Some Examples in Operator Theory

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1. An operator on an infinite dimensional space
2. C^* algebras
3. Crossed and semicrossed products
4. Discrete groups and operator algebras

An operator on an infinite dimensional space

Let H be a separable Hilbert space with orthonormal basis $\{e_n : n = 0, 1, 2 \dots\}$.
An operator is a bounded linear map $H \to H$.
If T is an operator, the adjoint of T is an operator T^* which satisfies:

$$\langle T^* x, y \rangle = \langle x, Ty \rangle .$$

Definition

The shift operator S is the operator on H defined by $Se_n = e_{n+1}$.

An operator on an infinite dimensional space

- The adjoint operator S^* satisfies $S^* e_n = e_{n-1}$ for $n = 1, 2, 3, \ldots$ and $S^* e_0 = 0$.
- the operator S is $1 - 1$ but not onto.
- the operator S^* is onto but not $1 - 1$.
- $S^* S = I$.
- $SS^* = P$ where $P(e_n) = e_n$ for $e_n : n = 1, 2, 3, 4, \ldots$ and $P(e_0) = 0$.
- $\|Sx\| = \|x\|$ for every $x \in H$.
- $(S^*)^n x \to 0$ for every $x \in H$.

An operator on an infinite dimensional space

The matrix of S is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the matrix of S^* is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

An operator on an infinite dimensional space

Definition

The spectrum of an operator T is the set

$$\{\lambda \in \mathbb{C} : T - \lambda I \ \text{ not } \ \text{invertible}\}.$$

Example

If T is an operator on a finite dimensional space over $\mathbb{C}$, the spectrum is the set of its eigenvalues.

An operator on an infinite dimensional space

Theorem

The spectrum of S is $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

If $\lambda \in \mathbb{C}$, $|\lambda| < 1$ then λ is an eigenvalue of S^*.

Remark

S has no eigenvalues.

Invariant subspaces

Definition

Let T be an operator on a Banach space X. A subspace V of X is invariant if $Tx \in V$ for all $x \in V$.

Example

If T is an operator on a finite dimensional space over $\mathbb{C}$, and v is an eigenvector of T, then the space $\{\mu v : \mu \in \mathbb{C}\}$ is an invariant subspace for T.

Question

Let X be a separable Banach space. Does every operator on X have a closed invariant subspace, different from $\{0\}$ and X?

Invariant subspaces

Theorem (Enflo, 1975)

There exists an infinite dimensional separable Banach space X, and an operator T on X with no invariant closed subspace other than $\{0\}$ and X.

Theorem (Argyros-Haydon, 2011)

There exists an infinite dimensional separable Banach space X, such that every operator on X has non trivial closed invariant subspace.

The answer to the following question is unknown.

Question

Let H be a separable Hilbert space (i.e. ℓ^2). Does every operator on H have a closed invariant subspace, different from $\{0\}$ and H?

An operator on an infinite dimensional space

Another representation of the operator S.

Definition

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : \|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{ix})|^2 dx < +\infty\}.$$

$L^2(\mathbb{T})$ is a Hilbert space for the scalar product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})}dx$$

An operator on an infinite dimensional space

Definition

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}.$$

Let $T : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the map

$$Tf = \zeta_1 f$$

where $\zeta_1(e^{ix}) = e^{ix}$.
We have

$$S\mathcal{F} = \mathcal{F}T$$

where $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is the Fourier transform.

An operator on an infinite dimensional space

Theorem (Beurling)

A closed nonzero subspace $E \subseteq H^2(\mathbb{T})$ is T-invariant if and only if there exists $\phi \in H^2(\mathbb{T})$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$. Moreover, ϕ is essentially unique in the sense that if $E = \psi H^2(\mathbb{T})$ where $|\psi| = 1$ a.e. then $\frac{\phi}{\psi}$ is (a.e. equal to) a constant (of modulus 1).

C^*-algebras

Definition

Let $\mathcal{A}$ be a Banach algebra. An involution on $\mathcal{A}$ is a map $a \to a^*$ on $\mathcal{A}$ such that

- $(a+b)^* = a^* + b^*$
- $(\lambda a)^* = \overline{\lambda} a^*, \ \ \lambda \in \mathbb{C}$
- $a^{**} = a$
- $(ab)^* = b^* a^*.$

C^*-algebras

Definition

A C^*-algebra is a Banach algebra with an involution satisfying

$$\|a^* a\| = \|a\|^2.$$

C^*-algebras

Examples

- $\mathbb{C}$
  $\|z\| = |z|$
  $z^* = \bar{z}$.
- $C(X)$ for X compact, $C_0(X)$ for X locally compact
  $\|g\| = \sup_{x\in X} |g(x)|$,
  $g^*(x) = \overline{g(x)}$.
- $\mathcal{B}(H)$, for a Hilbert space H
  $\|T\| = \sup_{x\in H, \|x\|\le 1} \|Tx\|$
  $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

C^*-algebras

If H is a Hilbert space, $\mathcal{B}(H)$ is the space of bounded linear operators on H.

Theorem

Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

Crossed and semicrossed products

X locally compact Hausdorff $\phi : X \to X$ homeomorphism.
Let $\alpha : C_0(X) \to C_0(X)$, $f \mapsto f \circ \phi$.
We obtain an action of $\mathbb{Z}$ on $C_0(X)$.

$$f \mapsto f \circ \phi^n.$$

Take the linear space of finite sums

$$\sum_{n \in \mathbb{Z}} U^n f_n$$

$f_n \in C_0(X)$.
Define a multiplication by

$$U^n f U^m g = U^{n+m}(\alpha^m(f)g).$$

Crossed and semicrossed products

Definition

The crossed product $\mathcal{A} = C_0(X) \rtimes_\alpha \mathbb{Z}$, is the completion of this space in a suitable norm. It admits a faithful representation in some $\mathcal{B}(H)$.

If

$$A = \sum_{\mathrm{n}\in\mathbb{Z}} U^{\mathrm{n}} f_{\mathrm{n}},$$

the f_n's are the Fourier coefficients of A.

Crossed and semicrossed products

Question

Study properties of the algebra in terms of the dynamical system.

This construction was used to construct factors of type II.

More general construction:

$\mathcal{A}$ a C^*-algebra and G a group acting on $\mathcal{A}$.

$$\mathcal{A} \rtimes_\alpha G.$$

Crossed and semicrossed products

Arveson introduced semicrossed products to study dynamical systems.
In the construction above, only positive powers are considered.

Question

Study properties of the algebra in terms of the dynamical system.

Example

Characterize the radical.

Discrete groups and operator algebras

Definition

Let G be a discrete, countable, $\ell^2(G)$ the Hilbert space of $f : G \to \mathbb{C}$, with inner product

$$\langle f, g \rangle = \sum_G f(x)\overline{g(x)}.$$

The representation λ defined by:

$$\lambda(y)f(x) = f(y^{-1}x)$$

is called the left regular representation of G.

Discrete groups and operator algebras

Definition

The reduced C^*-algebra $C_r^*(G)$ of G is the norm closure of the linear span of $\{\lambda(x) : x \in G\}$. It is a subalgebra of $\mathcal{B}(\ell^2(G))$.

Theorem (Pimsner-Voiculescu, 1982)

$C_r^*(\mathbb{F}_m)$ is not isomorphic to $C_r^*(\mathbb{F}_n)$ for $n \neq m$.

vN(G)

If H is a Hilbert space the weak operator topology (WOT) on $\mathcal{B}(H)$ is the topology defined by the family of seminorms $\{p_{x,y}\}_{x,y \in H}$ with $p_{x,y}(T) = |\langle Tx, y\rangle|$.

Definition

The von Neumann algebra vN(G) of G is the WOT closure of the linear span of $\{\lambda(x) : x \in G\}$. It is a subalgebra of $\mathcal{B}(\ell^2(G))$.

vN(G)

Question

Is vN$(\mathbb{F}_n)$ *isomorphic to* vN$(\mathbb{F}_m)$ *for $n \neq m$?*
In particular, is vN$(\mathbb{F}_2)$ *isomorphic to* vN$(\mathbb{F}_3)$*?*