Isoperimetric constants of metric probability spaces

Seminar on Functional Analysis and Operator Algebras

December 4, 2020

- Let (X, d, μ) be a metric probability space. The concentration function of X is defined on $(0, \infty)$ by

$$\alpha_\mu(t) := \sup\{1 - \mu(A_t) : \mu(A) \geqslant 1/2\}$$

where the supremum runs over all sets A in the Borel σ-algebra $\mathcal{B}(X)$ with $\mu(A) \geqslant 1/2$, and where $A_t = \{x : d(x, A) < t\}$ is the t-extension of A.

- We say that μ has exponential concentration on (X, d) if there exist constants $C, c > 0$ such that, for every $t > 0$,

$$\alpha_\mu(t) \leqslant Ce^{-ct}.$$

- Recall that a function $f : (X, d) \to \mathbb{R}$ is called Lipschitz if there exists $\sigma \geqslant 0$ such that $|f(x) - f(y)| \leqslant \sigma d(x, y)$ for all $x, y \in X$, and the smallest such constant σ is denoted by $\|f\|_{\mathrm{Lip}}$.
- We say that f is locally Lipschitz if for every $x \in X$ there exists a neighborhood U_x of x such that $f|_{U_x}$ is Lipschitz. For every locally Lipschitz function f we define (in the continuous case)

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Poincaré inequality and concentration

- We say that μ satisfies a Poincaré inequality with constant θ if

$$\mathrm{Var}_\mu(f) \leqslant \theta^2 \int |\nabla f|^2 d\mu$$

for every locally Lipschitz function $f : X \to \mathbb{R}$, where

$$\mathrm{Var}_\mu(f) = \mathbb{E}_\mu \left(f - \mathbb{E}_\mu(f)\right)^2 = \mathbb{E}_\mu(f^2) - (\mathbb{E}_\mu(f))^2.$$

Theorem (Gromov-Milman)

Let (X, d, μ) be a metric probability space. If μ satisfies a Poincaré inequality with constant θ, then μ has exponential concentration. More precisely,

$$\alpha_\mu(t) \leqslant \exp\Big(- \frac{t}{3\theta}\Big).$$

- We present an argument that uses the notion of the expansion coefficient of μ.
- This is defined for every $\varepsilon > 0$ as follows:

$$\mathrm{Exp}_\mu(\varepsilon) = \sup\{s \geqslant 1 : \mu(B_\varepsilon) \geqslant s\mu(B) \text{ for all } B \in \mathcal{B}(X) \text{ with } \mu(B_\varepsilon) \leqslant 1/2\}.$$

Theorem

Assume that for some $\varepsilon > 0$ we have $\mathrm{Exp}_\mu(\varepsilon) \geqslant s > 1$. Then, for every $t > 0$ we have $\alpha_\mu(t) \leqslant \frac{s}{2} s^{-t/\varepsilon}$.

- Let $A \subseteq X$ with $\mu(A) \geqslant \frac{1}{2}$ and let $t > 0$. There exists $k \geqslant 0$ such that $k\varepsilon \leqslant t < (k+1)\varepsilon$. Setting $B_0 = X \setminus A$ and $B_j = X \setminus A_{j\varepsilon}$, for every $1 \leqslant j \leqslant k$ we check that $(B_j)_\varepsilon \subseteq B_{j-1} \subseteq X \setminus A$.
- Applying the definition of the expansion coefficient to B_j (as $\mu(B_j) \leqslant 1/2$) and the assumption that $\mathrm{Exp}_\mu(\varepsilon) \geqslant s$ we get

$$\mu(B_j) = \mu(X \setminus A_{j\varepsilon}) \leqslant \frac{1}{s}\mu(X \setminus A_{(j-1)\varepsilon}) = \frac{1}{s}\mu(B_{j-1}),$$

  for all $1 \leqslant j \leqslant k$.
- Then, we have

$$\mu(X \setminus A_t) \leqslant \mu(X \setminus A_{k\varepsilon}) \leqslant \frac{1}{s}\mu(X \setminus A_{(k-1)\varepsilon}) \leqslant \frac{1}{s^2}\mu(X \setminus A_{(k-2)\varepsilon})$$
$$\leqslant \cdots \leqslant \frac{1}{s^k}\mu(X \setminus A) \leqslant \frac{1}{2}s^{-k} \leqslant \frac{1}{2}s^{-(\frac{t}{\varepsilon}-1)}$$

  where the last inequality follows from $t < (k+1)\varepsilon$.

Theorem (Gromov-Milman)

Let (X, d, μ) be a metric probability space. If μ satisfies a Poincaré inequality with constant θ, then μ has exponential concentration. More precisely, $\alpha_\mu(t) \leqslant \exp\big(-t/(3\theta)\big)$.

- For the proof of the Gromov-Milman theorem we shall show that if μ satisfies a Poincaré inequality with constant θ then $\mathrm{Exp}_\mu(\sqrt{2}\theta) \geqslant 2$.
- Let $\sqrt{2}\theta = \varepsilon > 0$ and consider $B \subseteq X$ such that $A = X \setminus B_\varepsilon$ satisfies $\mu(A) \geqslant 1/2$. We set $a = \mu(A)$, $b = \mu(B)$. Note that $d(A, B) \geqslant \varepsilon$.
- Define $f : X \to \mathbb{R}$ by $f(x) = \frac{1}{a} - \frac{1}{\varepsilon}\left(\frac{1}{a} + \frac{1}{b}\right)\min\{\varepsilon, d(x, A)\}$.
- Then, $f(x) = 1/a$ on A, $f(x) = -1/b$ on B and

$$|\nabla f|(x) \leqslant \frac{1}{\varepsilon}\left(\frac{1}{a} + \frac{1}{b}\right)$$

for all $x \in X$, while $|\nabla f|(x) = 0$ on a set of measure $a + b$.
- Consequently,

$$\int |\nabla f|^2 d\mu \leqslant \frac{1}{\varepsilon^2}\left(\frac{1}{a} + \frac{1}{b}\right)^2 (1 - a - b).$$

Poincaré inequality and concentration

- On the other hand, if $m=\mathbb{E}_\mu(f)$ we have

$$\mathrm{Var}_\mu(f)\geqslant \int_A (f-m)^2 d\mu+\int_B (f-m)^2 d\mu \geqslant a\left(\frac{1}{a}-m\right)^2+b\left(-\frac{1}{b}-m\right)^2\geqslant \frac{1}{a}+\frac{1}{b}.$$

- From the Poincaré inequality we get

$$\left(\frac{1}{a}+\frac{1}{b}\right)\leqslant \frac{\theta^2}{\varepsilon^2}\left(\frac{1}{a}+\frac{1}{b}\right)^2(1-a-b),$$

and hence $\frac{\varepsilon^2}{\theta^2}\leqslant \frac{a+b}{ab}(1-a-b)\leqslant \frac{1-a-b}{ab}=\frac{1-a}{ab}-\frac{1}{a}$.

- Solving for b we have

$$b\leqslant \frac{1-a}{a}\cdot\frac{1}{\frac{1}{a}+\frac{\varepsilon^2}{\theta^2}}=\frac{1-a}{1+\frac{a\varepsilon^2}{\theta^2}}\leqslant \frac{1-a}{1+\frac{\varepsilon^2}{2\theta^2}}.$$

- Having chosen $\varepsilon=\sqrt{2}\theta$, this implies

$$\mu(B)\leqslant \frac{1}{2}\mu(B_\varepsilon),$$

as claimed. Since B was arbitrary, we conclude that $\mathrm{Exp}_\mu(\sqrt{2}\theta)\geqslant 2$.

- Then,

$$\alpha_\mu(t)\leqslant \exp\left(-\frac{\ln 2}{\sqrt{2}\theta}t\right)\leqslant \exp\left(-\frac{t}{3\theta}\right).$$

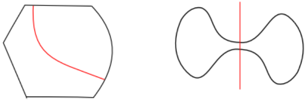
Cheeger constant

- Let μ be a Borel probability measure on $\mathbb{R}^n$. For every Borel subset A of $\mathbb{R}^n$, the Minkowski content of A with respect to μ is defined as

$$\mu^+(A) = \liminf_{t\to 0^+} \frac{\mu(A_t) - \mu(A)}{t}.$$

- The isoperimetric ratio of A is defined as follows:

$$\chi_\mu(A) := \frac{\mu^+(A)}{\min\{\mu(A), 1-\mu(A)\}}.$$



- Then, we define the Cheeger constant χ_μ of μ setting

$$\chi_\mu := \inf\{\chi_\mu(A) : A \text{ Borel } \subset \mathbb{R}^n\}.$$

Theorem (Rothaus, Cheeger, Maz'ya)

Let μ be a Borel probability measure on $\mathbb{R}^n$ with Cheeger constant χ_μ. Let α_1 be the largest constant with the following property: for every integrable, locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| \, d\mu(x) \leqslant \int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x).$$

Then, $\alpha_1 \leqslant \chi_\mu \leqslant 2\alpha_1$.

First we show that $\chi_\mu \leqslant 2\alpha_1$.

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable, locally Lipschitz function. We may assume that f is bounded from below and hence, by adding a suitable constant, that $f > 0$.
- The co-area formula shows that

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x) &\geqslant \int_0^\infty \mu^+(\{x : f(x) > s\}) \, ds \\ &\geqslant \chi_\mu \int_0^\infty \min\{\mu(A(s)), 1 - \mu(A(s))\} \, ds,\end{aligned}$$

where $A(s) = \{f > s\}$.

- Setting $A(s)=\{f>s\}$ we saw that
$$\int_{\mathbb{R}^n}|\nabla f(x)|\,d\mu(x)\geqslant \chi_\mu\int_0^\infty \min\{\mu(A(s)),1-\mu(A(s))\}\,ds.$$
- Using the fact that $\|\mathbf{1}_B-\mathbb{E}_\mu(\mathbf{1}_B)\|_1=2\mu(B)(1-\mu(B))$ for every Borel subset B of $\mathbb{R}^n$, and the simple identity $\mathbb{E}_\mu(f(g-\mathbb{E}_\mu(g)))=\mathbb{E}_\mu(g(f-\mathbb{E}_\mu(f)))$, we may write
$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla f(x)|\,d\mu(x)&\geqslant \chi_\mu\int_0^\infty \mu(A(s))(1-\mu(A(s)))\,ds\\
&=\frac{\chi_\mu}{2}\int_0^\infty \|\mathbf{1}_{A(s)}-\mathbb{E}_\mu(\mathbf{1}_{A(s)})\|_1\,ds\\
&\geqslant \frac{\chi_\mu}{2}\sup\left\{\int_0^\infty\int_{\mathbb{R}^n}(\mathbf{1}_{A(s)}-\mathbb{E}_\mu(\mathbf{1}_{A(s)}))g\,d\mu\,ds\;:\;\|g\|_\infty\leqslant 1\right\}\\
&=\frac{\chi_\mu}{2}\sup\left\{\int_0^\infty\int_{\mathbb{R}^n}\mathbf{1}_{A(s)}(g-\mathbb{E}_\mu(g))\,d\mu\,ds\;:\;\|g\|_\infty\leqslant 1\right\}\\
&=\frac{\chi_\mu}{2}\sup\left\{\int_{\mathbb{R}^n}f(g-\mathbb{E}_\mu(g))\,d\mu\;:\;\|g\|_\infty\leqslant 1\right\}\\
&=\frac{\chi_\mu}{2}\sup\left\{\int_{\mathbb{R}^n}g(f-\mathbb{E}_\mu(f))\,d\mu\;:\;\|g\|_\infty\leqslant 1\right\}=\frac{\chi_\mu}{2}\|f-\mathbb{E}_\mu(f)\|_1.\end{aligned}$$

This shows that $\chi_\mu\leqslant 2\alpha_1$.

- Recall that α_1 is the largest constant so that

$$\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)|\, d\mu(x) \leqslant \int_{\mathbb{R}^n} |\nabla f(x)|\, d\mu(x)$$

for locally Lipschitz functions. Now, we want to show that $\alpha_1 \leqslant \chi_\mu$.
- Consider any closed subset A of $\mathbb{R}^n$ and for small $\varepsilon > 0$ we define the function

$$f_\varepsilon(x) = \max\left\{0, 1 - \frac{d(x, A_{\varepsilon^2})}{\varepsilon - \varepsilon^2}\right\}.$$

- Then, $0 \leqslant f_\varepsilon \leqslant 1$, $f_\varepsilon \equiv 1$ on $A_{\varepsilon^2} \supseteq A$, $f \equiv 0$ on $\{x : d(x, A) > \varepsilon\}$, and $f_\varepsilon \longrightarrow \mathbf{1}_A$ as $\varepsilon \to 0$.
- Finally, f_ε is Lipschitz: we have

$$|f_\varepsilon(x) - f_\varepsilon(y)| \leqslant \frac{1}{\varepsilon(1-\varepsilon)}\Big|d(x, A_{\varepsilon^2}) - d(y, A_{\varepsilon^2})\Big| \leqslant \frac{|x-y|}{\varepsilon(1-\varepsilon)},$$

therefore $|\nabla f_\varepsilon(x)| \leqslant (\varepsilon - \varepsilon^2)^{-1}$.
- Since $\nabla f_\varepsilon(x) = 0$ on $C = \{x : d(x, A) > \varepsilon\} \cup \{x : d(x, A) < \varepsilon^2\}$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f_\varepsilon(x)|\, d\mu(x) &\leqslant \int_{\mathbb{R}^n \setminus C} |\nabla f_\varepsilon(x)|\, d\mu(x) \\ &\leqslant \frac{1}{1-\varepsilon}\,\frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\,\frac{\mu(A_{\varepsilon^2}) - \mu(A)}{\varepsilon^2}.\end{aligned}$$

- We have assumed that

$$\alpha_1 \int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)| \, d\mu(x) \leqslant \int_{\mathbb{R}^n} |\nabla f(x)| \, d\mu(x).$$

- Therefore,

$$\alpha_1 \int_{\mathbb{R}^n} |f_\varepsilon(x) - \mathbb{E}_\mu(f_\varepsilon)| \, d\mu(x) \leqslant \frac{1}{1-\varepsilon} \, \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon} - \frac{\varepsilon}{1-\varepsilon} \, \frac{\mu(A_{\varepsilon^2}) - \mu(A)}{\varepsilon^2}.$$

- Letting $\varepsilon \to 0^+$ we see that

$$\mu^+(A) \geqslant \alpha_1 \|\mathbf{1}_A - \mathbb{E}_\mu(\mathbf{1}_A)\|_1 = 2\alpha_1 \mu(A)(1-\mu(A)).$$

- This shows that $\chi_\mu \geqslant \alpha_1$.

Definition

$\psi_\mu = \frac{1}{\chi_\mu}$, the reciprocal Cheeger constant.

Poincaré constant and Cheeger constant

- Recall that a Borel probability measure μ on $\mathbb{R}^n$ satisfies the Poincaré inequality with constant $\vartheta > 0$ if
$$\mathrm{Var}_\mu(f) \leqslant \vartheta^2 \int |\nabla f|^2 \, d\mu,$$
for all smooth functions f on $\mathbb{R}^n$, where
$$\mathrm{Var}_\mu(g) = \mathbb{E}_\mu(g^2) - (\mathbb{E}_\mu(g))^2$$
is the variance of g with respect to μ.
- The Poincaré constant ϑ_μ of μ is the smallest constant $\vartheta > 0$ for which the Poincaré inequality is satisfied for all f.

Theorem (Maz'ya, Cheeger)

Let μ be a Borel probability measure with reciprocal Cheeger constant ψ_μ. Then its Poincaré constant ϑ_μ satisfies
$$\vartheta_\mu \leqslant 2\psi_\mu.$$

- By the co-area formula and the definition of the Cheeger constant, for every positive integrable locally Lipschitz function g we have

$$\begin{aligned}\chi_\mu \int_0^\infty \min\{\mu(\{g \geqslant s\}), 1-\mu(\{g \geqslant s\})\}\, ds &\leqslant \int_0^\infty \mu^+(\{g \geqslant s\})\, ds \\ &\leqslant \int_{\mathbb{R}^n} |\nabla g|\, d\mu.\end{aligned}$$

- Let f be an integrable locally Lipschitz function and set $m = \mathrm{med}(f)$. Then, we have $\mu(\{f \geqslant m\}) \geqslant \frac{1}{2}$ and $\mu(\{f \leqslant m\}) \geqslant \frac{1}{2}$.
- We set $f^+ = \max\{f-m, 0\}$ and $f^- = -\min\{f-m, 0\}$. Then, $f-m = f^+ - f^-$ and by the definition of m we have

$$\mu(\{(f^+)^2 \geqslant s\}) \leqslant \frac{1}{2} \quad \text{and} \quad \mu(\{(f^-)^2 \geqslant s\}) \leqslant \frac{1}{2}$$

for all $s > 0$.

- Using
$$\chi_\mu \int_0^\infty \min\{\mu(\{g \geqslant s\}), 1-\mu(\{g \geqslant s\})\}\, ds \leqslant \int_{\mathbb{R}^n} |\nabla g|\, d\mu$$
with $g=(f^+)^2$ and $g=(f^-)^2$ and applying integration by parts we see that
$$\begin{aligned}\chi_\mu \int_{\mathbb{R}^n} |f-m|^2 d\mu &= \chi_\mu \int_{\mathbb{R}^n} (f^+)^2 d\mu + \chi_\mu \int_{\mathbb{R}^n} (f^-)^2 d\mu \\ &= \chi_\mu \int_0^\infty \mu(\{(f^+)^2 \geqslant s\})\, ds + \chi_\mu \int_0^\infty \mu(\{(f^-)^2 \geqslant s\})\, ds \\ &\leqslant \int_{\mathbb{R}^n} |\nabla((f^+)^2)|\, d\mu + \int_{\mathbb{R}^n} |\nabla((f^-)^2)|\, d\mu \\ &= \int_{\mathbb{R}^n} (|\nabla((f^+)^2)| + |\nabla((f^-)^2)|)\, d\mu.\end{aligned}$$
- Note that
$$|\nabla((f^+)^2)| + |\nabla((f^-)^2)| \leqslant 2|f-m|\, |\nabla f|.$$
- Therefore, applying the Cauchy-Schwarz inequality we see that
$$\chi_\mu \int_{\mathbb{R}^n} |f-m|^2 d\mu \leqslant 2 \left(\int_{\mathbb{R}^n} |f-m|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla f|^2\, d\mu\right)^{1/2}.$$

- We saw that

$$\chi_\mu \int_{\mathbb{R}^n} |f-m|^2 d\mu \leqslant 2\left(\int_{\mathbb{R}^n} |f-m|^2 d\mu\right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla f|^2\, d\mu\right)^{1/2}.$$

- This gives

$$\frac{\chi_\mu^2}{4}\int_{\mathbb{R}^n} |f-m|^2 d\mu \leqslant \int_{\mathbb{R}^n} |\nabla f|^2\, d\mu.$$

Since

$$\int_{\mathbb{R}^n} |f-\mathbb{E}_\mu(f)|^2 d\mu = \min_{\alpha\in\mathbb{R}} \int_{\mathbb{R}^n} |f-\alpha|^2 d\mu \leqslant \int_{\mathbb{R}^n} |f-m|^2 d\mu$$

and f was arbitrary, we get $\vartheta_\mu^2 \leqslant 4\chi_\mu^{-2} = 4\psi_\mu^2$.

- A Borel probability measure μ on $\mathbb{R}^n$ is called log-concave if for all compact subsets A, B of $\mathbb{R}^n$ and all $0 < \lambda < 1$ we have

$$\mu((1-\lambda)A + \lambda B) \geqslant \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

Theorem (Buser, Ledoux)

Let μ be a log-concave probability measure on $\mathbb{R}^n$ with reciprocal Cheeger constant ψ_μ. Then its Poincaré constant ϑ_μ satisfies

$$\psi_\mu \leqslant c\,\vartheta_\mu.$$

Isotropic measures

- We say that a Borel probability measure μ on $\mathbb{R}^n$ is isotropic if $\mathrm{bar}(\mu) = \int_{\mathbb{R}^n} x d\mu(x) = 0$ and μ satisfies the isotropic condition

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \, d\mu(x) = 1, \qquad \theta \in S^{n-1}.$$

- Similarly, we shall say that a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with barycenter $\mathrm{bar}(f) = 0$ is isotropic if $\int f(x)dx = 1$ and the measure $d\mu(x) = f(x)dx$ is isotropic.
- A convex body K of volume 1 in $\mathbb{R}^n$ with barycenter at the origin is called isotropic if

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2$$

for some constant $L_K > 0$ (the isotropic constant of K) and all $\theta \in S^{n-1}$.
- One can check that K is isotropic if and only if the function $f_K := L_K^n \mathbf{1}_{\frac{1}{L_K} K}$ is an isotropic log-concave function.
- Every non-degenerate absolutely continuous probability measure μ has an isotropic image $\nu = \mu \circ S$, where $S : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map. Similarly, every log-concave $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$ has an isotropic image: there exist an affine isomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ and a positive number a such that $af \circ S$ is isotropic.

- Let f be a log-concave function with finite, positive integral. The covariance matrix $\mathrm{Cov}(f)$ is the matrix with entries

$$[\mathrm{Cov}(f)]_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f(x)\,dx}{\int_{\mathbb{R}^n} f(x)\,dx}.$$

- If f is the density of a measure μ we denote this matrix also by $\mathrm{Cov}(\mu)$. Note that if f is isotropic then $\mathrm{Cov}(f)$ is the identity matrix.
- The isotropic constant of f is defined by

$$L_f := \left(\frac{\sup_{x\in\mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(f)]^{\frac{1}{2n}}.$$

(and given a log-concave measure μ with density f_μ we let $L_\mu := L_{f_\mu}$).
- It is easy to check that the isotropic constant L_μ is an affine invariant.

- One can also prove that if $f:\mathbb{R}^n\to[0,\infty)$ is a log-concave density, then

$$nL_f^2=\inf_{\substack{S\in SL_n\\ y\in\mathbb{R}^n}}\Big(\sup_{x\in\mathbb{R}^n}f(x)\Big)^{2/n}\int_{\mathbb{R}^n}|S(x)+y|^2f(x)\,dx.$$

- If $f:\mathbb{R}^n\to[0,\infty)$ is an isotropic log-concave function then

$$L_f=\|f\|_\infty^{1/n}\geqslant c,$$

where $c>0$ is an absolute constant.

Conjecture 1

For any isotropic log-concave density $f:\mathbb{R}^n\to[0,\infty)$,

$$\|f\|_\infty^{1/n}\leqslant C,$$

where $C>0$ is an absolute constant.

- This would imply that a convex body of volume one, in any dimension, has at least one hyperplane section with volume bounded from below by an absolute constant (slicing problem).

- Define

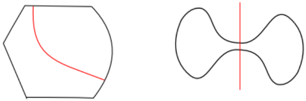
$$L_n := \sup\{L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\}.$$

- Then, Conjecture 1 states that $L_n \leqslant C$ for an absolute constant $C > 0$.
- Around 1985-6 (published in 1991), Bourgain introduced this conjecture and obtained the upper bound $L_n \leqslant c\sqrt[4]{n}\ln n$.
- In 2006 the estimate was improved by Klartag, who showed that the logarithmic factor can be omitted.

Theorem (Bourgain/Klartag)

There exists an absolute constant $c > 0$ such that $L_n \leqslant c\sqrt[4]{n}$ for all $n \geqslant 1$.

- Kannan, Lovász and Simonovits conjectured in 1994 that the isoperimetric ratio of any Borel set A with respect to the uniform measure μ_K on a convex body K in $\mathbb{R}^n$ (defined by $\mu_K(A)=\mathrm{vol}_n(K\cap A)/\mathrm{vol}_n(K))$ should be, up to an absolute constant, at least as large as the minimal isoperimetric ratio over all half-spaces.



Conjecture

One has

$$\chi(K)\geqslant c\cdot\inf_H\frac{\mu_K^+(H)}{\min\{\mu_K(H),\mu_K(\mathbb{R}^n\setminus H)\}}$$

for some absolute constant $c>0$, where the infimum is over all half-spaces H in $\mathbb{R}^n$.

- Their interest in this parameter was related to the study of randomized volume algorithms.
- Since the isoperimetric ratio of a half-space is basically a one-dimensional quantity, one can obtain an explicit formula for this infimum. Then, one arrives at the following conjecture:

Conjecture 2

$$\chi(K) \approx 1/\sqrt{\lambda(K)}$$

where $\lambda(K)$ is the largest eigenvalue of the matrix of inertia $M_{ij} := \int_K x_i x_j dx$ of K.

- They actually proved that one always has $\chi(K) \leqslant 10/\sqrt{\lambda(K)}$, therefore the question is about the lower bound.

Theorem (Kannan-Lovász-Simonovits)

For every convex body K in $\mathbb{R}^n$ one has

$$\chi(K) \geqslant \frac{\ln 2}{I_1(K)}.$$

- Here,

$$I_1(K) := \frac{1}{\mathrm{vol}_n(K)} \int_K |x - \mathrm{bar}(K)|\, dx.$$

- If K is isotropic this gives $\chi(K) \geqslant c/(\sqrt{n} L_K)$.
- In fact, one may find literature on the subject before their work, and there were known lower bounds for $\chi(K)$ of order $1/\mathrm{diam}(K)$.

- Another approach to the KLS-conjecture is due to Bobkov.

Theorem (Bobkov)

Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then we have

$$\chi_\mu \geqslant \frac{c}{\|f\|_{L_2(\mu)}},$$

where $f(x) = |x - \mathrm{bar}(\mu)|$ and $c > 0$ is an absolute constant.

- If μ is isotropic this gives $\chi_\mu \geqslant c/\sqrt{n}$.

KLS-conjecture and isotropic constant

Conjecture 2

$$\chi(K) \approx 1/\sqrt{\lambda(K)}$$

where $\lambda(K)$ is the largest eigenvalue of the matrix of inertia $M_{ij} := \int_K x_i x_j dx$ of K.

- For an isotropic convex body K this becomes $\chi(K) \approx 1/L_K$.

KLS-Conjecture

For every isotropic log-concave probability measure μ on $\mathbb{R}^n$ one has $\chi_\mu \geqslant c$, where $c > 0$ is an absolute constant.

Theorem (Eldan-Klartag)

$$L_n \leqslant C\psi_n = C/\chi_n.$$

- In other words, the KLS-conjecture is stronger than Conjecture 1 about the isotropic constant.

- The currently best known results are due to Lee and Vempala and are consequences of the following theorem:

Theorem (Lee-Vempala)

If μ is a log-concave probability measure on $\mathbb{R}^n$ with covariance matrix A then

$$\psi_\mu \leqslant c\big(\mathrm{tr}(A^2)\big)^{1/4}$$

where $c > 0$ is an absolute constant.

- If we make the additional assumption that μ is isotropic then we obtain the upper bound

$$\psi_\mu \leqslant c\sqrt[4]{n}.$$

- The approach of Lee and Vempala is based on Eldan's stochastic localization.

An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture

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November 30, 2020

Abstract

We prove an almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. The lower bound has dimension dependency $d^{-o_d(1)}$. When the dimension is large enough, our lower bound is tighter than the previous best bound which has dimension dependency $d^{-1/4}$. Improving the isoperimetric coefficient in the KLS conjecture has many implications, including improvements of the bounds in the thin-shell conjecture and in the slicing conjecture, better concentration inequalities for Lipschitz functions of log-concave measures and better mixing time bounds for MCMC sampling algorithms on log-concave measures.

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Quick Bio:

I am a postdoc fellow at ETH Foundations of Data Science (ETH-FDS) in ETH Zürich under the supervision of Prof. Peter Bühlmann. Previously, I obtained my PhD in the Department of Statistics at UC Berkeley in 2019. My Phd study was advised by Prof. Bin Yu. During my PhD, I am fortunate to also work with Prof. Martin Wainwright and Prof. Jack Gallant.
My main research interests lie on statistical machine learning, optimization and the applications in neuroscience. In particular, I am interested in domain adaptation, stability, MCMC sampling algorithms, convolutional neural networks and statistical problems that arise from computational neuroscience. Before my PhD study, I obtained my Diplome d'Ingénieur (Eng. Deg. in Applied Mathematics) at Ecole Polytechnique in France.