \infty$ and $(a_n) \in l^p$, then **Hilbert's inequality**:

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{1/p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=0}^{\infty}|a_n|^p\right)^{1/p},$$

Equivalently the operator $H : l^p \to l^p$, $(a_n) \to (A_n)$

$$A_n = \sum_{k=0}^{\infty}\frac{a_k}{n+k+1},$$

is bounded and

$$\|H\|_{l^p \to l^p} = \frac{\pi}{\sin(\frac{\pi}{p})}, \qquad 1 < p < \infty.$$

On spaces of analytic functions

Let $\mathbb{D}$ the unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

An analytic function on $\mathbb{D}$ has a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and we may let the Hilbert matrix act on the sequence (a_n), to formally obtain

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

The right series is not defined for all f, for example

$$f(z) = \frac{1}{1-z} = \sum z^n$$

is not good, so one has to restrict the domain of H.

We thus consider spaces of functions such Hardy, Bergman and other spaces. Each of them is defined by restricting the growth of an appropriate quantity of functions f in them, which implies restricted growth for (a_n).

H on Hardy spaces.

The Hardy space H^p may be defined as the closed subspace of $L^p(\mathbb{T}, d\theta)$

$$H^p = \text{closure of the span}\{e^{in\theta} : n = 0, 1, 2, ...\}$$

We will consider only $p \geq 1$ because the guilty function $\frac{1}{1-z}$ belongs to H^p, all $p < 1$.

Functions $\tilde{f} \in H^p$ have analytic extensions f on $\mathbb{D}$, and an equivalent definition then for membership to H^p is

$$\|f\|_p = \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

with $\|f\|_p = \|\tilde{f}\|_{L^p(\mathbb{T}, d\theta)}$. H^2 is a Hilbert space and

$$\|f\|_2 = \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} = \|(a_n)\|_{l^2},$$

which identifies H^2 with l^2.

Some properties we need:

• If $1 \leq p < q < \infty$ then $H^\infty \subset H^q \subset H^p \subseteq H^1$.

• The Riesz Projection

$$P_+ : L^p(\mathbb{T}, d\theta) \to H^p, \quad \sum_{-\infty}^{\infty} \hat{f}(n) e^{in\theta} \longrightarrow \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}.$$

is a bounded operator for $1 < p < \infty$.

• Hardy's inequality: If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$ then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_1.$$

• Fejér-Riesz inequality: If $f \in H^p$ then

$$\int_{-1}^{1} |f(t)|^p dt \leq \frac{1}{2} \|f\|_p^p.$$

• If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic then the composition operator $C_\varphi(f) = f \circ \varphi$ is bounded on H^p and

$$\|C_\varphi\|_{H^p \to H^p} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/p}$$

The Hilbert matrix on Hardy spaces

If $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H^1$ then the series

$$H(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n.$$

is well defined and $H(f)$ is analytic on $\mathbb{D}$ because

$$\sum_{k=0}^{\infty}\frac{|a_k|}{n+k+1}\le\sum_{k=0}^{\infty}\frac{|a_k|}{k+1}\le\pi\|f\|_1.$$

by Hardy's inequality. In addition, since

$$\frac{1}{n+k+1}=\int_0^1 t^{n+k}\,dt$$

we obtain

$$\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}=\int_0^1 t^n f(t)\,dt,$$

and we have an integral representation

$$H(f)(z)=\sum_{n=0}^{\infty}\left(\int_0^1 t^n f(t)\,dt\right)z^n=\int_0^1 f(t)\frac{1}{1-tz}\,dt.$$

The Fejer-Riesz inequality guarantees that the last integral is convergent for each $z\in\mathbb{D}$. Note that it can be seen as an "improper" line integral on the path $[0,1)$. We will need later to change the path of integration.

Observe at this point that H is not bounded on H^1, i.e. even though $H(f)$ is analytic on $\mathbb{D}$ for all $f\in H^1$, $H(f)$ does not always belong to H^1. This is the case for the functions

$$f_s(z)=\frac{1}{(1-z)(\frac{1}{z}\log\frac{1}{1-z})^s}\in H^1,\quad 1<s<2.$$

The image $H(f_s)$ is a function of the same form but with $0<s<1$, and when $s<1$, $f_s\notin H^1$.

Let $p > 1$.

In the integral representation

$$H(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt,$$

change the path of integration to

$$\gamma(s) = \gamma_z(s) = \frac{s}{(s-1)z+1}, \quad 0 \le s \le 1,$$

an arc of a circle joining 0 and 1, to get

$$\begin{aligned} H(f)(z) &= \int_0^1 \frac{1}{(s-1)z+1} f\left(\frac{s}{(s-1)z+1}\right) ds \\ &= \int_0^1 w_s(z) f(\varphi_s(z))\, ds, \end{aligned}$$

as an average of the weighted composition operators

$$T_s(f)(z) = w_s(z) f(\varphi_s(z)).$$

An involved calculation then gives:

(i) For $2 \le p < \infty$,

$$\|T_s\|_p \le s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}.$$

(ii) If $1 < p < 2$ and $f \in H^p$ with $f(0) = 0$ then

$$\|T_s(f)\|_{H^p} \le s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}\|f\|_{H^p}.$$

Using the above we find for $2 \le p < \infty$,

$$\begin{aligned} \|H\|_p &\le \int_0^1 \|T_s\|_p\, ds \le \int_0^1 s^{\frac{1}{p}-1}(1-s)^{-\frac{1}{p}}\, ds \\ &= B(\frac{1}{p}, 1-\frac{1}{p}) = \Gamma(\frac{1}{p})\Gamma(1-\frac{1}{p}) \\ &= \frac{\pi}{\sin(\frac{\pi}{p})}, \end{aligned}$$

and similarly for $1 < p < 2$.

Theorem [Diamantopoulos + S., (2000)]
If $1 < p < \infty$ then $H : H^p \to H^p$ is bounded and
(i) If $2 \le p < \infty$ then

$$\|H\|_{H^p \to H^p} \le \frac{\pi}{\sin(\frac{\pi}{p})}.$$

(ii) If $1 < p < 2$ and $f \in H^p$ with $f(0) = 0$ then

$$\|H(f))\|_{H^p} \le \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_{H^p}$$

H as a Hankel operator.

Recall the Riesz projection $P_+ : L^p(\mathbb{T}) \to H^p$

$$\sum_{-\infty}^{\infty} a_n e^{in\theta} \longrightarrow \sum_{n=0}^{\infty} a_n z^n,$$

is bounded when $1 < p < \infty$. Let

$$\phi(t) = i(\pi - t)e^{-it} \in L^\infty(\mathbb{T}),$$

with $\|\phi\|_{L^\infty(\mathbb{T})} = \pi$ and

$$M_\phi(f) = \phi f : L^p(\mathbb{T}) \to L^p(\mathbb{T}),$$

the corresponding multiplication operator. The Fourier coefficients of ϕ are

$$\hat{\phi}(n) = \frac{1}{n+1}, \quad n \ge 0.$$

If J is the isometric "flip"operator

$$J : H^p \to L^p(\mathbb{T}), \quad f(e^{it}) \to f(e^{-it})$$

then

$$H = P_+ \circ M_\phi \circ J, \qquad \begin{tikzcd} L^p(\mathbb{T}) \arrow[r, "M_\phi"] & L^p(\mathbb{T}) \arrow[d, "P_+"] \\ H^p \arrow[u, "J"] \arrow[r, "H"] & H^p \end{tikzcd}$$

It follows immediately that $H : H^p \to H^p$ is bounded for $1 < p < \infty$ and

$$\|H\|_p \le \|P_+\|_p \|M_\phi\| = \|\phi\|_\infty \|P_+\|_p = \pi \|P_+\|_p.$$

Now for $\|P_+\|$, it was known from 1968 that

$$\|P_+\|_p \geq \frac{1}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty,$$

and was conjectured that equality holds. This conjecture was proved by Hollenbeck and Verbitsky (2000). It follows that

$$\|H\|_{H^p \to H^p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty.$$

The lower estimate was obtained by M. Dostanić, M. Jevtić and D. Vukotić (2008) by using the test functions

$$f_\gamma(z) = \frac{1}{(1-z)^{\gamma/p}} \in H^p, \quad 0 < \gamma < 1,$$

then $\|f_\gamma\|_{H^p} \to \infty$ as $\gamma \to 1$. An involved calculation gives

$$H(f_\gamma)(z) = \frac{\pi}{\sin(\frac{\gamma\pi}{p})} z^{\gamma/p-1} f_\gamma(z) + R_\gamma(z)$$

with $R_\gamma(z)$ having a uniformly bounded $p-$integral on the circle for all γ close to 1. This gives

$$\begin{aligned} \|H\|_p \|f_\gamma\|_{H^p} &\geq \|H(f_\gamma)\|_{H^p} \\ &\geq \left| \frac{\pi}{\sin(\frac{\gamma\pi}{p})} \|f_\gamma\|_{H^p} - \|R_\gamma\|_{L^p(\mathbb{T})} \right|. \end{aligned}$$

so that

$$\|H\|_p \geq \left| \frac{\pi}{\sin(\frac{\gamma\pi}{p})} - \frac{\|R_\gamma\|_{L^p(\mathbb{T})}}{\|f_\gamma\|_{H^p}} \right|$$

and letting $\gamma \to 1^-$ gives

$$\|H\|_p \geq \frac{\pi}{\sin(\frac{\pi}{p})}.$$

Thus we have

Theorem. If $1 < p < \infty$ then $H : H^p \to H^p$ is bounded and

$$\|H\|_{H^p \to H^p} = \frac{\pi}{\sin(\frac{\pi}{p})}.$$

H on Bergman spaces.

An analytic $f : \mathbb{D} \to \mathbb{C}$ belongs to the Bergman space A^p if

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$. They are Banach spaces for $p \geq 1$, and A^2 is a Hilbert space with

$$\|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$$

Functions in A^p do not necessarily have boundary values on $\mathbb{T}$.

H cannot be defined on A^2: For the function

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{\log(n+1)} z^n \in A^2$$

one has

$$H(f)(0) = \sum_{n=0}^{\infty} \frac{1}{(n+1)\log(n+1)}, \quad \text{divergent.}$$

It can however be defined well on A^p for $p > 2$, and has the same integral representation, as an average of weighted composition operators. This representation was used, as in th Hardy space case, by Diamantopoulos 2004, Dostanic Jevtic Vukotic 2008, and V. Bozin and B. Karapetrovic 2017 to show,

Theorem. If $2 < p < \infty$ then $H : A^p \to A^p$ is bounded and,

$$\|H\|_{A^p \to A^p} = \frac{\pi}{\sin(\frac{2\pi}{p})}.$$

A more general representation.

In the integral representation we can use other paths of integration

$$H(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt$$

we can use various other paths of integration. For example

$$\gamma(s) = \phi_s(z)$$

where $\{\phi_s(z) : 0 \leq s \leq 1\}$ is a family of functions such that
(1) $\phi_s : \mathbb{D} \to \mathbb{D}$ analytic for each $0 \leq s < 1$,
(2) $\phi_0(z) = 0$, $\phi_1(z) = 1$ for each $z \in \mathbb{D}$,
(3) $\phi_s(z)$ is differentiable in s.

Then we obtain

$$H(f)(z) = \int_0^1 \frac{\frac{\partial \phi_s(z)}{\partial s}}{1-\phi_s(z)z} f(\phi_s(z))\,dt.$$

In particular if $h : \mathbb{D} \to \mathbb{C}$, $h(0) = 0$ is univalent and starlike and $\psi_s(z) = h^{-1}(sh(z))$, then $\psi_s(\mathbb{D}) \subset \mathbb{D}$ and $\psi_s(0) = 0$. The functions

$$\phi_s(z) = \frac{\psi_s(z)}{z},$$

are self-maps of $\mathbb{D}$ by Schwarz's Lemma, with the required properties. We then find

$$H(f)(z) = \frac{h(z)}{z}\int_0^1 w(z\phi_s(z))f(\phi_s(z))\,ds$$

where $w(z) = \frac{1}{(1-z)h'(z)}$.

For the choice $h(z) = \frac{1}{1-z}$ we recover the representation of H in terms of T_s, while the choice

$$h(z) = \log\frac{1}{1-z}$$

gives

$$H(f)(z) = \frac{1}{z}\log\frac{1}{1-z}\int_0^1 f\left(\frac{1-(1-z)^s}{z}\right)\,ds.$$

About the spectrum of H on H^p.

Let $F_n = \left(\frac{1}{i+j+1}\right)_{i,j=0}^{n-1}$ be the finite section of H of size n. Then

$$\det(F_n) = \frac{((n-1)!!)^4}{(2n-1)!!},$$

where $n!! = \prod_{k=1}^{n} k!$. For example

$$\det(F_3) = \frac{1}{2160},$$
$$\det(F_6) = \frac{1}{186313420339200000},$$
$$\det(F_9) \sim \frac{1}{10^{42}}.$$

In particular the inverse of F_n has very large integer entries and the computation of its eigenvalues is a very sensitive problem. In Numerical Analysis, F_n are typical examples of "ill-conditioned" matrices, difficult to use in numerical computation.

Various papers from 1950 - 1960, discuss H and more general versions of it like

$$H_\lambda = \left(\frac{1}{i+j+\lambda}\right), \quad \lambda \in \mathbb{C}.$$

concentrating mostly on the spectrum on l^2 and in the "latent roots".

A latent root is a phony eigenvalue in the sense that the corresponding eigenvector is not necessarily in the space under consideration. For H_λ, this will be a complex number c for which there is a nonzero sequence (x_n), not necessarily in l^2, such that

$$\sum_{k=0}^{\infty} \frac{x_k}{n+k+\lambda} = cx_n, \quad n = 0, 1, \cdots$$

Some typical results are,

Theorem [Hill, J. London Math. Soc. (1960)]

Suppose $x_n = x_n(\lambda, \mu)$ is defined by

$$x_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\Gamma(k+\mu)\Gamma(k+1-\mu)}{k!\Gamma(k+\lambda)}$$

where $0 < \mathrm{Re}(\mu) \leq \frac{1}{2}$. Then

$$\sum_{k=0}^{\infty} \frac{x_k}{n+k+\lambda} = \frac{\pi}{\sin(\pi\mu)} x_n, \quad n = 0, 1, 2, \cdots$$

i.e $\frac{\pi}{\sin(\pi\mu)}$ is a latent root of H_λ.

Theorem. [W. Magnus, (1950)]

The spectrum of $H = H_1$ on l^2 (and thus on H^2) is the interval $[0, \pi]$, and there are no eigenvalues.

A more complete study of the point spectrum on Hardy spaces was done recently by A. Aleman, A. Montes and A. Sarafoleanu, for

$$H_\lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{Z},$$

(note that $H = H_1$ is not included). They find that on H^p, $1 < p < 2$,

$$\left\{ \frac{\pi}{\sin(\pi a)} : \frac{1}{2} < \mathrm{Re}(a) \leq \frac{1}{p} \right\}$$

are eigenvalues with corresponding eigenfunctions

$$g_a(z) = (1-z)^a \, {}_2F_1(a+1, a+\lambda; \lambda; z)$$

These eigenvalues disappear when $p \geq 2$. In general the spectrum of H is not known except on $l^2 \equiv H^2$.

Some generalized versions of H

Given a measure μ on $[0,1)$ we can consider the moment sequence

$$\mu_j = \int_{[0,1)} t^j \, d\mu(t),$$

and the resulting Hankel matrix

$$H_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & . \\ \mu_1 & \mu_2 & \mu_3 & . \\ \mu_2 & \mu_3 & \mu_4 & . \\ . & . & . & . \end{pmatrix}.$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on $\mathbb{D}$ we have formally

$$H_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n.$$

If μ satisfies

$$(*) \quad \mu((t,1)) = \mathrm{O}((1-t)), \quad t \to 1,$$

then $\mu_n = \mathrm{O}(1/(n+1))$ and using Hardy's inequality it follows that the power series $H_\mu(f)$ is analytic on $\mathbb{D}$ for every $f \in H^1$ and

$$H_\mu(f)(z) = \int_{[0,1)} f(t) \frac{1}{1-tz} \, d\mu(t).$$

Condition $(*)$ is known to be equivalent to that the operator $H_\mu : H^2 \to H^2$ is bounded.

Theorem[P. Galanopoulos and J.A. Pelaez, 2010]

Suppose μ satisfies $(*)$, then

1. $H_\mu : H^1 \to H^1$ is bounded if and only if

$$\mu((t,1)) \log \frac{1}{1-t} = \mathrm{O}((1-t)), \quad t \to 1,$$

2. $H_\mu : H^1 \to H^1$ is compact if and only if

$$\mu((t,1)) \log \frac{1}{1-t} = \mathrm{o}((1-t)).$$

Similar Carleson measure conditions make H_μ bounded or compact on Hardy, Bergman and other spaces.

Changing the kernel function

The integral giving H can be written
$$H(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt = \int_0^1 f(t)g'(tz)\,dt$$
where $g(z) = \log(\frac{1}{1-z})$. For any g analytic on $\mathbb{D}$ we may consider
$$H_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt,$$
and ask to find for which symbols g the induced H_g has desired properties as a transformation on function spaces.

Observe that,

1. If g is a polynomial then H_g is a finite rank operator.
2. H_g is linear in g,
$$H_{\lambda g_1 + \mu g_2} = \lambda H_{g_1} + \mu H_{g_2}, \quad \lambda, \mu \in \mathbb{C},$$
thus if X is a space o functions then the set
$$V = \{g : H_g : X \to X \text{ is bounded}\}$$
is a linear space and contains the polynomials.

We can define a norm
$$\|g\|_V = |g(0)| + \|H_g\|_{X\to X}, \quad g \in V,$$
to make $(V, \|\ \|_V)$ into a normed space. The closure
$$V_0 = \text{closure of polynomials in V}$$
is a linear subspace and $H_g : X \to X$ is a compact operator, for $g \in V_0$.

3. We have
$$\begin{aligned} H_g(f)(z) &= \sum_{n=0}^{\infty}\left((n+1)\hat{g}(n+1)\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{n+k+1}\right)z^n \\ &= \Lambda_{g'} \circ H(f)(z) \end{aligned}$$
where $\Lambda_{g'}$ is the coefficient multiplication operator by the sequence $\lambda_n = (n+1)\hat{g}(n+1)$. In particular if g has gaps, then $H_g(f)$ has the same gaps for every f.

Mean Lipschitz spaces.

If $1 \le p < \infty$ and $f \in L^p(\mathbb{T})$

$$\omega_p(f,t) = \sup_{0<h\le t} \left(\int_0^{2\pi} |f(\theta+h) - f(\theta)|^p d\theta \right)^{1/p},$$

the integral modulus of continuity of f. For $0 < \alpha \le 1$ the mean Lipschitz space $\Lambda\,(p,\alpha)$ is

$$\Lambda\,(p,\alpha) = \{f : \omega_p(f,t) = \mathrm{O}(t^\alpha),\ t \to 0\}\,.$$

$\Lambda\,(p,\alpha)$ decreases in size as either of p,α increase. If $\alpha > \frac{1}{p}$ then $\Lambda\,(p,\alpha)$ consists of continuous functions. The borderline space $\Lambda\left(p,\frac{1}{p}\right)$ contains unbounded functions, in fact

$$\log\frac{1}{1-z} \in \Lambda\left(p,\frac{1}{p}\right), \quad \text{all } p > 1,$$

and these spaces increase with p but they stay always inside $BMOA$:

$$\Lambda\left(q,\frac{1}{q}\right) \subset \Lambda\left(p,\frac{1}{p}\right) \subset BMOA, \quad 1 \le q < p < \infty,$$

Theorem.(P. Galanopoulos, D. Girela, J.A. Pelaez, and A. S. 2011)

Suppose g is analytic on $\mathbb{D}$. Then
1. If $1 < p \le 2$ then $H_g : H^p \to H^p$ is bounded if and only if $g \in \Lambda\left(p,\frac{1}{p}\right)$
2. For $2 < p < \infty$, if $H_g : H^p \to H^p$ is bounded then $g \in \Lambda\left(p,\frac{1}{p}\right)$.
If $g \in \Lambda\left(q,\frac{1}{q}\right)$ for some $q < p$ then $H_g : H^p \to H^p$ is bounded.
3. If $2 < p < \infty$. Then $H_g : A^p \to A^p$ is bounded if and only if $g \in \Lambda\left(p,\frac{1}{p}\right)$.

Continuous analogue

(Work in progress with A. Aleman and D. Vukotic)

Let $\mathbb{U} = \{z = x + iy : y > 0\}$, the upper half-plane. For $0 < p < \infty$, $H^p(\mathbb{U})$ contains the analytic functions $f : \mathbb{U} \to \mathbb{C}$ such that

$$\|f\|_p^p = \sup_{0<y<\infty} M_p(y, f) < \infty$$

where

$$M_p(y, f) = \int_{-\infty}^{\infty} |f(x + iy)|^p \, dx < \infty.$$

They are Banach spaces for $1 \leq p < \infty$. For $f \in H^p(\mathbb{U})$, the limit

$$\lim_{y \to 0} f(x + iy) = f^*(x)$$

exists a.e. on $\mathbb{R}$ and

$$\|f\|_p^p = \int_{-\infty}^{\infty} |f^*(x)|^p dx.$$

Thus $H^p(\mathbb{U})$ can be viewed as a subspace of $L^2(\mathbb{R})$.

The continuous Hilbert operator is

$$\mathcal{H}(f)(z) = \int_0^{\infty} \frac{f(\zeta)}{\zeta + z} \, d\zeta$$

Choosing the path of integration to be the straight line from 0 to ∞ passing through z and parametrizing it as

$$\gamma(s) = \frac{s}{1 - s} z, \; 0 < s < 1$$

we find

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{1 - s} f\left(\frac{s}{1 - s} z\right) \, ds$$

i.e. $\mathcal{H}$ is an average of composition operators.

Using Minkowski's inequality in its continuous form, and making a change of variable we obtain

$$\begin{aligned}
\|\mathcal{H}(f)\|_p &= \sup_{y>0}\left(\int_{-\infty}^{\infty} |\mathcal{H}(f)(x+iy)|^p\, dx\right)^{1/p} \\
&\leq \int_0^1 \left(\sup_{y>0}\int_{-\infty}^{\infty}\left|\frac{1}{1-s}f\left(\left(\frac{s}{1-s}\right)(x+iy)\right)\right|^p\, dx\right)^{1/p} ds \\
&= \int_0^1 s^{1-\frac{1}{p}}(1-s)^{-\frac{1}{p}}\left(\sup_{v>0}\int_{-\infty}^{\infty}|f(u+iv)|^p\, du\right)^{1/p} ds\,, \\
&= \left(\frac{\pi}{\sin(\frac{\pi}{p})}\right)\|f\|_p\,.
\end{aligned}$$

Thus $\|\mathcal{H}\|_{H^p(\mathbb{U})\to H^p(\mathbb{U})} \leq \frac{\pi}{\sin(\frac{\pi}{p})}$.

In order to go further we need some facts from Harmonic Analysis. Let X be a Banach space and $L(X)$ the space of bounded operators on X. View $L(X)$ as a Banach algebra, $(L(X),\circ)$.

Let $\{T_t\}_{t\in\mathbb{R}}$ be a strongly continuous group of isometries on X. For $\phi\in L^1(\mathbb{R})$ define

$$T_\phi(f) = \int_{-\infty}^{\infty}\phi(t)T_t(f)\, dt, \quad f\in X.$$

It is clear that $T_\phi\in L(X)$ and

$$\|T_\phi\|_{X\to X} \leq \int_{-\infty}^{\infty}|\phi(t)|\, dt = \|\phi\|_{L^1(\mathbb{R})}.$$

Then the map $T: L^1(\mathbb{R})\to L(X), \quad T(\phi)=T_\phi$ satisfies

$$T(\phi*\psi) = T(\phi)T(\psi)$$

for $\phi,\psi\in L^1(\mathbb{R})$ so it is a homomorphism between $(L^1(\mathbb{R}),*)$ and $(L(X),\circ)$.

With the above setting, using results from Banach algebras (mainly the work of W. Arveson) the following spectral theorem holds,

Theorem. *Let X a Banach space, $\{T_t\}$ a strongly continuous group of isometries on X with infinitesimal generator Γ. For $\phi \in L^1(\mathbb{R})$ let*

$$T_\phi(f) = \int_{-\infty}^{\infty} \phi(t) T_t(f)\, dt, \quad f \in X.$$

Then T_ϕ has spectrum

$$\sigma(T_\phi) = \overline{\widehat{\phi}(\sigma(\Gamma))}$$

where $\widehat{\phi}$ is the Fourier transform of ϕ,

$$\widehat{\phi}(s) = \int_{-\infty}^{\infty} \phi(t) e^{-ist}\, dt.$$

In our case, we start with the integral defining $\mathcal{H}$ and evaluate it again on a straight line through z, now parametrized as $\gamma(t) = e^t z$, $-\infty < t < \infty$.

$$\begin{aligned}
\mathcal{H}(f)(z) &= \int_0^\infty \frac{f(\zeta)}{\zeta + z}\, d\zeta = \int_{-\infty}^{\infty} \frac{e^t}{e^t+1} f(e^t z)\, dt \\
&= \int_{-\infty}^{\infty} \frac{e^{(1-\frac{1}{p})t}}{e^t+1}\, e^{\frac{t}{p}} f(e^t z)\, dt \\
&= \int_{-\infty}^{\infty} \phi_p(t) T_t(f)(z)\, dt,
\end{aligned}$$

where

$$T_t(f)(z) = e^{\frac{t}{p}} f(e^t z)$$

is a group of isometries of $H^p(\mathbb{U})$, and

$$\phi_p(t) = \frac{e^{(1-\frac{1}{p})t}}{e^t+1}.$$

The group $\{T_t\}$ is strongly continuous on $H^p(\mathbb{U})$. Its infinitesimal generator is the unbounded operator

$$\Gamma(f)(z) = -izf'(z) - \frac{i}{p} f(z),$$

with spectrum $\sigma(\Gamma) = \mathbb{R}$. Further $\phi_p \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \phi_p(t)\, dt = \frac{\pi}{\sin(\frac{\pi}{p})}$$

The Fourier transform of ϕ_p is found to be

$$\begin{aligned}
\widehat{\phi}_p(s) &= \int_{-\infty}^{\infty} e^{-ist}\,\frac{e^{(1-\frac{1}{p})t}}{e^t+1}\,dt \\
&= \frac{2\pi i e^{\frac{\pi i}{p}} e^{\pi s}}{e^{\frac{2\pi i}{p}} - e^{2\pi s}} = \frac{2\pi i}{e^{\frac{\pi i}{p}} e^{-\pi s} - e^{-\frac{\pi i}{p}} e^{\pi s}} \\
&= \frac{2\pi i}{e^{i(\frac{\pi}{p}+i\pi s)} - e^{-i(\frac{\pi}{p}+i\pi s)}} \\
&= \frac{\pi}{\sin(\frac{\pi}{p}+i\pi s)}
\end{aligned}$$

Applying the theorem we find that the spectrum of $\mathcal{H}$ on $H^p(\mathbb{U})$ is

$$\sigma(\mathcal{H}) = \overline{\{\frac{\pi}{\sin(\frac{\pi}{p}+i\pi s)} : s \in \mathbb{R}\}}$$

Note that taking $s = 0$ says that $\frac{\pi}{\sin(\frac{\pi}{p})}$ belongs to the spectrum so that $\|\mathcal{H}\| \geq \frac{\pi}{\sin(\frac{\pi}{p})}$. This together with the upper estimate gives

$$\|\mathcal{H}\|_p = \frac{\pi}{\sin(\frac{\pi}{p})}, \quad 1 < p < \infty.$$

Similar arguments cab be used to study $\mathcal{H}$ on Bergman and other spaces of the upper half plane.

Thanks for your attention

References

[1] A. Aleman, A. Montes-Rodriguez and A. Sarafoleanu, *The eigenfunctions of the Hilbert matrix*, Constr.Approx. 36 (2012), 353–374.

[2] V. Bozin, B. Karapetrovic, *Norm of the Hilbert matrix on Bergman spaces*, J. Funct. Anal. 274 (2018), 525–543.

[3] P. S. Bourdon, J. H. Shapiro and W. T. Sledd, *Fourier series, mean Lipschitz spaces and bounded mean oscillation*, Analysis at Urbana, **1** (1986), LMS Lecture Note Series n. 137, 81–110

[4] M. D. Choi, *Tricks or treats with the Hilbert matrix*, Amer. Math. Monthly 90 (1983), 301–312.

[5] E. Diamantopoulos and A. G. Siskakis, *Composition operators and the Hilbert matrix*, Studia Math. 140 (2000), 191–198.

[6] E. Diamantopoulos, *Hilbert matrix on Bergman spaces*, Illinois J. of Math. 48, (2004), 1067–1078.

[7] M. Dostanic, M. Jevtic and D. Vukotic, *Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type*, J. Funct. Anal. 254 (2008), 2800–2815.

[8] P. Galanopoulos and J. A. Peláez, *A Hankel matrix acting on Hardy and Bergman spaces*, Studia Math. 200 (2010), 201–220.

[9] P. Galanopoulos, D. Girela, J.A. Pelaez and A. G. Siskakis, *Generalized Hilbert operators* Ann. Acad. Sci. Fenn. Math. 39 (2014), 231–258.

[10] M. Gonzalez, *The Spectrum of the Hilbert Matrix as an Operator on l^p*, Integral Equations Operator Theory 74 (2012), 3–4.

[11] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge Mathematical Library, Reprint of the 1952 edition, Cambridge University Press, (1988).

[12] C. K. Hill *On the singly-infinite Hilbert matrix*, J. London Math. Soc. 35 (1960) 17–29.

[13] C. K. Hill *On the doubly infinite Hilbert matrix*, J. London Math. Soc. 36 (1961) 403–423.

[14] B. Hollenbeck and I. Verbitsky, *Best constants for the Riesz projection*, J. Funct. Anal. 175 (2000), 370–392.

[15] M. Jevtic, B. Karapetrovic, *Hilbert matrix on spaces of Bergman-type*, J. Math. Anal. Appl. 453 (2017), 241–254.

[16] B. Karapetrovic, *Norm of the Hilbert matrix operator on the weighted Bergman spaces*, Glasg. Math. J. 60 (2018), 513–525.

[17] B. Lanucha, M. Nowak and M. Pavlovic, *Hilbert matrix operators on spaces of analytic functions*, Ann. Acad. Sci. Fenn. Math. 37 (2012), 161–174.

[18] M. Lindstrom, S. Miihkinen, N. Wikman *Norm estimates of weighted composition operators pertaining to the Hilbert matrix* Proc. Amer. Math. Soc. 147 (2019), 2425–2435.

[19] W. Magnus, *On the spectrum of Hilbert's matrix*, Amer. J. Math. 72, (1950) 699–704.

[20] S. C. Power, *Hankel operators on Hilbert space*. Bull. London Math. Soc. 12 (1980), 422–442.

[21] M. Rosenblum *On the Hilbert matrix. I*, Proc. Amer. Math. Soc. 9 (1958) 137–140.

[22] M. Rosenblum *On the Hilbert matrix. II*, . Proc. Amer. Math. Soc. 9 (1958) 581–585.