Introduction to C* Algebras II

Aristides Katavolos

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C^*-algebras

Definition

(a) A Banach algebra $\mathcal{A}$ is a complex algebra equipped with a complete norm which is sub-multiplicative:

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } \ a, b \in \mathcal{A}.$$

(b) An involution is a map on $\mathcal{A}$ such that
$(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.
(c) A C^*-algebra $\mathcal{A}$ is a Banach algebra equipped with an involution $a \to a^*$ satisfying the C^*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \ a \in \mathcal{A}.$$

A morphism Φ between C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ which preserves multiplication and involution.
A representation π of $\mathcal{A}$ on a Hilbert space H is a morphism $\pi : \mathcal{A} \to \mathcal{B}(H)$.

Gelfand theory for commutative C*-algebras

Theorem (Gelfand-Naimark 1)

Every commutative C-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is the set of nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. For each $a \in \mathcal{A}$ the function $\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} : \phi \to \phi(a)$ is in $C_0(\hat{\mathcal{A}})$. The Gelfand transform:*

$$\mathcal{A} \to C_0(\hat{\mathcal{A}}) : \ a \to \hat{a}$$

*is an isometric *-isomorphism. The space $\hat{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ is unital.*

Positivity

Recall The spectrum $\sigma(a)$ of an element a of a C*-algebra $\mathcal{A}$ is

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ not invertible in } \mathcal{A}\}$$

(calculated in the unitisation for non-unital $\mathcal{A}$.)

Definition

An element $a \in \mathcal{A}$ is **positive** if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$.
We write $\mathcal{A}_+ = \{a \in \mathcal{A} : a \geq 0\}$.
If a, b are selfadjoint, we define $a \leq b$ by $b - a \in \mathcal{A}_+$.

Examples

- In $C(X)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.
- In $M_n(\mathbb{C})$: $T \geq 0$ iff T is diagonalisable and has nonnegative e-values, equivalently iff it is positive semidefinite, i.e. $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{C}^n$.
- In $\mathcal{B}(\mathcal{H})$: $T \geq 0$ iff $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.

Positivity

Proposition

Every positive element has a unique positive square root. In fact

$$a \in \mathcal{A}_+ \quad \Longleftrightarrow \quad \textit{there exists } \; b \in \mathcal{A}_+ \; \textit{such that} \; a = b^2 \quad (= b^* b).$$

Theorem

In any C-algebra, any element of the form $b^* b$ is positive.*

For the proof of the Theorem, we need

Positivity

Proposition

For any C-algebra, the set $\mathcal{A}_+$ is a* cone*:*

$$a, b \in \mathcal{A}_+,\ \lambda \geq 0 \quad \Rightarrow \quad \lambda a \in \mathcal{A}_+, a + b \in \mathcal{A}_+.$$

Lemma

In a unital C-algebra, if $x = x^*$ and $\|x\| \leq 1$, then*

$$x \geq 0 \quad \Longleftrightarrow \quad \|1 - x\| \leq 1.$$

Positivity

Definition

A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is positive if $a \in \mathcal{A}_+ \Rightarrow \Phi(a) \in \mathcal{B}_+$.

Remark

Any morphism $\pi : \mathcal{A} \to \mathcal{B}$ between C-algebras is positive.*

Indeed, $\pi(a^*a) = \pi(a)^*\pi(a) \geq 0$.

Remark

If $a = a^*$ then $\qquad -\|a\| 1 \leq a \leq \|a\| 1.$

States

Proposition

If ϕ is a positive linear form, then $\phi(x^*) = \overline{\phi(x)}$ for all $x \in \mathcal{A}$.
The Cauchy-Schwarz inequality holds:

$$|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b) \text{ for all } a, b \in \mathcal{A}.$$

Hence the map $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C} : \langle a, b \rangle = \phi(b^*a)$ is a semi-inner product; it is an inner product iff ϕ is faithful.

Proposition

Every positive linear form is continuous. When $\mathcal{A}$ is unital and ϕ is positive, $\|\phi\| = \phi(1)$.

Definition

A **state** on a C*-algebra $\mathcal{A}$ is a positive linear map of norm 1, i.e. $\phi : \mathcal{A} \to \mathbb{C}$ linear such that $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\|\phi\| = 1$. A state is called **faithful** if $\phi(a^*a) > 0$ whenever $a \neq 0$.

States

Examples

• On $\mathcal{B}(\mathcal{H})$, $\phi(T) = \langle T\xi, \xi\rangle$ for a unit vector $\xi \in \mathcal{H}$,
or $\phi(T) = \sum_i p_i \langle T\xi_i, \xi_i\rangle$ where $\{\xi_i\}$ orthonormal and $p_i \geq 0$ with $\sum p_i = 1$ ('density matrix').

• On $C(K)$, $\phi(f) = f(t)$ for $t \in K$,
or $\phi(f) = \int f d\mu$ for a probability measure μ.

• For a C*-algebra $\mathcal{A}$, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation and $\xi \in \mathcal{H}$ a unit vector, $\phi(a) = \langle \pi(a)\xi, \xi\rangle$.

The GNS construction

Conversely to the last example,

Theorem (Gelfand, Naimark, Segal)

For every state f on a C-algebra $\mathcal{A}$ there is a triple $(\pi_f, \mathcal{H}_f, \xi_f)$ where π_f is a representation of $\mathcal{A}$ on $\mathcal{H}_f$ and $\xi_f \in \mathcal{H}_f$ a* cyclic [1] *unit vector such that*

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle \quad \textit{for all } \ a \in \mathcal{A}.$$

The GNS triple $(\pi_f, \mathcal{H}_f, \xi_f)$ is uniquely determined by this relation up to unitary equivalence.

[1] i.e. $\pi_f(\mathcal{A})\xi_f$ is dense in $\mathcal{H}_f$.

GNS: sketch of proof

(for unital $\mathcal{A}$).

1. Consider the linear space $\mathcal{A}$.
2. Equip it with the semi-inner product $\langle a, b\rangle_0 := \phi(b^* a)$.
   When $\mathcal{A} = C(X)$ we have $\langle a, b\rangle_0 = \int_X a(t)\overline{b(t)}d\mu(t)$.
3. Since ϕ is positive, $\langle a, a\rangle_0 = \phi(a^* a) \geq 0$.
   By Cauchy-Schwarz the set $\mathcal{N}_\phi = \mathcal{N} := \{u \in \mathcal{A} \ : \langle u, u\rangle_0 = 0\}$ is a linear space.
4. Let $H_{0\phi} := \mathcal{A}/\mathcal{N}$ and define H_ϕ $(= L^2(\mu))$ to be the completion of $H_{0\phi}$ with respect to $\|[a]\|_\phi := \sqrt{\langle a, a\rangle_0}$.
   (here $[a] = a + \mathcal{N}$, $a \in \mathcal{A}$).

GNS: sketch of proof II

5 $\mathcal{A}$ acts on the linear space $\mathcal{A}$ as: $\pi_0(a)(b) = ab$.

6 Since $\pi_0(a)(\mathcal{N}) \subseteq \mathcal{N}$, the map $\pi_0(a)$ induces $\pi_1(a)$ on $H_{0\phi} = \mathcal{A}/\mathcal{N}$.

7 Show that $\|\pi_1(a)([b])\|_\phi \leq \|a\| \, \|[b]\|_\phi$.
[When $\mathcal{A} = C(X)$, $\|ab\|_2 \leq \|a\|_\infty \|b\|_2$.]
It follows that $\pi_1(a)$ extends to a bounded operator $\pi_\phi(a)$ on H_ϕ.

Easy to verify: $\pi_\phi : a \to \pi_\phi(a) : \mathcal{A} \to \mathcal{B}(H_\phi)$ is a *-representation.
[When $\mathcal{A} = C(X)$, the map $\pi_\phi(a)$ is a multiplication operator on $L^2(\mu)$, i.e. $(\pi_\phi(a)b)(t) = a(t)b(t)$ for μ-almost all $t \in X$.]

8 Let $\xi_\phi = [\mathbf{1}_\mathcal{A}]$. Then

$$\begin{aligned} \left\langle \pi_\phi(a)\xi_\phi, \xi_\phi \right\rangle_{H_\phi} &= \left\langle \pi_\phi(a)[\mathbf{1}], [\mathbf{1}] \right\rangle_{H_\phi} \\ &= \langle a, \mathbf{1} \rangle_{H_\phi} = \phi(\mathbf{1}^* a) = \phi(a) . \qquad \square \end{aligned}$$

The universal representation

Theorem (Gelfand, Naimark)

For every C-algebra $\mathcal{A}$ there exists a representation $(\pi, \mathcal{H})$ which is one to one (called* faithful*).*

Idea of proof Enough to assume $\mathcal{A}$ unital. Let $\mathcal{S}(\mathcal{A})$ be the set of all states. For each $f \in \mathcal{S}(\mathcal{A})$ consider $(\pi_f, \mathcal{H}_f)$ and ‘add them up’ to obtain $(\pi, \mathcal{H})$. Why is this faithful? Because

Lemma

For each nonzero $a \in \mathcal{A}$ there exists $f \in \mathcal{S}(\mathcal{A})$ such that $f(a^*a) > 0$.

... and then

$$\left\|\pi(a)\xi_f\right\|^2 = \left\langle \pi(a^*a)\xi_f, \xi_f \right\rangle = \left\langle \pi_f(a^*a)\xi_f, \xi_f \right\rangle = f(a^*a) > 0$$

so $\pi(a) \neq 0$.

Completely positive maps

Reminder A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C* algebras is positive iff

$$a \geq 0 \;\Rightarrow\; \Phi(a) \geq 0.$$

For $n \in \mathbb{N}$, let $\Phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ where $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$.

It is NOT always true that Φ_n is positive.

Example Let $\Phi(a) = a^\dagger$ (transpose) on $\mathcal{A} = M_2$: clearly positive.
However, in $M_2(\mathcal{A})$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ positive, but } \Phi_2(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ not positive.}$$

Completely positive maps

Definition

A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C* algebras is called completely positive if Φ_n is positive for all $n \in \mathbb{N}$.

Examples of completely positive (cp) maps:

Every *-morphism π is positive ($\pi(a^*a) = \pi(a)^*\pi(a) \geq 0 \ \forall a$).

Hence every *-morphism is completely positive (because every π_n is a *-morphism).

Every map of the form $a \to V^*aV$ is completely positive (here $\mathcal{A} \subseteq \mathcal{B}(H)$ and $V \in \mathcal{B}(H)$).

Hence every $a \to V^*\pi(a)V : a \to \pi(a) \to V^*\pi(a)V$ is completely positive.

There are no others:

Stinespring's dilation theorem

Theorem (Stinespring)

If $\Phi : \mathcal{A} \to \mathcal{B}(H)$ is a completely positive [unital] map from a [unital] C-algebra $\mathcal{A}$ to $\mathcal{B}(H)$, then*

$$\Phi(a) = V^*\pi(a)V \quad \textit{for all } \ a \in \mathcal{A}.$$

*where π is a *-representation of $\mathcal{A}$ on the Hilbert space H_π and $V : H \to H_\pi$ is bounded.*

When $\mathcal{A}$ and ϕ are unital, V is an isometry and the representation π is called a dilation of Φ via the 'embedding' $V : \mathcal{H} \to \mathcal{K}$.
[The dilation is unique under a minimality condition.]

Remark When $H = \mathbb{C}$ this reduces to the GNS construction (with $V : \mathbb{C} \to H_\pi : 1 \to \xi_\Phi$).