Noncommutative Choquet theory

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Classical Choquet Theory

K is a compact convex subset of a locally convex space.
∂K is the set of extreme points of K.
$A(K)$ is the space of affine functions on K.
$S(A(K)) = \{f : A(K) \to \mathbb{C} : 1 = f(1) = \|f\|\}$ states.

Hahn-Banach extension of $f \in S(A(K))$ to $\mathrm{C}(K)$ is given by a probability measure μ on K (a representing measure for f.) s.t.

$$f(a) = \int_K a \, d\mu \quad \text{for all} \quad a \in A(K).$$

Theorem (Choquet-Bishop-de Leeuw)

Every $f \in S(A(K))$ has representing measure μ supported on ∂K.

Technical point. If K not metrizable, ∂K may not be Borel. One deals with this by asserting that μ is a Borel measure on K s.t. $\int f \, d\mu = 0$ if f is a Baire function and $f|_{\partial K} = 0$.

Example

$1 \in A = A^* \subset \mathrm{C}(X)$ function system that separates points (so $\mathrm{C}^*(A) = \mathrm{C}(X)$). Let

$$K = S(A) = \{f : A \to \mathbb{C} : f \geq 0,\ f(1) = 1\} \quad \text{state space.}$$

Theorem (Kadison (1951))

Given function system A and $K = S(A)$, then A is affinely isometrically order isomorphic to $A(K)$.
The map $A \to S(A)$ is a contravariant equivalence of categories from (Function systems with unital order homomorphisms) to (Compact convex sets with continuous affine maps).

There are many applications of Choquet theory in various areas of analysis: approximation theory, ergodic theory, group representations, direct integral theory for C*-algebras, etc.

Noncommutative Convexity

Let E be an operator space.
For any cardinal n, let $M_n(E)$ denote the $n \times n$ arrays of elements of E such that all finite subarrays are uniformly bounded.
Let κ be the minimum cardinality of a dense subset of E.

$$K = \coprod_{1 \le n \le \kappa} K_n, \quad K_n \subset M_n(E).$$

K is nc convex if it is closed under direct sums and compressions:

- $x \in K_n,\ y \in K_m \implies x \oplus y \in K_{n+m}$
- $x \in K_n,\ \alpha \in \mathcal{M}_{nm}$ isometry, $\implies \alpha^* x \alpha \in K_m$.

Equivalently,

$$x_i \in K_i,\ \alpha_i \in \mathcal{M}_{n_i,n},\ \sum_i \alpha_i^* \alpha_i = 1_n \implies \sum \alpha_i^* x_i \alpha_i \in K.$$

If E is a dual space with predual E_*, say that K is compact if each K_n is compact in the weak-$*$ topology.

Essential point: we need the infinite cardinals!
K is determined by $\coprod_{n<\infty} K_n$ but need higher levels. This is important when considering nc functions, and extreme points.

An important nc convex set is

$$\mathcal{M} = \coprod_{1 \leq n \leq \kappa} M_n \quad \text{where } M_n = \mathcal{B}(H_n).$$

$\theta : K \to \Delta$ is nc affine if

1. $\theta(K_n) \subset \Delta_n$ (graded)
2. $\theta(\sum \oplus x_i) = \sum \oplus \theta(x_i)$ (respects direct sums)
3. $\theta(\alpha^* x \alpha) = \alpha^* \theta(x) \alpha$ for α isometry. (respects compressions)

Definition

$\mathrm{A}(K)$ is the set of continuous nc affine functions $\theta : K \to \mathcal{M}$.
$\mathrm{BA}(K)$ is the set of bounded nc affine functions.

Example

$1 \in A = A^* \subset \mathcal{B}(H)$ operator system.

$$K = S(A) = \coprod_{1 \le n \le \kappa} \mathrm{UCP}(A, \mathcal{B}(H_n))$$

where $\dim H_n = n$. κ is a cardinal large enough for all cyclic representations of $\mathrm{C}^*(A)$.

Theorem (Webster-Winkler 1999)

$A \simeq \mathrm{A}(S(A))$ via $a \to \hat{a}$, $\hat{a}(x) = x(a)$. The map from A to $S(A)$ is a contravariant equivalence of categories from (Operator systems with unital complete order homomorphisms) to (Compact nc convex sets with continuous nc affine maps).

Noncommutative Functions

An nc function: $f : K \to \mathcal{M}$ is graded, respects $\oplus$, $\mathcal{U}$-equivariant:

1. $f(K_n) \subset \mathcal{M}_n$
2. $f(\sum \oplus x_i) = \sum \oplus f(x_i)$
3. $f(uxu^*) = uf(x)u^*$ for $x \in K_n$, $u \in \mathcal{M}_n$ unitary.

The set $\mathrm{B}(K)$ of all bounded nc functions is a C*-algebra.
Let $\mathrm{C}(K) := \mathrm{C}^*(A(K))$ be the 'continuous' nc functions.

Theorem (Takesaki-Bichteler 1969)

C-algebra C, then $C \simeq \mathrm{C}(\mathrm{Rep}(C, H))$ and $C^{**} \simeq \mathrm{B}(\mathrm{Rep}(C, H))$.*

If A is an operator system, Kirchberg-Wassermann defined $\mathrm{C}^*_{\max}(A)$ as the universal C*-algebra s.t. every u.c.p. map $x \in K$ extends to a $*$-repn. δ_x of $\mathrm{C}^*_{\max}(A)$.

It has the universal property that if $j : A \to \mathcal{B} = C^*(jA)$ is a u.c.p.complete isometry (order iso), then there is a (unique) $*$-homomorphism $\pi : \mathrm{C}^*_{\max}(A) \to \mathcal{B}$ s.t. $\pi|_A = j$.

Theorem

$\mathrm{C}^*_{\max}(A) \simeq \mathrm{C}(K)$ and $\mathrm{C}^*_{\max}(A)^{**} \simeq \mathrm{B}(K)$ and $\mathrm{A}(K)^{**} \simeq \mathrm{BA}(K)$.

Noncommutative Convexity Again

Each $x \in K_n$ determines the 'point evaluation' δ_x in $\mathrm{Rep}(C(K))$, and $\delta_x|_{\mathrm{A}(K)} = x$. Conversely every repn. of $C(K)$ has this form.

A representing map for $x \in K_n$ is $\mu \in \mathrm{UCP}(\mathrm{C}(K), \mathcal{M}_n(\mathcal{M}))$ such that $\mu|_{\mathrm{A}(K)} = x$; and x is the barycenter of μ.
By Stinespring, $\mu = \alpha^* \delta_y \alpha$ for $y \in K_m$ and isometry $\alpha \in \mathcal{M}_{mn}$.
Say (y, α) represents x and y dilates x.

x has unique representing map iff δ_x is only u.c.p.extension of x.
x is maximal if (y, α) represents $x \implies y = x \oplus z$.

Proposition

x has a unique representing map iff x is maximal.

Theorem (Dritschel-McCullough 2005)

$x \in K$ has a maximal dilation y.

x is pure if whenever $x = \sum \alpha_i^* x_i \alpha_i$, then

- $\alpha_i^* x_i \alpha_i \in \mathbb{R}x$, and
- α_i is a positive scalar multiple of an isometry β_i satisfying $\beta_i^* x_i \beta_i = x$.

x is extreme if whenever $x = \sum \alpha_i^* x_i \alpha_i$, then

- $\alpha_i^* x_i \alpha_i \in \mathbb{R}x$, and
- $x_i \simeq x \oplus y_i$ w.r.t. range of α_i.

Theorem

x is extreme iff x is pure and maximal.

Arveson's C*-envelope or $(\mathrm{C}^*_{\min}(A), i)$ of an operator system A with $i : A \to \mathrm{C}^*_{\min}(A) = C^*(iA)$ u.c.p. order iso. has the universal property that if $j : A \to \mathcal{B} = C^*(jA)$ is a u.c.p.order iso, then there is a (unique) $*$-homomorphism $\pi : \mathcal{B} \to \mathrm{C}^*_{\min}(A)$ s.t. $\pi|_{jA} = i$.

Theorem (DK 2015)

Every pure point in K dilates to an extreme point.
$\sigma = \sum^{\oplus}_{x \in \partial K} \delta_x$ is a complete order iso on $A(K)$ and
$\sigma(C(K)) = \mathrm{C}^*_{\min}(A(K))$.

Remark

When $K = S(A)$ for an operator system A, then x is pure if and only if δ_x is irreducible. Thus x extreme if and only if δ_x is a boundary representation for A.

Theorem (NC Krein-Milman theorem)

K is the closed nc convex hull of ∂K.

Theorem (NC Milman converse)

1. If $X \subset K$ closed
2. $x \in X_n$ and isometry $\alpha \in \mathcal{M}_{mn}$ implies that $\alpha^* x \alpha \in X$
3. and $\overline{\mathrm{ncconv}(X)} = K$

then $X \supset \partial K$.

Webster-Winkler have a Krein-Milman Theorem for matrix convex sets. Its weakness is that the notion of extreme point only considers combinations of elements on the same or lower levels. As a result, this is a cumbersome notion with many 'extreme' points which are not extreme in our sense. However many (and sometimes all) of our extreme points live at infinite levels.

Example

Let $K = \coprod_n K_n$ denote the noncommutative complex unit ball, i.e. the nc convex set of row contractions in M_n^d

$$K_n = \left\{\alpha = \begin{bmatrix}\alpha_1 & \dots & \alpha_d\end{bmatrix} : \alpha_i \in M_n,\ \left\|\begin{bmatrix}\alpha_1 & \dots & \alpha_d\end{bmatrix}\right\| \le 1\right\}.$$

Then K is a compact nc convex set. The extreme points are precisely the row unitaries:

$$(\partial K)_n = \{\alpha \in K_n : \alpha^*\alpha = 1_d \otimes 1_n \text{ and } \alpha\alpha^* = 1_n\}.$$

Let $\mathcal{O}_n$ be the Cuntz algebra generated by isometries $s_1, \dots, s_n$ such that $\sum s_i s_i^* = 1$; i.e. $[s_1 \ \dots \ s_n]$ is a row unitary.
The operator system $A = \operatorname{span}\{1, s_1, s_1^*, \dots, s_n, s_n^*\}$ generates $\mathcal{O}_n$.
A u.c.p. map x is determined by $[x(s_1), \dots, x(s_n)]$, an arbitrary row contraction. So $S(A) = K$. The extreme points arise from the irreducible representations of $\mathcal{O}_n$ via $\pi \to \begin{bmatrix}\pi(s_1) & \dots & \pi(s_n)\end{bmatrix}$.

NC convexity versus matrix convexity

If $K = \coprod_n K_n$ is an nc convex set, then the finite part $\coprod_{n\in\mathbb{N}} K_n$ is a matrix convex set. Conversely, a matrix convex set determines a unique nc convex set. Also an nc affine function is determined by its restriction to the finite portion. Thus categorically they are equivalent.

However the finite part need not have any extreme points in our sense. (e.g. the Cuntz system has none.)

There are many nc functions that are not determined by their finite part. For example, for the Cuntz system, the characteristic function of the extreme points vanishes on all finite levels.

Noncommutative Convex Functions

Classically, convex functions play a central role.
Note that a scalar function $f \in C(X)$ is convex iff

$$\{(x,t) : x \in X,\ t \geq f(x)\} \quad \text{is convex}$$

and is l.s.c. if this set is closed.

Definition

Say $f \in \mathrm{C}(K)^{sa}$ is nc convex if its epigraph

$$\mathrm{Epi}(f) = \coprod \{(x,\alpha) : x \in K,\ \alpha \geq f(x)\} \subset \coprod K_n \times M_n$$

is nc convex. It is l.s.c. if $\mathrm{Epi}(f)$ is closed.

Equivalently, f is nc convex if

$$f(\alpha^* x \alpha) \leq \alpha^* f(x) \alpha \quad \text{for} \quad x \in K,\ \alpha \ \text{ isometry.}$$

Example

Let $I = [a, b] \subset \mathbb{R}$.
Let $K = \coprod K_n$ where $K_n = \{\alpha \in M_m^{sa} : \sigma(\alpha) \subset I\}$.

$f \in \mathrm{C}(K)$ are in bijective correspondence with $C(I)$ by $f \to f|_{K_1}$ (usual functional calculus).

f is nc convex if and only if f is operator convex in the classical sense: for $0 \le t \le 1$, $\alpha, \beta \in K_n$, $n \ge 1$

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta).$$

This follows from the Hansen-Pedersen inequality. The proof is simple and natural in our framework, and holds in general.

Theorem

If $f \in \mathrm{C}(K)$ and $f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta)$ for all $0 \le t \le 1$, $\alpha, \beta \in K_n$, $n \ge 1$, then f is nc convex.

Classically, if $f \in \mathrm{C}(K)$, the convex (lower) envelope is

$$\check{f}(x) = \sup\{a(x) : a \in \mathrm{A}(K),\ a \le f\}.$$

Choquet theory is based crucially on convex functions.

However in the nc situation, one cannot take the supremum of operators, so we use multivalued functions.

A multivalued s.a. nc function is upward directed:
if $F : K \to \mathcal{M}$, then $F(x) = F(x) + \mathcal{M}_p^+$ for $x \in K_p$.

Note that if $f \in \mathrm{C}(K)^{sa}$ is nc convex and l.s.c., then $\mathrm{Epi}(f)$ is the graph of a multivalued nc function.

The nc convex envelope of $f \in \mathrm{C}(K)$ is the multivalued function $\bar{f}$

$$\mathrm{Graph}(\bar{f}) = \overline{\mathrm{ncconv}(\mathrm{Epi}(f))}.$$

In classical Choquet theory, a routine separation theorem shows that the largest convex function less than f coincides with the maximum of all affine functions smaller than f. This is not easy at all in the non-commutative setting. But it is still true.

Theorem

If $f \in \mathcal{M}_n(\mathrm{C}(K))$ and $x \in K_p$,

$$\overline{f}(x) = \bigcap_{m} \bigcap_{a \leq 1_m \otimes f} \{\alpha \in (\mathcal{M}_n(\mathcal{M}_p))_{sa} : a(x) \leq 1_m \otimes \alpha\}.$$

Classically, $\check{f}(x) = \inf_{\mu \sim x} \mu(f)$ and the inf is attained.

Theorem

For $f \in M_n(\mathrm{C}(K))^{sa}$,

$$\overline{f}(x) = \bigcup_{\mu : \mu|_{\mathrm{A}(K)} = x} [\mu(f), \infty).$$

Noncommutative Orders on UCP maps

Classical Choquet order: $\mu \prec_c \nu$ if $\mu(f) \le \nu(f)$ for all f convex. Relates measures with same barycenter x. Intuitively $\mu \prec_c \nu$ means that ν is supported closer to the extreme boundary.

Definition

Nc Choquet order: $\mu \prec_c \nu$ if $\mu(f) \le \nu(f)$ for all $f \in M_n(\mathrm{C}(X))$ nc convex matrix valued functions.

Note that $\mu \prec_c \nu$ means that they have the same barycenter. If a is nc affine, then $\pm a$ are nc convex, so that $\pm\mu(a) \le \pm\nu(a)$. Hence $\mu|_{\mathrm{A}(K)} = \nu|_{\mathrm{A}(K)}$.

Proposition

$\prec_c$ is a partial order.

If a is s.a. and nc affine, and f is operator convex, then $f(a)$ is nc convex. There are enough such functions to separate points.

Definition

Dilation order: $\mu \prec_d \nu$ if

1. (x, α) represents μ; i.e. $\mu = \alpha^* \delta_x \alpha$
2. (y, β) represents ν; i.e. $\nu = \beta^* \delta_y \beta$, and
3. $\exists_\gamma$ s.t. $x = \gamma^* y \gamma$ and $\beta = \gamma\alpha$; i.e. (y, β) dilates (x, α).

This dilation order comes out of comparison of the Stinespring dilations of μ and ν, and doesn't have a classical parallel.

Proposition

If μ has minimal representation (x, α), then μ is maximal in $\prec_d$ iff x has a unique representing map.

The following result is crucial.

Theorem

$\mu \prec_c \nu$ if and only if $\mu \prec_d \nu$.

Noncommutative Choquet-Bishop-de Leeuw theorems

The classical Bishop-de Leeuw was established by showing that x always has a representing measure which is maximal in the Bishop-de Leeuw order (now replaced with the Choquet order).

The Baire-Pedersen algebra $\mathfrak{B}(K)$ is the monotone sequential completion of $\mathrm{C}(K)$ in $\mathrm{B}(K)$.
This algebra was studied by Pedersen for arbitrary C*-algebras.
In the case of $C(X)$, it produces the Baire functions.

Theorem (nc Choquet-Bishop-de Leeuw)

If $x \in K$, then there is a dilation maximal μ representing x.
If $f \in \mathfrak{B}(K)$ with $f|_{\partial K} = 0$, then $\mu(f) = 0$.

In the separable case, Choquet showed (without using Choquet order!) that there is a bona fide integral representation for x over the extreme boundary.

Theorem (nc Choquet representation theorem)

If A is separable and $x \in K$, there is an nc probability measure λ on ∂K that represents x. i.e.

$$a(x) = \int_{\partial K} a \, d\lambda \quad \text{for} \quad a \in \mathrm{A}(K).$$

Here λ is a measure with values in the cp maps.
The integration theory for cp maps was developed by Fujimoto.
We use the disintegration of representations of C*-algebras into an integral of irreducibles as presented in Takesaki (which is based on Choquet theory).

A new proof of existence of extreme points (boundary reps).

The Dritschel-McCullough result: if $x \in K_n$, an easy Zorn's lemma argument yields a maximal u.c.p. map $\mu : C(K) \to \mathcal{M}_n$ with barycentre x.

DK: every pure point $x \in K_n$ has an extreme dilation.

In $L = S(C(K))$, $F = \{\mu : \mu|_{A(K)} = x\}$ is a hereditary nc face; i.e. if $\mu \in F$ and $\mu \prec_d \nu$, then $\nu \in F$.
Use Zorn to obtain a minimal herediatry nc face F_0.
Show $F_0 = \{\mu_0\}$. Hereditary, so μ is maximal.
If (y, β) minimal representation of μ_0, then δ_y is irreducible, maximal, hence a boundary rep. and y is an extreme dilation of x.

The end.