Typical properties of contractions on ℓ_p spaces

by S. Grivaux

Joint work with É. Matheron and Q. Menet

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- A Polish space is a topological space which is separable and completely metrizable, i.e. its topology can be defined by a distance which turns it into a complete metric space.

  Examples:
  - any separable complete metric space;
  - $(0,1)$ endowed with the topology induced by that of $\mathbb{R}$ is a Polish space.

- In a Polish space, the Baire Category Theorem applies: if (U_n) is a sequence of open dense subsets of X, then $\bigcap_n U_n$ is dense in X.

  A countable intersection of open sets is called a G_δ set. A set containing a dense G_δ set is called a comeager set.

- **Philosophy:** In a Polish space, the comeager sets (i.e. the sets containing an intersection of dense open sets) are big sets in the Baire category sense.

  By the Baire Category Theorem, countable intersections of big sets in this sense are again big.

  This motivates the following definition:

  Definition. Let X be a Polish space, and let (P) be a certain property of elements of X. We say that property (P) is typical, or that a typical $x \in X$ has property (P) if the set

  $$\{x \in X \, ; \, x \text{ has } (P)\}$$

  is comeager in X.

- **Examples:**
  - a typical $x \in (\mathbb{R}, |\,.\,|)$ is irrational;
  - a typical probability measure $\mu \in \mathcal{P}_1(\,[0,1], w^*)$ is continuous and purely singular with respect to Lebesgue measure.

Definition. Let X be a (complex) separable Banach space.

$$\mathcal{B}_1(X) := \{T \in \mathcal{B}(X) \,;\, ||T|| \leq 1\}.$$

- **Aim:** put on $\mathcal{B}_1(X)$ a topology τ which turns it into a Polish space, and study whether some properties of elements of $(\mathcal{B}_1(X), \tau)$ are typical, or not.
- **Natural choices of topologies on $\mathcal{B}_1(X)$:**
  - the operator-norm topology: $\big(\mathcal{B}_1(X),\ |||\,.\,|||\big)$ is usually not separable $\longrightarrow$ not Polish;
  - the WOT (Weak Operator Topology): $T_i \to T$ for the WOT if $T_i x \to Tx$ weakly for every $x \in X$;
  - the SOT (Strong Operator Topology): $T_i \to T$ for the SOT if $T_i x \to Tx$ in norm for every $x \in X$;
  - the SOT* (Strong* Operator Topology): $T_i \to T$ for the SOT* if $T_i \to T$ and $T_i^* \to T^*$ for the SOT.

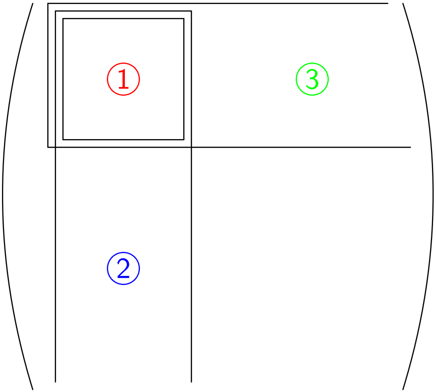
- ℓ_p**-spaces:** They are sequence spaces defined for $1 \leq p < +\infty$ as
$$\ell_p(\mathbb{N}) := \{x = (x_j)_{j\geq 0} \; ; \; \sum_j |x_j|^p < +\infty\}$$
endowed with the norm
$$||x||_p = \left(\sum_j |x_j|^p\right)^{\frac{1}{p}}$$
which turns them into Banach spaces.
- **Canonical basis:** for $j \geq 0$,
$$e_j := (0, \dots, 0, 1, 0, \dots)$$
where 1 appears in the j-th place. Any $x \in \ell_p(\mathbb{N})$ can be written in a unique way as
$$x = \sum_j x_j e_j$$
where the series converges in $\ell_p(\mathbb{N})$.

Example. Let $(e_j)_{j\geq 0}$ be the canonical basis of $\ell_p(\mathbb{N})$, and set $E_N = [e_0, \ldots, e_N]$, $N \geq 0$. Let $T_0, T \in \mathcal{B}_1(\ell_p)$.

- T is WOT-close to T_0 if $\|P_{E_N}(T - T_0)P_{E_N}\| < \varepsilon$.
- T is SOT-close to T_0 if $\|(T - T_0)P_{E_N}\| < \varepsilon$.
- T is SOT*-close to T_0 if

  $\|(T - T_0)P_{E_N}\| < \varepsilon$ and $\|P_{E_N}(T - T_0)\| < \varepsilon$.

$T =$



1 $= P_{E_N} T P_{E_N}$

2 $= T P_{E_N}$

3 $= P_{E_N} T$

Proposition.

- *If X is separable, $\big(\mathcal{B}_1(X), \mathrm{SOT}\big)$ is a Polish space.*
- *If X^* is separable, $\big(\mathcal{B}_1(X), \mathrm{SOT}^*\big)$ is a Polish space.*
- *If X is reflexive, $\big(\mathcal{B}_1(X), \mathrm{WOT}\big)$ is a Polish space.*

$$* * *$$

Let (x_n) be dense in the unit sphere of X. The distance d on $\mathcal{B}_1(X)$ defined by

$$T, S \in \mathcal{B}_1(X)\ ;\ d(S, T) = \sum_n 2^{-n} \,||\,(T - S)x_n\,||$$

is an equivalent distance for the SOT on $\mathcal{B}_1(X)$ which turns it into a complete metric space.

$$* * *$$

If $X = \ell_p(\mathbb{N})$, $1 \le p < +\infty$, or $X = c_0(\mathbb{N})$,

- *$\big(\mathcal{B}_1(X), \mathrm{SOT}\big)$ is a Polish space;*
- *if $p > 1$, $\big(\mathcal{B}_1(X), \mathrm{SOT}^*\big)$ and $\big(\mathcal{B}_1(X), \mathrm{WOT}\big)$ are Polish spaces.*

Question. Find some interesting typical properties of operators $T \in \mathcal{B}_1(\ell_p)$ or $T \in \mathcal{B}_1(c_0)$ for one of these Polish topologies.

For instance, one may ask: is it true

- that a typical contraction on ℓ_p or c_0 for one of these topologies has an eigenvector?
- that it has a non-trivial invariant subspace, i.e. a closed subspace M of X with $M \neq \{0\}$ and $M \neq X$ such that $T(M) \subseteq M$?

Recall that saying for instance that a typical $T \in \big(\mathcal{B}_1(\ell_p), \text{SOT}\big)$ has a non-trivial invariant subspace means that the set

$$\{T \in \mathcal{B}_1(\ell_p)\ ;\ T \text{ has a non-trivial invariant subspace}\}$$

contains a comeager subset of $\big(\mathcal{B}_1(\ell_p), \text{SOT}\big)$.

The Invariant Subspace Problem. Given a bounded operator T on a separable (infinite-dimensional) Banach space X, does there exists a closed subspace M of X which is *non-trivial* ($M \neq \{0\}$ and $M \neq X$), and *T-invariant* (i.e. $T(M) \subseteq M$)?

- Enflo, '70: No. Counterexamples on some strange Banach spaces.
- Read, '80: more counterexamples, much simpler, on some classical Banach spaces like ℓ_1, c_0, $\bigoplus_{\ell_2} J$.

  Examples of operators on ℓ_1 without non-trivial invariant closed *subset*.
- Grivaux-Roginskaya: unification of Read's type constructions on *non-reflexive* Banach spaces.

- Argyros-Haydon '11: there exist Banach spaces X such that every $T \in \mathcal{B}(X)$ has a non-trivial invariant subspace.

The Invariant Subspace Problem is widely open for reflexive spaces and, most importantly, for Hilbert spaces.

ℓ_p - spaces: very interesting class of Banach spaces in this context.

This motivates the question we asked a few slides ago:

Question. Given a "natural" Polish topology on $\mathcal{B}_1(\ell_p)$, is it true that a typical contraction T for this topology has a non-trivial invariant subspace?

- **The Hilbertian case:** typical properties of contractions on ℓ_2 for the WOT and the SOT were studied by Eisner and Eisner-Mátrai.

  Theorem. *A typical* $T \in \big(\mathcal{B}_1(\ell_2), \text{WOT}\big)$ *is unitary.*

  Theorem. *A typical* $T \in \big(\mathcal{B}_1(\ell_2), \text{SOT}\big)$ *is unitarily similar to the backward shift of infinite multiplicity* B_∞ on $\bigoplus_{\ell_2} \ell_2$:

$$B_\infty : \bigoplus_{\ell_2} \ell_2 \longrightarrow \bigoplus_{\ell_2} \ell_2, \quad (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots)$$

- **Consequences:**

  A typical $T \in \big(\mathcal{B}_1(\ell_2), \text{WOT}\big)$ has a non-trivial invariant subspace.

  A typical $T \in \big(\mathcal{B}_1(\ell_2), \text{SOT}\big)$ has eigenvectors (and hence invariant subspaces), and is such that T^* is an isometry.

Why should such results be true?

- the case of WOT-typical contractions: given $x \in \ell_2(\mathbb{N})$, the maps
$$T \mapsto ||Tx|| \quad \text{and} \quad T \mapsto ||T^*x||$$
are not continuous. A WOT-typical T is such that T and T^* are isometries.
- the case of SOT-typical contractions: given $x \in \ell_2(\mathbb{N})$, the maps
$$T \mapsto ||Tx||$$
are indeed continuous, but the maps
$$T \mapsto ||T^*x||$$
are not continuous. An SOT-typical T is such that T^* is an isometry.

Typical properties of contractions for the the SOT* topology are not so well understood. For instance a typical $T \in \big(\mathcal{B}_1(\ell_2), \text{SOT}^*\big)$ has no eigenvalue [Grivaux-Matheron-Menet].

It is still true that a typical $T \in \big(\mathcal{B}_1(\ell_2), \text{SOT}^*\big)$ has non-trivial invariant subspaces, but for much less "trivial" reasons than for the SOT and the WOT.

- **Brown-Chevreau-Pearcy Theorem:**

If $T \in \mathcal{B}_1(\ell_2)$ is such that its spectrum contains the whole unit circle $\mathbb{T}$, then T has a non-trivial invariant subspace.

One can show without too much trouble that a typical $T \in \mathcal{B}_1(\ell_p)$ or $\mathcal{B}_1(c_0)$ for one of the topologies SOT or SOT* is such that $\sigma(T) = \overline{\mathbb{D}(0,1)} \supseteq \mathbb{T}$.

- **What happens for SOT-typical contractions on $X = \ell_p$, $1 \leq p \neq 2 < +\infty$?**

For $p = 1$, the situation is surprisingly similar to the Hilbertian one:

Theorem. *A typical $T \in \big(\mathcal{B}_1(\ell_1), \text{SOT}\big)$ is such that any $\lambda \in \mathbb{D}(0,1)$ is an eigenvalue of T of infinite multiplicity, and T^* is an isometry.*

Consequently, a typical $T \in \big(\mathcal{B}_1(\ell_1), \text{SOT}\big)$ has a non-trivial invariant subspace.

- **What happens for SOT-typical contractions on $X = \ell_p$, $1 < p \neq 2 < +\infty$?**

We do not know really...

Theorem. *Let $p > 2$. A typical $T \in \big(\mathcal{B}_1(\ell_p), \mathrm{SOT}\big)$ has no eigenvalue.*

Recall that it follows from the Eisner-Mátrai result that a typical $T \in (\mathcal{B}_1(\ell_2), \mathrm{SOT})$ is such that any $\lambda \in \mathbb{D}$ is an eigenvalue of T.

We can actually prove the following stronger result:

Theorem. *Let $p > 2$. A typical $T \in (\mathcal{B}_1(\ell_p), \mathrm{SOT})$ is such that $2T^*$ is a hypercyclic operator.*

Definition. An operator $S \in \mathcal{B}(X)$ is hypercyclic if there exists $x \in X$ whose orbit $\{S^n x \,;\, n \geq 0\}$ is dense in X.

If S is hypercyclic, S^* has no eigenvalue:

$Sx^* = \lambda x^* \Rightarrow \forall\, x \in X,\ \forall\, n \geq 0, \quad \langle x^*, S^n x\rangle = \lambda^n \langle x^*, x\rangle.$

If x is a hypercyclic vector and if $x^* \neq 0$, the set $\{\langle x^*, S^n x\rangle \,;\, n \geq 0\}$ is dense in $\mathbb{C} \Rightarrow$ contradiction.

Recall again: a typical $T \in (\mathcal{B}_1(\ell_2), \mathrm{SOT})$ is such that T^* is an *isometry*.

Theorem. *Let* $p > 2$. *A typical* $T \in (\mathcal{B}_1(\ell_p), \mathrm{SOT})$ has no eigenvalue.

A key step in the proof of this theorem is given by the following rather intriguing result:

Theorem. *Let* $p > 2$, *and let* G *be a subset of* $\mathcal{B}_1(\ell_p)$. *If* G *is comeager in* $(\mathcal{B}_1(\ell_p), \mathrm{SOT}^*)$, *then* G *is comeager in* $(\mathcal{B}_1(\ell_p), \mathrm{SOT})$.

This statement is non-trivial: SOT^*-$G_\delta \not\Rightarrow$ SOT-G_δ.

Proposition. *Let* $p > 1$. *A typical* $T \in (\mathcal{B}_1(\ell_p), \mathrm{SOT}^*)$ has no eigenvalue.

Theorem. *Let $p > 2$ and let G be a subset of $\mathcal{B}_1(\ell_p)$. If G is comeager in $(\mathcal{B}_1(\ell_p), \mathrm{SOT}^*)$, then G is comeager in $(\mathcal{B}_1(\ell_p), \mathrm{SOT})$.*

Idea of proof: *Banach-Mazur game*.

Let $\mathcal{E}$ be a Polish space, $\mathcal{A} \subseteq \mathcal{E}$. Two players I and II play alternatively non-empty open sets $\mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \dots$ Player II wins the run if $\bigcap_n \mathcal{U}_n \subseteq \mathcal{A}$.

The set $\mathcal{A}$ is comeager in $\mathcal{E}$ if and only if player II has a winning strategy in this game.

WLOG, the sets $\mathcal{U}_n$ can be required to be picked from a given basis of the topology of $\mathcal{E}$.

In our context, let $\mathcal{E} = (\mathcal{B}_1(\ell_p), \mathrm{SOT})$. Let $(e_j)_{j\geq 0}$ be the canonical basis of ℓ_p, and let $E_N = [e_0, \dots, e_N]$ for $N \geq 0$.

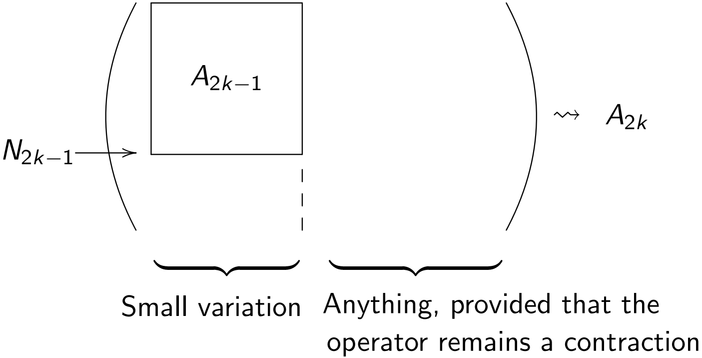
The open sets played in the game will be of the form

$$\mathcal{U}(N, A, \varepsilon) = \{\, T \in \mathcal{B}_1(\ell_p) \; ; \; \|Te_j - Ae_j\| < \varepsilon \quad \text{for } j = 0 \dots N \}$$

where $N \geq 0$, $A \in \mathcal{B}_1(E_N)$, and $0 < \varepsilon \leq 1$.

Player I plays $\mathcal{U}_{2k} = \mathcal{U}(N_{2k}, A_{2k}, \varepsilon_{2k})$,
Player II plays $\mathcal{U}_{2k+1} = \mathcal{U}(N_{2k+1}, A_{2k+1}, \varepsilon_{2k+1})$.

At step $2k$:



"Anything, provided that the operator remains a contraction": It is here that the condition that $p > 2$ appears, through the following elementary inequality (to be found for instance in a paper by C. H. Kan):

Lemma. *Let $u, v \in \mathbb{C}^*$. If $2 < p < +\infty$,*

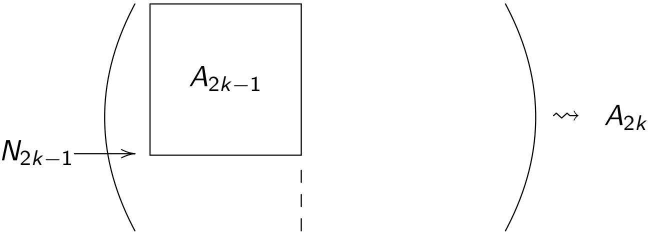
$$|u+v|^p + |u-v|^p > 2\,|u|^p + p\,|u|^{p-2}|v|.$$

Corollary. Let $T \in \mathcal{B}_1(\ell_p)$, $p > 2$. If $f \in S_{\ell_p}$ is such that $||Tf|| = 1$, then $\forall g \in \ell_p$,

$$\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing \implies \operatorname{supp}(Tf) \cap \operatorname{supp}(Tg) = \varnothing.$$

Corollary. Let $T \in \mathcal{B}_1(\ell_p)$, $p > 2$. If $f \in S_{\ell_p}$ is such that $||Tf|| = 1$, then $\forall g \in \ell_p$,

$$\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing \implies \mathrm{supp}(Tf) \cap \mathrm{supp}(Tg) = \varnothing.$$



If A_{2k-1} has norm 1 and attains its norm on $E_{N_{2k-1}}$ at f such that

$$\mathrm{supp}(f) = \mathrm{supp}(A_{2k-1}f) = [0, N_{2k-1}],$$

then in order for A_{2k} to remain a contraction one must have

$$\mathrm{supp}(A_{2k}(e_l)) \subseteq (N_{2k-1}, +\infty) \quad \text{for every } l > N_{2k-1}.$$

Proposition. *Let $p > 2$, and let $A \in \mathcal{B}_1(E_N)$ be such that*

> $(*)$ *the norming vectors $x \in S_{E_N}$ of the operator A consist of unimodular multiples of a single vector $x_0 \in S_{E_N}$, and $\langle e_j^*, x_0 \rangle \neq 0$ and $\langle e_j^*, A x_0 \rangle \neq 0$ for every $j = 0, \ldots, N$.*

Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $T \in \mathcal{B}_1(\ell_p)$,

$$\| P_{E_N} T P_{E_N} - A \| < \delta \Longrightarrow \| P_{E_N} T (I - P_{E_N}) \| < \varepsilon.$$

$$T = \begin{pmatrix} \underset{N \to}{\boxed{\begin{matrix} (1) = \\ P_{E_N} T P_{E_N} \end{matrix}}} & \boxed{(2) = P_{E_N} T (I - P_{E_N})} \\ & \end{pmatrix}$$

(1) δ-close to A

$\Downarrow$

(2) ε-small

T SOT-close to A

$\Downarrow$

T SOT*-close to A

The proof breaks down for $1 < p < 2$, and we do not know the answer to the following question:

Question.
If $1 < p < 2$, is it true that a typical $T \in (\mathcal{B}_1(\ell_p), \mathrm{SOT})$ has no eigenvalue?

Well... we are still unable to answer the following question:

Question. If $p > 1$, $p \neq 2$, is it true that a typical $T \in \big(\mathcal{B}_1(\ell_p), \mathrm{SOT}\big)$ has a non-trivial invariant subspace?

Theorem. *If $p > 1$, $p \neq 2$, a typical $T \in \big(\mathcal{B}_1(\ell_p), \mathrm{SOT}\big)$ has a non-trivial invariant* cone.

$$(C \text{ is a cone if } C + C \subseteq C \quad \text{and} \quad tC \subseteq C \text{ for every } t \geq 0)$$

This theorem follows from results of V. Müller.

Summary: typical properties of $T \in (\mathcal{B}_1(\ell_p), \mathrm{SOT})$:

	$p=1$	$1<p<2$	$p=2$	$2<p<\infty$	c_0
$\sigma(T)$	$\overline{\mathbb{D}}$	$\overline{\mathbb{D}}$	$\overline{\mathbb{D}}$ (Eisner-Mátrai)	$\overline{\mathbb{D}}$	$\overline{\mathbb{D}}$
T^* is an isometry	Yes	No	Yes (Eisner-Mátrai)	No	No
T has a non-trivial invariant closed cone	Yes	Yes	Yes (Eisner-Mátrai)	Yes	?
$\sigma_p(T)$	$\mathbb{D}$	?	$\mathbb{D}$ (Eisner-Mátrai)	$\emptyset$	$\emptyset$
T has a non-trivial invariant subspace	Yes	?	Yes (Eisner-Mátrai)	?	?

Thanks!