Amenability, proximality and higher order syndeticity

Matthew Kennedy
joint work with Sven Raum and Guy Salomon

University of Waterloo, Waterloo, Canada

April 16, 2021

Proximality and strong proximality

Let G be a topological group. A G-**flow** is a compact Hausdorff space X with a continuous action $G \curvearrowright X$. Get affine G-flow $G \curvearrowright \mathrm{Prob}(X)$.

Definition

1. A G-flow X is *proximal* if $\overline{G\mu} \cap X \neq \emptyset$ for all finitely supported $\mu \in \mathrm{Prob}(X)$.
2. A G-flow X is *strongly proximal* if $\overline{G\mu} \cap X \neq \emptyset$ for all $\mu \in \mathrm{Prob}(X)$.

Note: we have identified X with $\partial\,\mathrm{Prob}(X)$.

A G-flow X is **minimal** if $\overline{Gx} = X$ for all $x \in X$.

Theorem (Furstenberg 1973, Glasner 1976)

There is a unique universal minimal proximal flow $\partial_p G$ and a unique universal minimal strongly proximal flow $\partial_{sp} G$. For every minimal proximal flow X there is a surjective G-map $\partial_p G \to X$. Similarly for $\partial_{sp} G$.

Statements about specific flows translate to statements about universal flows. E.g. G has a free proximal flow iff $\partial_p G$ is free.

Strong proximality

Example

Let $\Gamma \leq G$ be a lattice in a locally compact group G. Then $G \curvearrowright \Gamma$ is minimal and strongly proximal. Typically $\Gamma \neq \partial_{sp} G$.

Example

Let G be a non-elementary hyperbolic group (e.g. $G = \mathbb{F}_n$ for $n \geq 2$). The hyperbolic boundary ∂G is minimal and strongly proximal. It is also topologically free (i.e. for $g \in G \setminus \{e\}$, $\mathrm{Fix}(g)$ has empty interior). Also, $\partial G \neq \partial_{sp} G$.

Theorem (Furstenberg 1973)

The group G is non-amenable iff $\partial_{sp} G$ is non-trivial.

Proof.

For a minimal G-flow X, choose an irreducible affine G-subflow $K \subseteq \mathrm{Prob}(X)$. Then $\overline{\partial K}$ is a strongly proximal G-flow and $\overline{\partial K}$ is trivial iff K is trivial iff there is an invariant measure in $\mathrm{Prob}(X)$. □

Strong proximality

Example (Connected groups)

For locally compact G with $P \leq G$ amenable and co-compact,

$$\partial_{sp} G = G/Q \quad \text{for} \quad \text{closed } Q \geq P.$$

For G connected semi-simple real Lie group with finite center and no compact factors,

$$\partial_{sp} G = G/P \quad \text{for} \quad G = KAN, \ P = AN.$$

Example (Discrete groups)

For non-amenable discrete G, $\partial_{sp} G$ is non-trivial and extremally disconnected (Stone space!). Hence non-metrizable.

Strong proximality

Theorem (K-Kalantar 2017)

For discrete G, the reduced C-algebra $\mathrm{C}^*_\lambda G$ is simple iff G has a topologically free strongly proximal flow.*

Say $H \le G$ is **confined** if

$$\mathbf{1} \notin \overline{\{gHg^{-1} : g \in G\}} \subseteq 2^G.$$

Equivalent to existence of finite $F \subseteq G \setminus \{e\}$ such that

$$gHg^{-1} \cap F \neq \emptyset \quad \text{for all} \quad g \in G.$$

Theorem (K 2018)

The reduced C-algebra $\mathrm{C}^*_\lambda G$ is simple iff G has no amenable confined subgroups.*

Proximality

Definition (Glasner 1976)

A group G is **strongly amenable** if every minimal proximal flow is trivial.

Note: Equivalent to $\partial_p G$ being trivial. Strongly amenable implies amenable.

Proposition

Discrete groups with no non-trivial ICC quotient (i.e. FC-hypercentral) are strongly amenable.

Proof.

Suffices to show $G \curvearrowright X$ is faithful, minimal and proximal implies G is ICC.

Suppose otherwise $G \in G \setminus \{e\}$ has a finite conjugacy class. The centralizer $C_G(g)$ has finite index, so $C_G(g) \curvearrowright X$ is also minimal and proximal.

For $x \in X$, there is $h_i \in C_G(g)$ with $\lim h_i x = \lim h_i g x = x$. Then $gx = \lim g h_i x = \lim h_i g x = x$, contradicting faithfulness. $\square$

Proximality

Problem: Characterize strongly amenable groups. Note that strongly amenable implies amenable (as name suggests).

Theorem (Frisch-Tamuz-Vahidi Ferdowsi 2019)

The group G is strongly amenable if and only if it has no non-trivial ICC quotient.

Sketch of proof: Equivalent to constructing minimal proximal flow for $ICC\ G$. Construction is "probabilistic." For each n, construct a dense open family of $\frac{1}{n}$-minimal and $\frac{1}{n}$ -proximal subshifts of 2^G. Then take intersection.

Theorem (Glasner-Tsankov-Weiss-Zucker 2019)

The von Neumann algebra LG is a factor iff $\partial_p G$ is free.

Key point: If $\partial_p G$ is non-trivial then it is free. This is not true for $\partial_{sp} G$. (Reminiscent of the fact that LG has a unique trace iff LG is a factor, but $C^*_\lambda G$ can have a unique trace without being simple.)

Injectivity

A G-C*-algebra B is G-**injective** if for every inclusion of G-C*-algebras $A \subseteq B$ and equivariant ucp map $\phi : A \to C$, there is an equivariant extension $\tilde{\phi} : B \to C$.

Theorem (Kalantar-K 2017)

The C*-algebra $C(\partial_{sp} G)$ is G-injective.

Theorem (K-Raum-Salomon 2020)

The C*-algebra $C(\partial_p G)$ is G-injective.

By results of Hamana and Gleason, $\partial_{sp} G$ and $\partial_p G$ are extremally disconnected. The algebras $C(\partial_{sp} G)$ and $C(\partial_p G)$ are generated by their Boolean algebras of projections.

Correspondence

Affinely "G-projective" flows correspond to translation invariant Boolean subalgebras of subsets of G.

Goal: Identify these Boolean algebras, thereby giving "concrete" descriptions of $\partial_p G$ and $\partial_{sp} G$.

Higher order syndeticity

Definition

A subset $A \subseteq G$ is **completely syndetic** if for every n there is finite $F \subseteq G$ such that $FA^n = G^n$.

Note: Syndetic is the case $n = 1$.

Definition

A subset $A \subseteq G$ is **strongly completely syndetic** if for every $\epsilon > 0$ there is finite $F \subseteq G$ such that for every finite multiset $K \subseteq G$,

$$\sup_{f \in F} \frac{|fK \cap A|}{|K|} \geq 1 - \epsilon.$$

Note: Says there is finite $F \subseteq G$ such that $FA^n \approx G^n$ for every n.

Theorem (KRS 2020)

The universal minimal (strongly) proximal flow is isomorphic to the spectrum of any invariant Boolean algebra of (strongly) completely syndetic subsets of G.

Characterizations of complete syndeticity

Proposition (KRS 2020)

Let X be a proximal flow. For open $U \subseteq X$ and $x \in X$, the return set $U_x = \{g \in G : gx \in U\}$ is completely syndetic.

Theorem (KRS 2020)

The following are equivalent for $A \subseteq G$:

1. The set A is completely syndetic, i.e. for each n, there is finite $F \subseteq G$ such that $FA^n = G^n$.
2. For every n and every mean m on G^n that is invariant under left translation by G, $m(A^n) > 0$
3. The closur $\overline{A} \subseteq \beta G$ contains a closed right ideal of βG.

Similar characterizations hold for strongly completely syndetic subsets.

Examples of completely syndetic subsets

Example

A subset $A \subseteq \mathbb{Z}$ is syndetic if and only if it has "bounded gaps," meaning there is $k \in \mathbb{N}$ such that for all $a \in \mathbb{Z}$,

$$\{a, a+1, \ldots, a+k\} \cap A \neq \emptyset.$$

Example

A subset $A \subseteq Z$ is completely syndetic if and only if for every n, A^n has "bounded diagonal gaps," meaning there is $k \in \mathbb{N}$ such that for any $(a_1, \ldots, a_n) \in \mathbb{Z}^n$,

$$\{(a_1, \ldots, a_n), (a_1 + 1, \ldots, a_n + 1), \ldots, (a_1 + k, \ldots, a_n + k)\} \cap A^n \neq \emptyset.$$

Fact: The group $\mathbb{Z}$ does not contain disjoint completely syndetic subsets.

Example 1 (the integers)

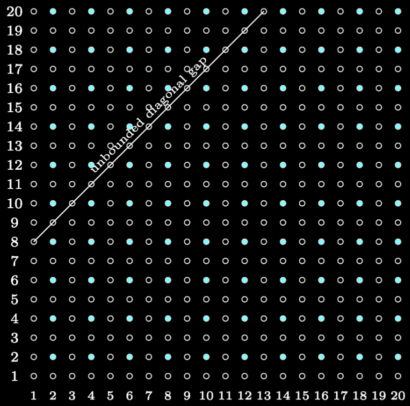

FIGURE 1. The subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$ is syndetic but not 2-syndetic since there are arbitrarily long diagonal segments in $\mathbb{Z} \times \mathbb{Z}$ that do not intersect $2\mathbb{Z} \times 2\mathbb{Z}$.

Example 2 (the integers)

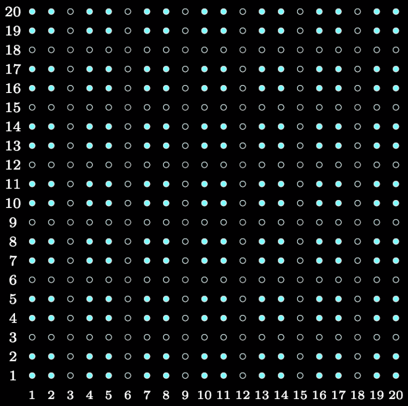

FIGURE 2. The subset $\mathbb{Z} \setminus 3\mathbb{Z} \subseteq \mathbb{Z}$ is 2-syndetic but not 3-syndetic since for $k \in \mathbb{N}$, every element in the set $\{(1,2,3),(2,3,4),(4,5,6),\ldots,(1+k,2+k,3+k)\}$ has an entry that is a multiple of 3, implying that the set does not intersect A^3.

Example 3 (the integers)

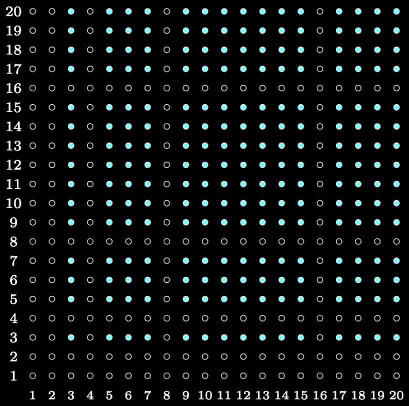

FIGURE 3. The complement of the set of powers of 2 in $\mathbb{Z}$ is completely syndetic, and in particular is 2-syndetic.

Example

Example

Consider the free group $\mathbb{F}_2 = \langle a, b \rangle$. For $w \in \mathbb{F}_2$, let

$$B_w = \{g \in G : g = wg' \text{ in reduced form}\}.$$

Can show by hand that B_a and B_b are strongly completely syndetic. Alternatively, B_a is the return set $B_a = U_{a^\infty}$ where U is the set of infinite reduced words beginning with a in the hyperbolic boundary $\partial\mathbb{F}_2$.

Characterization of (strong) amenability

Theorem (KRS 2020)

A discrete group G is not strongly amenable if and only if there is a proper normal subgroup $H \leq G$ such that for every finite subset $F \subsetneq G \setminus H$, there is a completely syndetic subset $A \subseteq G$ satisfying $FA \cap A = \emptyset$.

Theorem (KRS 2020)

A discrete group G is not amenable if and only if there is a subset $A \subseteq G$ such that both A and A^c are strongly completely syndetic.

Note: Does not seem easy to derive from existing criteria (e.g. Følner condition, paradoxicality, etc).

Upshot: Measures the difference between strong amenability and amenability.

“Coloring” criterion for strong amenability

For finite $F \subseteq G$, an F-**coloring** of G is a pair (K, k) consisting of finite colors $K \subseteq G$ and $k : 2^G_2 \to K$ such that

$$Fk(g_1, g_2)\{g_1, g_2\} \cap k(h_1, h_2)\{h_1, h_2\} = \emptyset \quad \text{for all} \quad g_1, g_2, h_1, h_2 \in G.$$

Here 2^G_2 denotes the the subsets of G of size 2.

Theorem (KRS 2020)

The group G is not strongly amenable if and only if for every finite $F \subseteq G$, G has an F-coloring.

Observation: Proof of FTV can be translated to give a construction of F-colorings.

Symmetric higher order syndeticity

Definition

A subset $A \subseteq G$ is **symmetrically (completely, strongly completely) syndetic** if for any finite subsets $F_1, F_2 \subseteq G$, the set

$$(\cap_{f_1 \in F_1} f_1 A) \cap (\cap_{f_2 \in F_2} f_2 A^c)$$

is either (completely, strongly completely) syndetic or empty.

Fact

A subset $A \subseteq G$ is symmetrically (completely, strongly completely) syndetic iff it generates a non-trivial Boolean algebra of (completely, strongly completely) syndetic subsets.

Theorem

A subset $A \subseteq G$ is symmetrically (completely, strongly completely) syndetic iff for every pair of finite subsets $F_1 \subseteq A$ and $F_2 \subseteq A^c$, the set

$$(\cap_{f_1 \in F_1} f_1^{-1} A) \cap (\cap_{f_2 \in F_2} f_2^{-1} A^c)$$

is (completely, strongly completely) syndetic.

Second characterization of (strong) amenability

Theorem (KRS 2020)

A discrete group G is not strongly amenable if and only if it contains a symetrically completely syndetic subset.

Theorem (KRS 2020)

A discrete group G is not amenable if and only if it contains a symetrically strongly completely syndetic subset.

Dense Orbit Sets

Definition

A subset $A \subseteq G$ is a **dense orbit set** if Ax is dense in X for every minimal flow X and every point $x \in X$.

Problem (Glasner-Tsankov-Weiss-Zucker)

Characterize dense orbit sets.

Theorem (KRS 2020)

A subset $A \subseteq G$ is a dense orbit set iff A^c does not contain a symmetrically syndetic subset.

Problem

Problem

Let G be a discrete non-trivial ICC group (e.g. S_∞). Give a direct proof using the factoriality of LG that G is not strongly amenable.

Frisch, Hartman, Tamuz and Vahidi Ferdowi showed in 2019 that discrete G is Choquet-Deny (i.e. has no non-trivial Poisson boundary) iff G has a non-trivial ICC quotient.

Problem

Is there a direct proof that G is Choquet-Deny iff G is strongly amenable?

Thanks!