Introduction to C* Algebras III - Tensor products

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October 30, 2020

Definition of the algebraic tensor product

The algebraic tensor product of two linear spaces is a linear space with the property that it linearises bilinear maps to all linear spaces:

$$bil(E_1 \times E_2, F) \simeq \mathcal{L}(E_1 \odot E_2, F)$$

Proposition

Given two $\mathbb{K}$-linear spaces E_1 and E_2, there exists a linear space $E_1 \odot E_2$ equipped with a bilinear map $\otimes : E_1 \times E_2 \to E_1 \odot E_2$ having the 'universal property':

For every $\mathbb{K}$-linear space F and every bilinear map $b : E_1 \times E_2 \to F$ there is a unique linear map $B : E_1 \odot E_2 \to F$ satisfying $B(x \otimes y) = b(x, y)$ for every $x \in E_1, y \in E_2$.

The linear space $E_1 \odot E_2$ is unique up to linear isomorphisms (see below).

Definition

The Algebraic tensor product of E_1 and E_2 is $(E_1 \odot E_2, \otimes)$.

Universal property of $(E_1 \odot E_2, \otimes)$

For every $\mathbb{K}$-linear space F and every bilinear map $b : E_1 \times E_2 \to F$ there is a unique linear map $B : E_1 \odot E_2 \to F$ satisfying $B(x \otimes y) = b(x, y)$ for every $x \in E_1, y \in E_2$:
Thus if $\otimes : E_1 \times E_2 \to E_1 \odot E_2 : (x, y) \to x \otimes y$, the diagram

$$\begin{tikzcd} E_1 \times E_2 \arrow[r, "\otimes"] \arrow[d, "b"'] & E_1 \odot E_2 \arrow[dl, "B"] \\ F & \end{tikzcd}$$

commutes.

$$bil(E_1 \times E_2, F) \simeq \mathcal{L}(E_1 \odot E_2, F) .$$

Uniqueness of $(E_1 \odot E_2, \otimes)$

Let G_1, G_2 be two $\mathbb{K}$-linear spaces and $\pi_i : E_1 \times E_2 \to G_i$ $(i = 1, 2)$ bilinear maps. Assume that: for every linear space F and bilinear map $b : E_1 \times E_2 \to F$, there are unique linear maps $B_i : G_i \to F$ $(i = 1, 2)$ so that $B_i \circ \pi_i = b$ $(i = 1, 2)$.

$$
\begin{tikzcd}
G_1 \arrow[dr, "\exists!\, B_1"'] & E_1 \times E_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "b"'] & G_2 \arrow[dl, "\exists!\, B_2"] \\
& F &
\end{tikzcd}
$$

Then $\exists$ a linear isomorphism $\phi : G_1 \to G_2$ such that $\phi \circ \pi_1 = \pi_2$:

$$
\begin{tikzcd}
& E_1 \times E_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 \arrow[rr, "\phi"'] & & G_2
\end{tikzcd}
$$

Tensor products of linear spaces

If E_1, E_2 are $\mathbb{K}$-linear spaces, can consider $E_i \hookrightarrow \mathbb{K}^{X_i}$ where X_i is a set (e.g. an algebraic basis, or an o.n. basis, of E_i). Define

$$\xi \otimes \eta : X_1 \times X_2 \to \mathbb{K} : (s,t) \to \xi(s)\eta(t).$$

A Realisation of the algebraic tensor product

$$E_1 \odot E_2 \simeq \operatorname{span}\{\xi \otimes \eta : \xi \in E_1, \eta \in E_2\} \subseteq \mathbb{K}^{X_1 \times X_2}.$$

Remark $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,
$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, $(\lambda x) \otimes y = \lambda(x \otimes y) = x \otimes (\lambda y)$.

Remark If $\{x_i : i \in I\} \subseteq E_1$ and $\{y_j : j \in J\} \subseteq E_2$ are linearly independent, then so is $\{x_i \otimes y_j : (i,j) \in I \times J\} \subseteq E_1 \odot E_2$.

Remark $\mathbb{K} \odot E \simeq E : \lambda \otimes x \mapsto \lambda x$ and
$\mathbb{K}^n \odot E \simeq E^n : e_i \otimes x \mapsto (0 \ldots, 0, x, 0 \ldots, 0)$. Also
$M_n(\mathbb{K}) \odot E \simeq M_n(E) : e_{ij} \otimes x \mapsto [x_{kl}]$ with $x_{ij} = x$ and the rest=0.

Another realisation: Finite rank operators

Notation: for $x \in H$ (Hilbert) let $x^* \in H^*$ be $x^* : z \to \langle z, x \rangle$.
Every $u = \sum_i x_i^* \otimes y_i \in H^* \odot K$ (here $x^* : H \ni z \to \langle z, x \rangle \in \mathbb{C}$) defines a map

$$\tilde{u} : H \to K : z \mapsto \sum_i \langle z, x_i \rangle y_i := \sum_i y_i x_i^*(z).$$

Conversely every *bounded finite rank* operator $T : H \to K$ is of the form $T = \tilde{u}$ where $u \in H^* \odot K$. The map $u \to \tilde{u}$ is a linear space isomorphism.

Conclusion: $H^* \odot K \simeq \mathcal{BF}(H, K)$.

Also, (for $H = K$) if $\omega_{y,x} : \mathcal{B}(H) \to \mathbb{C} : T \mapsto \langle Ty, x \rangle$ and $\mathcal{B}_\sim(H)$ is their span, then $H^* \odot H \simeq \mathcal{B}_\sim(H)$, via the mapping
$\sum_i x_i^* \otimes y_i \mapsto \sum_i \omega_{y_i, x_i}$
Note that if $u = x^* \otimes y \in H^* \odot H$ then $\omega_{y,x}(T) = \mathrm{Tr}(T\tilde{u})$ for all $T \in \mathcal{B}(H)$.

Tensor products of Hilbert spaces

Definition

Let H_1, H_2 be Hilbert spaces. On $H_1 \odot H_2$ put

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{hs} = \langle x_1, x_2 \rangle_{H_1} \cdot \langle y_1, y_2 \rangle_{H_2} .$$

This gives a well-defined scalar product. (Exercise!) We define

$$\begin{aligned} H_1 \otimes H_2 &:= \overline{(H_1 \odot H_2, \|\cdot\|_{hs})} \\ &:= H_1 \otimes_2 H_2 . \end{aligned}$$

If $\{e_i\}_I$ is an orthonormal basis of H_1 and $\{f_j\}_J$ an o.n. basis of H_2, then $H_1 \otimes H_2$ has o.n. basis $\{e_i \otimes f_j\}_{I \times J}$.

Remark If $\dim H_1 < \infty$ and $\dim H_2 < \infty$, then $H_1 \odot H_2 = H_1 \otimes H_2$.

Example $L^2(\mu) \otimes L^2(\nu) = L^2(\pi)$ where π is the product measure.

Example $\mathbb{C}^k \otimes \mathbb{C}^n = \mathbb{C}^k \odot \mathbb{C}^n = \mathbb{C}^{nk}$.

Operators on tensor products of Hilbert spaces

If $A \in \mathcal{B}(H_1)$ and B $\in \mathcal{B}(H_2)$ want to define an operator $A \hat{\otimes} B : H_1 \otimes H_2 \to H_1 \otimes H_2$:
First define

$$A \hat{\otimes} B : H_1 \odot H_2 \to H_1 \odot H_2 : \sum_i x_i \otimes y_i \to \sum_i Ax_i \otimes By_i$$

Check this is well-defined.

Now show that $\left\|\sum_i Ax_i \otimes By_i\right\| \leq \|A\| \, \|B\| \left\|\sum_i x_i \otimes y_i\right\|$.

Hence $A \hat{\otimes} B$ extends to $A \hat{\otimes} B : H_1 \otimes H_2 \to H_1 \otimes H_2$ with $\left\|A \hat{\otimes} B\right\| \leq \|A\| \, \|B\|$. (In fact equality)

Thus have embedding

$$\mathcal{B}(H_1) \odot \mathcal{B}(H_2) \to \mathcal{B}(H_1 \otimes H_2) : A \otimes B \mapsto A \hat{\otimes} B \, .$$

Verify this is 1-1.
(Henceforth identify $A \otimes B$ with $A \hat{\otimes} B$).

Tensor products of *-algebras

If A and B are *-algebras, make $A \odot B$ into a *-algebra by $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ and $(a \otimes b)^* := a^* \otimes b^*$.

For example if $A = M_n = M_n(\mathbb{C})$ then $M_n \odot B \simeq M_n(B)$ as follows:

$$M_n(B) \ni [b_{ij}] \mapsto \sum_{i,j} e_{ij} \otimes b_{ij} \in M_n \odot B \,.$$

(In particular, when B is a C* algebra, $M_n \odot B$ inherits the norm of $M_n(B)$ and becomes a C*-algebra - but notice that $\dim M_n < \infty$.)

Thus if $\Phi : B \to C$ is a linear map, the map $\Phi_n : M_n(B) \to M_n(C)$ is just

$$\mathrm{id}_{M_n} \otimes \Phi : M_n \odot B \to M_n \odot C : a \otimes b \to a \otimes \Phi(b) \,.$$

Thus the map Φ is completely positive if $\mathrm{id}_{M_n} \otimes \Phi$ is positive for all $n \in \mathbb{N}$ and is completely bounded if $\sup_n \left\| \mathrm{id}_{M_n} \otimes \Phi \right\| < \infty$. We write $\Phi \in \mathcal{CB}(B, C)$.

Reminder: Stinespring's dilation theorem

Theorem (Stinespring)

If $\Phi : \mathcal{A} \to \mathcal{B}(H)$ is a completely positive [unital] map from a [unital] C-algebra $\mathcal{A}$ to $\mathcal{B}(H)$, then*

$$\Phi(a) = V^*\pi(a)V \quad \textit{for all } \; a \in \mathcal{A}.$$

*where π is a *-representation of $\mathcal{A}$ on the Hilbert space H_π and $V : H \to H_\pi$ is bounded.*

When $\mathcal{A}$ and ϕ are unital, V is an isometry and the representation π is called a dilation of Φ via the 'embedding' $V : H \to H_\pi$.
[The dilation is unique under a minimality condition.]

Remark When $H = \mathbb{C}$ this reduces to the GNS construction
(with $V : \mathbb{C} \to H_\pi : 1 \to \xi_\Phi$).

Extension theorems

Definition

A (concrete) operator system $\mathcal{S}$ is a linear subspace of a unital C* algebra $\mathcal{A}$ which is unital and selfdajoint, i.e. $\mathbf{1} \in \mathcal{S}$ and $s \in \mathcal{S} \Rightarrow s^* \in \mathcal{S}$.

Theorem (Arveson)

If $\Phi : \mathcal{S} \to \mathcal{B}(H)$ is a completely positive unital map defined on an operator system $\mathcal{S} \subseteq \mathcal{A}$, then Φ has a completely positive extension $\Psi : \mathcal{A} \to \mathcal{B}(H)$.

Theorem (Wittstock)

Let $\mathcal{M}$ be a subspace of a unital C-algebra $\mathcal{A}$. If $\Phi : \mathcal{M} \to \mathcal{B}(H)$ is completely bounded map, then Φ has a completely bounded extension $\Psi : \mathcal{A} \to \mathcal{B}(H)$ with $\|\Psi\|_{cb} = \|\Phi\|_{cb}$.*

Factorisation theorem

Theorem (Haagerup, Paulsen, Wittstock)

If $\Phi : \mathcal{A} \to \mathcal{B}(H)$ is a completely bounded map from a unital C-algebra $\mathcal{A}$ to $\mathcal{B}(H)$, then*

$$\Phi(a) = V^*\pi(a)W \quad \textit{for all } \; a \in \mathcal{A}.$$

*where π is a *-representation of $\mathcal{A}$ on the Hilbert space H_π and $V, W : H \to H_\pi$ are bounded, with $\|\Phi\|_{cb} = \|V\|\,\|W\|$..*

$$\begin{tikzcd} A \arrow[r, "\pi"] & \mathcal{B}(H_\pi) \arrow[d, "x \to V^*xW"] \\ A \arrow[u, "\mathrm{id}"] \arrow[r, "\Phi"] & \mathcal{B}(H) \end{tikzcd}$$

Back to tensor products: C*-cross norms

Let A and B be C*-algebras.

A cross-norm on $A \odot B$ is a norm $\|\cdot\|_\gamma$ s.t. $\|a \otimes b\|_\gamma = \|a\| \, \|b\|$.

A C*-cross-norm is a cross norm $\|\cdot\|_\gamma$ satisfying $\|xy\|_\gamma \leq \|x\|_\gamma \|y\|_\gamma$ and the C*-property $\|x^* x\|_\gamma = \|x\|_\gamma^2$.

Do such norms exist?

Minimal tensor norm

Take faithful reps $\pi : A \to \mathcal{B}(H_1)$ and $\rho : B \to \mathcal{B}(H_2)$ and define $\pi \otimes \rho : A \odot B \to \mathcal{B}(H_1 \otimes H_2) : a \otimes b \mapsto \pi(a) \otimes \rho(b)$.

This is a faithful rep of the *-algebra, so defines a norm $\|x\|_{\min} := \|(\pi \otimes \rho)(x)\|_{\mathcal{B}(H_1 \odot H_2)}$. Clearly a cross norm.

It is independent of the reps π, ρ and is minimal:

Theorem (Takesaki)

If $\|\cdot\|_\gamma$ is a C-cross norm on $A \odot B$, then $\|x\|_\gamma \geq \|x\|_{\min}$.*

The completion is denoted $A \otimes_{\min} B$ or $A \otimes B$.

The min norm is *injective:* If $A \subseteq C$ and $B \subseteq D$ then $A \otimes B \subseteq C \otimes D$.

Thus the notion extends to subspaces:

Minimal tensor norm

Definition

Consider two (concrete) operator spaces $X \subseteq \mathcal{B}(\mathcal{H})$ and $Y \subseteq \mathcal{B}(\mathcal{K})$ for $\mathcal{H}, \mathcal{K}$ Hilbert spaces. Their spatial or minimal tensor product is defined to be the completion of their algebraic tensor product in the norm of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ through the inclusion:

$$X \otimes Y \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$$

i.e. $X \otimes_{\min} Y := \overline{X \otimes Y}^{norm}$.

The space $X \otimes_{\min} Y$ is independent of the embeddings $X \subseteq \mathcal{B}(\mathcal{H})$ and $Y \subseteq \mathcal{B}(\mathcal{K})$.

Minimal tensor norm

Proposition

For any $x = \sum a_1 \otimes b_i \in X \otimes Y$ *we have*

$$\|x\|_{\min} = \sup \left\{ \left\| \sum v(a_i) \otimes w(b_i) \right\|_{M_{nm}} \right\}$$

where the supremum runs over $n, m \geq 1$ *and all pairs of* $v \in Ball(\mathcal{CB}(X, M_n))$ *and* $w \in Ball(\mathcal{CB}(Y, M_m))$.

Maximal tensor norm

If A and B are C* algebras, define a norm on $A \odot B$ by

$$\|x\|_{\max} := \sup\{\|\pi(x)\| : \pi : \text{*-rep. of } A \odot B\}\,.$$

This is finite (why?), and is a norm because it dominates $\|\cdot\|_{\min}$.

The completion is denoted $A \otimes_{\max} B$.

If $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : B \to \mathcal{B}(\mathcal{H})$ are *-reps (same H) with commuting ranges, then the map

$$\sum a_i \otimes b_i \mapsto \sum \pi(a_i)\rho(b_i) : A \odot B \to \mathcal{B}(H)$$

extends to a *-rep of $A \otimes_{\max} B$ on same H.

References [1, Chapter 3], [2, Appendix T].

Bibliography

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